SPLINES WITH NON-NEGATIVE

B-SPLINE COEFFICIENTS

by

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Abstract

We consider the question of the approximation of non-negative functions by non-negative splines of order k (degree < k) compared with approximation by that subclass of non-negative splines of order k consisting of all those whose B-spline coefficients are non-negative; while approximation by the former gives errors of order h^k, the latter may yield only h^2. These results are related to certain facts about quasi-interpolants.

Key words: non-negative splines, one-sided approximation, splines, approximation

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1. Introduction

In approximating or representing certain data by a spline function s subject to continuous inequality constraints such as $s(x) \geq g(x)$ on $[0,1]$ for a given function g, the continuous constraints are often difficult to treat numerically. To avoid this problem, one might replace g by an accurate spline approximation s_g and then enforce $s \geq s_g$ by enforcing the still stronger condition $A_j(s) \geq A_j(s_g)$, where for any spline s we denote by $A_j(s)$ the j^{th} coefficient in the representation of s as a linear combination of B-splines; since each B-spline is non-negative, this condition is at least as strong as the condition $s \geq s_g$. One is immediately forced to ask whether or not the approximation properties of the class of splines satisfying $A_j(s) \geq A_j(s_g)$ are as good as those for the class satisfying $s \geq s_g$. In this short note we take the simple case of $s_g \equiv g \equiv 0$ and demonstrate that much approximation power is lost by this computational simplification. As usual, the results can be extended to higher dimension via tensor products.

2. A Constrained Approximation Problem

We shall consider the problem of approximating a given non-negative function f on a closed and bounded interval [a,b] by splines of positive integer order k

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(degree $< k$); unbounded and open intervals can be treated similarly. Let $\pi := \{x_i\}_0^N$ be a k-extended partition of $[a,b]$, so that $a = x_0 < x_1 \leq x_2 \leq \ldots \leq x_{N-1} < x_N = b$ with no more than k consecutive x_j's coinciding, i.e., $x_j < x_{j+k}$. Let S_π^k denote the linear space of polynomial splines of order k on π, so that each s in S_π^k is a polynomial of degree less than k in each interval (x_i, x_{i+1}) and $s^{(r)}$ is continuous at x_i for all $r < k - d_i$, where d_i is the frequency with which the number $x = x_i$ appears among the x_j's. Then the sequence of normalized B-splines $\{N_{j,k} | j = -k+1, -k+2, \ldots, N-1\}$ is a basis for S_π^k, where we have augmented π by additional points $x_{-k+1} = \ldots = x_{-1} = a$ and $b = x_{N+1} = \ldots = x_{N+k-1}$. As usual [Curry-Schoenberg (1966), de Boor (1968)], $N_{i,k}$ denotes the normalized B-spline defined by divided differences: $N_{i,k}(x) := (x_{i+k} - x_i) g_k(x_i, \ldots, x_{i+k}; x)$ with $g_k(t;x) = (t-x)_+^{k-1}$. These B-splines are so normalized that $\sum_i N_{i,k}(x) \equiv 1$. Since these B-splines are also non-negative, it follows that

$$(2.1) \qquad \left| \sum_i A_i N_{i,k}(x) \right| \leq \max_i |A_i| .$$

We wish to approximate the non-negative function f on $[a,b]$ by an element s in S_π^k with the constraint that $A_j(s) \geq 0$ for all j, where $s = \sum_j A_j(s) N_{j,k}$. This, of course, implies that $s(x) \geq 0$ on $[a,b]$ since every $N_{i,k}$ is non-negative; in fact, since for $k > 2$ the B-splines do not give all the extreme points in the cone of non-negative splines [Burchard (1973)], the condition $A_j(s) \geq 0$ for all j is appreciably stronger than $s \geq 0$. We shall see just how much stronger it is.

For any f in the Sobolev space $W_\infty^k[a,b]$, we recall the definition [de Boor-Fix (1973)] of the quasi-interpolant $F_\pi f \in S_\pi^k$ of f,

(2.2) $$F_\pi f = \sum_i \lambda_i(f)N_{i,k}$$

where

(2.3) $$\lambda_j(f) = \sum_{r<k} (-1)^{k-1-r}\psi_j^{(k-1-r)}(\tau_j)f^{(r)}(\tau_j)$$

and

(2.4) $$\psi_j(x) = (x_{j+1}-x)\ldots(x_{j+k-1}-x)/(k-1)!$$

If τ_j has been chosen to satisfy $x_j^+ \leqq \tau_j \leqq x_{j+k}^-$ as well as $\tau_j \in [a,b]$, then one has [de Boor-Fix (1973)]

(2.5) $$\|f-f_\pi f\|_\infty \leqq K_0\|f^{(k)}\|_\infty|\pi|^k, \quad |\pi| = \max_i |x_{i+1}-x_i|$$

for some constant K_0 independent of f and π, and

(2.6) $$\lambda_j(s) = A_j(s), \text{ implying } F_\pi s = s \text{ for } s \text{ in } S_\pi^k.$$

From Equation 2.3 for $k \geqq 2$, we have

(2.7) $$\begin{cases} \lambda_j(f) = f(\tau_j) + (-1)^{k-2}\psi_j^{(k-2)}(\tau_j)f^{(1)}(\tau_j) + \sum_{r=2}^{k-1}(-1)^{k-1-r}\psi^{(k-1-r)}(\tau_j)f^{(r)}(\tau_j) \\ \qquad = f(\tau_j) + (-1)^{k-2}\psi_j^{(k-2)}(\tau_j)f^{(1)}(\tau_j) + O(|\pi|^2) \end{cases}$$

Since ψ vanishes at $x_{j+1},\ldots,x_{j+k-1}$, we know that $\psi^{(k-2)}$ vanishes at the point $\xi_j = (x_{j+1} + \ldots + x_{j+k-1})/(k-1)$ in (x_{j+1},x_{j+k-1}); choose $\tau_j = \xi_j$. Then from Equation 2.7 we have $\lambda_j(f) = f(\tau_j) + O(|\pi|^2)$ if $f^{(2)},\ldots,f^{(k)}$ are all bounded.

Since f is non-negative, a shift in the B-spline coefficients by $O(|\pi|^2)$ produces a new approximation to f with non-negative B-spline coefficients and, by Equation 2.1, only $O(|\pi|^2)$ away from the original high-order approximation. This proves the following.

(2.8) Proposition. Let $f \in W_\infty^k[a,b]$ be non-negative, $k \geq 2$. Then there exists a spline s^P in S_π^k with non-negative B-spline coefficients and such that $\|s^P - f\|_\infty = O(|\pi|^2)$. Equivalently, there exists a quasi-interpolant s whose B-spline coefficients A_j converge to values of f in $[x_j, x_{j+k}]$ at least as fast as $O(|\pi|^2)$.

We now proceed to show that the result in Proposition 2.8 is essentially best possible; since this is clear for $k = 2$, we consider $k > 2$. Consider the function $f(x) = x^2$ on the interval $[-1,1]$, and consider uniform partitions π_n of width $|\pi_n| = h = \frac{2}{n}$ for integers n. If k is even we consider partitions π_n for which n is also even; for k odd we consider only n odd.

We now provide details for the case in which $k = 2\ell+2$ is even and $n = 2m$; the argument in the remaining case is similar. Let $j = m - \ell - 1$ and consider $\lambda_j(f)$ with $\tau_j = 0$. Since $f(\tau_j) = f^{(1)}(\tau_j) = f^{(r)}(\tau_j) = 0$ for $r > 2$ and $f^{(2)}(\tau_j) = 2$, we conclude from Equation 2.5 that $\lambda_j(f) = (-1)^{k-3} \cdot 2 \cdot \psi_j^{(k-3)}(0)$. By simple induction, one finds that $\psi_j^{(k-3)}(0) = \frac{h^2}{2\ell(2\ell-1)} \sum_{i=1}^{\ell} i^2 = \frac{\ell+1}{12} h^2$, so that

$$\lambda_{m-\ell-1}(f) = -\frac{\ell+1}{6} h^2. \tag{2.9}$$

It is known [de Boor (1968)] that there exists a constant $D_k < \infty$ and independent of π such that $|\lambda_i(s)| \leq D_k \|s\|_\infty$ for all i and for all s in S_π^k. Note that

$f(x) = x^2$ is in S_π^k for $k > 2$. Let s^P be any spline in S_π^k with non-negative B-spline coefficients, i.e., satisfying $\lambda_i(s^P) \geq 0$ for all i. Then

$$0 < -\lambda_{m-\ell-1}(f) = \lambda_{m-\ell-1}(-f) \leq \lambda_{m-\ell-1}(-f) + \lambda_{m-\ell-1}(s^P) = \lambda_{m-\ell-1}(s^P-f) \leq D_k\|s^P-f\|_\infty.$$

Therefore $\|s^P-f\|_\infty \geq -D_k^{-1}\lambda_{m-\ell-1}(f)$; using Equation 2.9, we now see that Proposition 2.8 is essentially best possible.

(2.10) Proposition. For $f(x) = x^2$ on $[-1,1]$, π the uniform partition of width $\frac{1}{m}$, and $k = 2\ell+2$, every spline s^P in S_π^k with positive B-spline coefficients satisfies $\|s^P-f\|_\infty \geq D_k^{-1}|\pi|^2 \frac{\ell+1}{6}$. Equivalently, the B-spline coefficients converge to a value of f no faster than $O(|\pi|^2)$.

Since it is easy to approximate non-negative functions with non-negative splines to $O(|\pi|^k)$ simply by translating the quasi-interpolant by $O(|\pi|^k)$, we see that approximation by splines with non-negative B-spline coefficients may lose much of the approximation power of splines. By arguing essentially as we did for $f(x) = x^2$, it is easy to show that for general non-negative f the precise order of best approximation by elements of S_π^k with non-negative B-spline coefficients is $O(|\pi|^k)$ plus the order of the most negative B-spline coefficient in a quasi-interpolant of f; as we saw for $f(x) = x^2$, this total error can be as large as $O(|\pi|^2)$, but it is no larger.

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