MATH 521A: Abstract Algebra

Homework 9 Solutions

1. Calculate the multiplication table for $\mathbb{Z}_5[x]/(x^2+4x+1)$.

   The elements of this ring are $\{[ax+b] : a, b \in \mathbb{Z}_5\}$, and the key property is that $[x^2+4x+1] = [0]$, i.e. $[x^2] = [-4x-1]$. Hence the multiplication table is given by:

   $[ax+b][a'x+b'] = [aa'x^2 + (ab'+ba')x + bb'] = [(ab'+ba'-4aa')x + (bb'-aa')]$.

2. Calculate the multiplication table for $\mathbb{Z}_5[x]/(x^2+3x+1)$.

   The elements of this ring are $\{[ax+b] : a, b \in \mathbb{Z}_5\}$, and the key property is that $[x^2+3x+1] = [0]$, i.e. $[x^2] = [-3x-1]$. Hence the multiplication table is given by:

   $[ax+b][a'x+b'] = [aa'x^2 + (ab'+ba')x + bb'] = [(ab'+ba'-3aa')x + (bb'-aa')]$.

3. Calculate the multiplication table for $\mathbb{Z}_5[x]/(x^2)$.

   The elements of this ring are $\{[ax+b] : a, b \in \mathbb{Z}_5\}$, and the key property is that $[x^2] = [0]$. Hence the multiplication table is given by:

   $[ax+b][a'x+b'] = [aa'x^2 + (ab'+ba')x + bb'] = [(ab'+ba')x + bb']$.

4. Calculate the multiplication table for $\mathbb{Q}[x]/(x^2+2)$.

   The elements of this ring are $\{[ax+b] : a, b \in \mathbb{Q}\}$, and the key property is that $[x^2+2] = [0]$, i.e. $[x^2] = [-2]$. Hence the multiplication table is given by:

   $[ax+b][a'x+b'] = [aa'x^2 + (ab'+ba')x + bb'] = [(ab'+ba')x + (bb'-2aa')]$.

5. Calculate the multiplication table for $\mathbb{Q}[x]/(x^2-2)$.

   The elements of this ring are $\{[ax+b] : a, b \in \mathbb{Q}\}$, and the key property is that $[x^2-2] = [0]$, i.e. $[x^2] = [2]$. Hence the multiplication table is given by:

   $[ax+b][a'x+b'] = [aa'x^2 + (ab'+ba')x + bb'] = [(ab'+ba')x + (bb'+2aa')]$.

6. Calculate the multiplication table for $\mathbb{Q}[x]/(x^2-1)$.

   The elements of this ring are $\{[ax+b] : a, b \in \mathbb{Q}\}$, and the key property is that $[x^2-1] = [0]$, i.e. $[x^2] = [1]$. Hence the multiplication table is given by:

   $[ax+b][a'x+b'] = [aa'x^2 + (ab'+ba')x + bb'] = [(ab'+ba')x + (bb'+aa')]$.

7. * For each of the rings in problems 1-6, calculate the (multiplicative) inverse of $[x-1]$, or prove it does not exist.

   $\mathbb{Z}_5[x]/(x^2+4x+1)$: $[0x+1] = [ax+b][1x-1] = [(-a+b-4a)x+(-b-a)]$. Equating coefficients, we see that $b=0$ and $-a=1$. We verify that $[-x+0][x-1] = [-x^2+x] = [(4x+1)+x] = [1]$, so the inverse we seek is $[-x]$.

   $\mathbb{Z}_5[x]/(x^2+3x+1)$: $[0x+1] = [ax+b][1x-1] = [(-a+b-3a)x+(-b-a)]$. Equating coefficients, we see that $b+a=0$ and $-b-a=1$. Adding, we have $0=1$; since this is impossible, there is no inverse to $[x-1]$ in this ring.

   $\mathbb{Z}_5[x]/(x^2)$: $[0x+1] = [ax+b][1x-1] = [(-a+b)x+(-b)]$. Equating coefficients, we see that $-a+b=0$ and $-b=1$. This gives $a=b=-1$. We verify that $[-x-1][x-1] = [-x^2+1] = [1]$, so the inverse we seek is $[-x-1]$.

   $\mathbb{Q}[x]/(x^2+2)$: $[0x+1] = [ax+b][1x-1] = [(-a+b)x+(-b-2a)]$. Equating coefficients, we see that $-a+b=0$ and $-b-2a=1$. This gives $a=b=-\frac{1}{3}$. We verify that

$[-\frac{1}{3}x - \frac{1}{3}][x-1] = [-\frac{1}{3}x^2 + \frac{1}{3}] = [1]$, so the inverse we seek is $[-\frac{1}{3}x - \frac{1}{3}]$.
$\mathbb{Q}[x]/(x^2-2)$: $[0x+1] = [ax+b][1x-1] = [(-a+b)x + (-b+2a)]$. Equating coefficients, we see that $-a+b=0$ and $-b+2a=1$. This gives $a=b=1$. We verify that $[x+1][x-1] = [x^2-1] = [1]$, so the inverse we seek is $[x+1]$.
$\mathbb{Q}[x]/(x^2-1)$: $[0x+1] = [ax+b][1x-1] = [(-a+b)x + (-b+a)]$. Equating coefficients, we see that $-a+b=0$ and $-b+a=1$. Adding, we have $0=1$; since this is impossible, there is no inverse to $[x-1]$ in this ring.

8. Let $f(x), g(x), p(x) \in F[x]$, where all three polynomials are nonconstants. Suppose that $f(x)g(x) = p(x)$. Prove that $[f(x)]$ is a zero divisor in $F[x]/(p(x))$.

We have $[f(x)][g(x)] = [p(x)] = [0]$. Also, $\deg f(x) < \deg p(x)$, so $[f(x)]$ is already in standard form. Since $f(x)$ is not a constant, $[f(x)] \neq [0]$. Similarly, $[g(x)] \neq [0]$. Hence $[f(x)]$ is a zero divisor.

9. Let $f(x), p(x) \in F[x]$, where both polynomials are nonconstants. Set $g(x) = \gcd(f(x), p(x))$. Suppose that $[g(x)] \neq [0]$. Prove that $[f(x)]$ is a unit in $F[x]/(p(x))$, if and only if $g(x)$ is a constant polynomial.

Suppose first that $g(x) \in F$ is a constant polynomial. By the extended Euclidean algorithm we can find $a(x), b(x) \in F[x]$ with $g = af + bp$. Reducing this modulo p, we get $[g] = [af+bp] = [af] + [b][p] = [af] + [b][0] = [af] = [a][f]$. Since $[g] \neq [0]$, there is some $\alpha \in F$ such that $[\alpha][g(x)] = [1]$, and we have $[1] = [\alpha][g(x)] = [\alpha][a(x)][f(x)] = [\alpha a(x)][f(x)]$. Hence $[f(x)]$ is a unit in $F[x]/(p(x))$.

If instead $[f(x)]$ is a unit, then there is some $[a(x)]$ where $[fa] = [f][a] = [1]$. Hence there is some $b(x) \in F[x]$ such that $p(x)b(x) = 1 - f(x)a(x)$. Rearranging, we see that $f(x)a(x) + p(x)b(x) = 1$. Hence $\deg(g) = \deg(\gcd(a,p)) \leq \deg(1) = 0$, so $g(x)$ is constant.

10. Determine, with proof, which of the rings in problems 1-6 are integral domains, and which are fields.

$\mathbb{Z}_5[x]/(x^2+4x+1)$: We verify that x^2+4x+1 is irreducible (it has no roots in $\mathbb{Z}_5$). Let $f(x)$ be a nonzero polynomial, of degree at most 1. Since $\gcd(f, x^2+4x+1) = 1$, $[f]$ is a unit by problem 9. Hence this ring is a field.
$\mathbb{Z}_5[x]/(x^2+3x+1)$: We see that $x^2+3x+1 = (x-1)^2$. Hence, by problem 8, $[x-1]$ is a zero divisor. Hence this ring is not an integral domain, much less a field.
$\mathbb{Z}_5[x]/(x^2)$: We see that $x^2 = (x)^2$. Hence, by problem 8, $[x]$ is a zero divisor. Hence this ring is not an integral domain, much less a field.
$\mathbb{Q}[x]/(x^2+2)$: We verify that x^2+2 is irreducible, by the rational root test. Let $f(x)$ be a nonzero polynomial, of degree at most 1. Since $\gcd(f, x^2+2) = 1$, $[f]$ is a unit by problem 9. Hence this ring is a field.
$\mathbb{Q}[x]/(x^2-2)$: We verify that x^2-2 is irreducible, by the rational root test. Let $f(x)$ be a nonzero polynomial, of degree at most 1. Since $\gcd(f, x^2-2) = 1$, $[f]$ is a unit by problem 9. Hence this ring is a field.
$\mathbb{Q}[x]/(x^2-1)$: We see that $x^2-1 = (x+1)(x-1)$. Hence, by problem 8, $[x+1]$ is a zero divisor. Hence this ring is not an integral domain, much less a field.