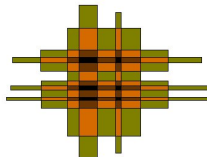


# Singular integrals in Fluid Mechanics



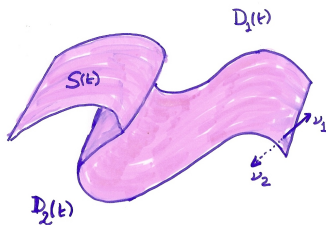
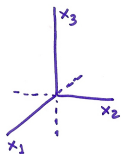
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(A.C., D. CÓRDOBA AND F. GANCEDO)

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Two fluids  
velocity  $u^j$   
pressure  $p^j$   
viscosity  $\mu^1 \neq \mu^2$   
density  $\rho^1 \neq \rho^2$

- **Porous media:** porosity  $\kappa > 0$  (homogeneous, isotropic).

- **Incompressibility** (mass conservation):

$$\operatorname{div}(u) = 0, \quad u = u^1 \chi_{D_1} + u^2 \chi_{D_2}$$

- **Darcy's Law** (conservation of momentum):

$$\frac{\mu^j}{\kappa} u^j = -\nabla p^j - (0, 0, \rho^j g) = -\nabla(p^j + g \rho^j x_3) \quad (g = \text{gravity}).$$

# Immediate consequences

- $\operatorname{div}(u) = 0 \implies (u^1 - u^2) \cdot \nu_j = 0.$
- $\operatorname{curl}(u) = \Omega \, dS$ ,  $\Omega$  is a tangent vector field at  $S$ .
- $p^1|_S = p^2|_S.$
- $\exists$  velocity potentials:  $u^j = \nabla \phi^j$

$$\frac{\mu^j}{\kappa} \phi^j = -p^j - \rho^j g x_3$$

$$u(x) = \int_S \frac{x-z}{|x-z|^3} \wedge \Omega(z) dS(z), \quad x \notin S \quad (\text{Biôt-Savart}).$$

Taking limits in each domain:

$$u^j(x) = p.v. \int_S \frac{x-z}{|x-z|^3} \wedge \Omega(z) dS(z) + \text{tangential terms}, \quad x \in S.$$

Choosing a parametrization (isothermal coordinates)

$$X(\alpha) = X(\alpha_1, \alpha_2) \quad \begin{cases} X_{\alpha_1} \cdot X_{\alpha_2} = 0 \\ \|X_{\alpha_1}\| = \|X_{\alpha_2}\| \end{cases}$$

Two scenarios:

- Periodic in the horizontal variables;
- asymptotically flat.



$$\begin{aligned}\frac{dX(\alpha)}{dt} &= p.v. \int \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge \omega(\alpha) d\alpha \\ &\quad + C_1(X(\alpha, t), t)X_{\alpha_1} + C_2(X(\alpha, t), t)X_{\alpha_2} \\ &= BR(X, \omega) + C_1X_{\alpha_1} + C_2X_{\alpha_2}.\end{aligned}$$

where

$$\text{curl}(u) = \omega(\alpha) \delta_{X-X(\alpha, t)}$$

Darcy's law:

$$\frac{\mu^j}{\kappa} \phi^j = -p^j - \rho^j g x_3, \quad u^j = \nabla \phi^j$$

We define

$$\Pi(\alpha) = \phi^1(X(\alpha, t), t) - \phi^2(X(\alpha, t), t).$$

Then we have

- $\omega(\alpha) = \frac{\partial \Pi}{\partial \alpha_2} X_{\alpha_1} - \frac{\partial \Pi}{\partial \alpha_1} X_{\alpha_2}$
- $\Pi(\alpha) + \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1} \mathcal{D}(\Pi)(\alpha) = g \frac{\rho^2 - \rho^1}{\mu^2 + \mu^1} X_3(\alpha)$

where

$$\mathcal{D}(\Pi)(\alpha) = \int_S \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} N(\beta) \Pi(\beta) d\beta$$

and

$$N(\beta) = X_{\alpha_1} \wedge X_{\alpha_2}.$$

It is then crucial to estimate

$$\|(I + \lambda \mathcal{D})^{-1}\|, \quad |\lambda| = \left| \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1} \right| < 1.$$

The problem

$$\left\{ \begin{array}{l} \frac{dX}{dt} = BR(X, \omega) + C_1 X_{\alpha_1} + C_2 X_{\alpha_2} \\ \omega = \frac{\partial \Pi}{\partial \alpha_2} X_{\alpha_1} - \frac{\partial \Pi}{\partial \alpha_1} X_{\alpha_2} \\ \Pi(\alpha) = \left( I + \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1} \mathcal{D} \right)^{-1} \left( g \frac{\rho^2 - \rho^1}{\mu^2 + \mu^1} X_3(\alpha, t) \right) \end{array} \right.$$

is not well-posed.

Rayleigh-Taylor:



$$\begin{aligned} \sigma(\alpha, t) &= (\nabla p^2 - \nabla p^1) \cdot (\nu^2 - \nu^1) \\ &= (\mu^2 - \mu^1) BR(X, \omega) \cdot (X_{\alpha_1} \wedge X_{\alpha_2}) + (\rho^2 - \rho^1) (X_{\alpha_1} \wedge X_{\alpha_2})_3 \\ &> 0 \end{aligned}$$

Geometric constraint: not self-intersecting surface

$$F(x) = \sup_{\alpha \neq \beta} \frac{\|\alpha - \beta\|}{\|X(\alpha) - X(\beta)\|} < \infty.$$

### Theorem

*If*

- $X_0(\alpha) \in H^4(\mathbb{R}^2)$ ;
- $\sigma(\alpha, 0) > 0$ ;
- $F(X_0) < \infty$ ;

*then  $\exists T > 0$  and a unique solution  $X \in C^1([0, T]; H^4)$ .*

$$\begin{aligned} \frac{d}{dt} \|X\|_{H^k}^2 &\leq P(\|X\|_{H^k} + \|F(X)\|_{L^\infty}) \\ &\quad - \sum_{j=0}^k \frac{2^{3/2}}{\mu^1 + \mu^2} \int \frac{\sigma(\alpha)}{|\nabla X(\alpha)|^3} \frac{\partial^{k-j}}{\partial \alpha_1} \frac{\partial^j}{\partial \alpha_2} X(\alpha) \\ &\quad \cdot \Lambda \left( \frac{\partial^{k-j}}{\partial \alpha_1} \frac{\partial^j}{\partial \alpha_2} X(\alpha) \right) d\alpha \end{aligned}$$

where

$$\Lambda = \sqrt{-\Delta}$$

Then we use the pointwise inequality

$$\theta(x) \cdot \Lambda\theta(x) \geq \frac{1}{2}\Lambda\theta^2(x)$$

to get a polynomial estimate in the norm  $H^k$ .

However, we need to control the evolution of all the involved quantities, namely,

- $\sigma > 0$ .
- $F < \infty$ .
- $C_j$  (isothermal).
- $\|I + \lambda\mathcal{D}\|^{-1}, \quad |\lambda| < 1$ .