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Algebraic Properties of the Category of Q-P Quantale Modules

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Abstract

In this paper, the definition of a Q-P quantale module and some relative concepts were introduced. Based on which, some properties of the Q-P quantale module, and the structure of the free Q-P quantale modules generated by a set were obtained. It was proved that the category of Q-P quantale modules is algebraic.

Keywords: Q-P quantale quantale modules, equalizer, forgetful functor, algebraic category

1. Introduction

Quantale was proposed by C.J.Mulvey in 1986 for studying the foundations of quantum logic and for studying non-commutation C*-algebras. The term quantale was coined as a combination of "quantum logic" and "locale" by C.J.Mulvey. The systematic introduction of quantale theory came from the book ≪Quantales and their applications≫, which written by K.I.Rosenthal in 1990.

Since quantale theory provides a powerful tool in studying noncommutative structures, it has a wide applications, especially in studying noncommutative C*-algebra theory, the ideal theory of commutative ring, linear logic and so on. So, the quantale theory has aroused great interests of many scholar and experts, a great deal of new ideas and applications of quantale have been proposed in twenty years.

Since the ideal of quantale module was proposed by S.Abramsky and S.Vickers, the quantale module has attracted many scholars eyes. With the development of the quantale theory, the theory of quantale module was studied deeply in the past years. In this paper, some properties of the category of Q-P quantale modules was discussed, especially that the category of Q-P quantale modules is algebraic was proved.

2. Preliminaries

Definition 2.1. A *quantale* is a complete lattice Q with an associative binary operation "&" satisfying:

$a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i)$ and $(\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a)$,
for all $a, b_i \in Q$, where I is a set, 0 and 1 denote the smallest element and the greatest element of Q respectively.

Definition 2.2. A nonzero element a in a quantale Q is said to be a *nonzero divisor* if for all nonzero element $b \in Q$ such that $a\&b \neq 0$, $b\&a \neq 0$. Q is *nonzero divisor* if every $a \in Q$ is a *nonzero divisor*.

Definition 2.3. Let Q, P be a quantale, a Q-P quantale module over Q, P (briefly, a Q-P-module) is a complete lattice M, together with a mapping $T : Q \times M \times P \longrightarrow M$ satisfies the following conditions:

(1) $T(\bigvee_{i\in I} a_i, m, \bigvee_{j\in J} b_j) = \bigvee_{i\in I}\bigvee_{j\in J} T(a_i, m, b_j)$;

(2) $T(a, (\bigvee_{k\in K} m_k), b) = \bigvee_{k\in K} T(a, m_k, b)$;

(3) $T(a\&b, m, c\&d) = T(a, T(b, m, c), d)$.
for all $a_i, a, b \in Q$,$b_j, c, d \in P$, $m_k, m \in M$. We shall denote the Q-P quantale module M over Q, P by (M, T).

If Q is unital quantale with unit e, we define $T(e, m, e) = m$ for all $m \in M$.

Example 2.4. (1) Let $Q = P = \{0, a, b, c, 1\}$ be a set, $M = \{0, d, e, 1\}$ is a complete lattice. The order relations of Q and M are given by the following figure 1 and 2, we give a binary operator "&" on Q satisfying the diagram 1.

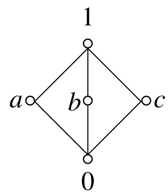

Figure 1

&	0	a	b	c	1
0	0	0	0	0	0
a	0	b	c	a	1
b	0	c	a	b	1
c	0	a	b	c	1
1	0	1	1	1	1

Diagram 1

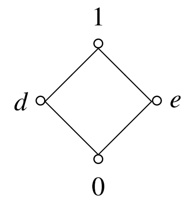

Figure 2

We can prove that Q is a quantale.

Now, define a mapping $T : Q \times M \times Q \longrightarrow M$ such that $T(x, m, y) = m$ for all $x, y \in Q$, $m \in M$. Then $(M,\ T)$ be a Q-P quantale module.

(2) Let $Q = P = \{0, a, b, 1\}$ be a complete lattice. The order relation on Q satisfies the following Figure 3 and the binary operation of Q satisfies the diagram 2:

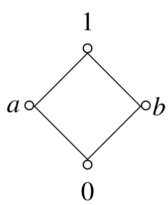

Figure 3

&	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Diagram 2

It is easy to show that $(Q, \&)$ is a quantale. Let $M = \{0, a, 1\} \subseteq Q$, then M is a complete lattice with the inheriting order on Q. Now, we define $T : Q \times M \times Q \longrightarrow M$ satisfies $T(x, m, y) = x\&m\&y$ for all $x, y \in Q$, $m \in M$. Then $(M,\ T)$ is a Q-P quantale module.

Definition 2.5. Let Q, P be a quantale, (M_1, T_1) and (M_2, T_2) are Q-P quantale modules. A mapping f : $M_1 \longrightarrow M_2$ is said to be a *$Q - P$ quantale module homomorphism* if f satisfies the following conditions:

(1) $f(\bigvee\limits_{i \in I} m_i) = \bigvee\limits_{i \in I} f(m_i)$;

(2) $f(T_1(a, m, b)) = T_2(a, f(m), b)$ for all $a \in Q$,$b \in P$, $m_i, m \in M$.

Definition 2.6. Let (M, T_M) be a $Q-P$ quantale module over Q, P, N be a subset of M, N is said to be a *submodule* of M if N is closed under arbitrary join and $T_M(a, n, b) \in N$ for all $a \in Q$,$b \in P$, $n \in N$.

Definition 2.7.[26] A concrete category $(\mathcal{A}, U)$ is called *algebraic* provied that it satisfies the following conditions:

(1) $\mathcal{A}$ has coequalizers;

(2) U has a left adjoint;

(3) U preserves and reflects regular epimorphisms.

3. The Category of Q-P Quantale Modules is Algebraic

Definition 3.1. Let Q, P be a quantale, $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ be the category whose objects are the Q-P quantale modules of Q, P, and morphisms are the Q-P quantale module homomorphisms, i.e.,

$Ob(_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}})$={M : M is Q-P quantale modules},

$\mathcal{M}or(_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}})$={$f : M \longrightarrow N$ is the Q-P quantale modules homorphism}.
Hence, the category $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ is a concrete category.

Definition 3.2. Let Q, P is a quantale, (M, T_M) is a Q-P quantale module, $R \subseteq M \times M$. The set R is said to be a *congruence* of Q-P quantale module on M if R satisfies:

(1) R is an equivalence relation on M;

(2) If $(m_i, n_i) \in R$ for all $i \in I$, then $(\vee_{i\in I} m_i, \vee_{i\in I} n_i) \in R$;

(3) If $(m, n) \in R$, then $(T_M(a, m, b), T_M(a, n, b)) \in R$ for all $a \in Q$,$b \in P$.

We denote the set of all congruence on M by $Con({}_QM_P)$, then $Con({}_QM_P)$ is a complete lattice with respect to the inclusion order.

Let Q, P be a quantale, M is a Q-P quantale module, R is a congrence of Q-P quantale module on M, define the order relation on M/R such that $[m] \leq [n]$ if and only if $[m \vee n] = [n]$ for all $[m], [n] \in M/R$.

Theorem 3.3. Let Q, P be a quantale, M be a Q-P quantale module, R be a congrence of double quantale module on M. Define $T_{M/R} : Q \times M/R \times P \longrightarrow M/R$ such that $T_{M/R}(a, [m], b) = [T_M(a, m, b)]$ for all $a \in Q$,$b \in P$, $[m] \in M/R$, then $(M/R, T_{M/R})$ is a Q-P quantale module and $\pi : m \mapsto [m] : M \longrightarrow M/R$ is a Q-P quantale module homomorphisms.

Proof. (1) We will prove that " $\leq$ "is a partial order on M/R, and $T_{M/R}$ is well defined. In fact, for all $[m], [n], [l] \in M/R$,

(i) It's clearly that $[m] \leq [m]$;

(ii) Let $[m] \leq [n]$, $[n] \leq [m]$, then $[m \vee n] = [n] and [n \vee m] = [m]$, thus $[m] = [n]$;

(iii) Let $[m] \leq [n]$, $[n] \leq [l]$, then $[m \vee n] = [n] and [n \vee l] = [l]$, therefore $[m \vee l] = [m \vee (n \vee l)] = [(m \vee n) \vee e] = [n \vee l] = [l]$.

If $[m_1] = [m_2]$, then $(m_1, m_2) \in R$, $(T_M(a, m, b), T_M(a, n, b)) \in R$ for all $a, b \in Q$, i.e., $[T_M(a, m, b)] = [T_M(a, n, b)]$, thus $T_{M/R}$ is well defined.

(2) We will prove that $(M/R, \leq)$ is a complete lattice. Let $\{[m_i] \mid i \in I\} \subseteq M/R$, we have

(i) Since $[m_i \vee (\bigvee_{i\in I} m_i)] = [\bigvee_{i\in I} m_i]$ for all $i \in I$, then $[m_i] \leq [\bigvee_{i\in I} m_i]$;

(ii) Let $[m] \in M/R$ and $[m_i] \leq [m]$ for all $i \in I$, then $[m_i \vee m] = [m]$ for all $i \in I$, hence, $[(\bigvee_{i\in I} m_i) \vee m] = [\bigvee_{i\in I}(m_i \vee m)] = [m]$, i.e., $[\bigvee_{i\in I} m_i] \leq [m]$.

Thus $\overset{M/R}{\bigvee_{i\in I}} [m_i] = [\bigvee_{i\in I} m_i]$.

(3) For all $\{a_i \mid i \in I\} \subseteq Q$, $\{b_j \mid j \in J\} \subseteq Q$, $\{[m_l] \mid l \in H\} \subseteq M/R$, $a, b \in Q$,$c, d \in P$, $[m] \in M/R$, we have that

(i) $T_{M/R}(\bigvee_{i\in I} a_i, [m], \bigvee_{j\in J} b_j) = [T_M(\bigvee_{i\in I} a_i, m, \bigvee_{j\in J} b_j)] = [\bigvee_{i\in I} \bigvee_{j\in J} T_M(a_i, m, b_j)]$
$= \bigvee_{i\in I} \bigvee_{j\in J} T_M[a_i, m, b_j] = \bigvee_{i\in I} \bigvee_{j\in J} T_{M/R}(a_i, [m], b_j)$;

(ii) $T_{M/R}(a, (\bigvee_{j\in J} [m_j]), b) = T_{M/R}(a, [\bigvee_{j\in J} m_j], b) = [T_M(a, (\bigvee_{j\in J} m_j), b)] = [\bigvee_{j\in J} T_M(a, m_j, b)]$
$= \bigvee_{j\in J} [T_M(a, m_j, b)] = \bigvee_{j\in J} T_{M/R}(a, [m_j], b)$;

(iii) $T_{M/R}(a\&b, [m], c\&d) = [T_M(a\&b, m, c\&d)] = [T_M(a, T_M(b, m, c), d)]$
$= T_{M/R}(a, [T_M(b, m, c)], d) = T_{M/R}(a, T_{M/R}(b, [m], c), d)$.

Then $(M/R, T_{M/R})$ is a Q-P quantale module.

(4) For all $\{[m_i] \mid i \in I\} \subseteq M/R$, $a \in Q$,$b \in P$, $[m] \in M/R$,

$\pi(\bigvee_{i\in I} m_i) = [\bigvee_{i\in I} m_i] = \bigvee_{i\in I}[m_i] = \bigvee_{i\in I} \pi(m_i)$;

$\pi(T_M(a, m, b)) = [T_M(a, m, b)] = T_{M/R}(a, [m], b) = T_{M/R}(a, \pi(m), b)$.
So $\pi : m \mapsto [m] : M \longrightarrow M/R$ is a Q-P quantale module homomorphisms. □

Theorem 3.4. Let Q, P be a quantale, M a double quantale module, then $\triangle = \{(x, x) \mid x \in M\}$ is a congrence of Q-P quantale module on M.

Theorem 3.5. Let Q, P be a quantale, M and N be Q-P quantale modules, $f : M \longrightarrow N$ a Q-P quantale module homphorism, R a Q-P quantale module congrence on N. Then $f^{-1}(R) = \{(x, y) \in M \times M \mid (f(x), f(y)) \in R\}$ is a Q-P quantale module congrence on M.

Theorem 3.6. Let Q, P be a quantale, M and N are Q-P quantale modules,
$f : M \longrightarrow N$ be a Q-P quantale module homphorism. Then $f^{-1}(\triangle) = \{(x, y) \in M \times M \mid f(x) = f(y)\}$ be a Q-P quantale module congrence on M, where $\triangle = \{(a, a) \mid a \in N\}$.

Let Q, P be a quantale, M be a Q-P quantale module, $R \subseteq M \times M$, since $Con({}_QM_P)$ is a complete lattice, there exists a smallest Q-P quantale congrence containing R , which is the intersection all the Q-P quantale module congrence containing R on M. We said that this congrence is generated by R.

Theorem 3.7. The category $\mathbf{{}_QMod_P}$ has coequalizer.

$$\begin{tikzcd} M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N \arrow[r, "h"] \arrow[d, "\pi"'] & E' \\ & E \arrow[ur, "\overline{h}"'] & \end{tikzcd}$$

Proof. Let Q, P be a quantale, (M, T_M) and (N, T_N) be Q-P quantale modules, f and g be Q-P quantale module homomorphisms, Suppose R is the smallest congrence of the Q-P quantale modules on N, which contain $\{(f(x), g(x)) \mid x \in M\}$. Let $E = N/R$, $\pi : N \longrightarrow N/R$ is the canonical mapping, then $(N/R, T_{N/R})$ is a Q-P quantale module and π is a Q-P quantale module homomorphisms by theorem 3.3 . We will prove that (π, E) is the coequalier of f and g. In fact,

(1) $\pi \circ f = \pi \circ g$ is clear£

(2) Let $(E', T_{E'})$ be a Q-P quantale module, $h : N \longrightarrow E'$ be a Q-P quantale module homomorphisms such that $h \circ f = h \circ g$. Let $R_1 = h^{-1}(\triangle)$, where $\triangle = \{(x, x) \mid x \in E'\}$. By theorem 3.5, we can see that R_1 is a congrence of Q-P quantale module on N. Since $h(f(x)) = h(g(x))$ for all $x \in M$, then $(f(x), g(x)) \in R_1$. Define $\overline{h} : N/R \longrightarrow E'$such that $\overline{h}([n]) = h(n)$ for all $[n] \in Q/R$. Let $n_1, n_2 \in N$ and $(n_1, n_2) \in R$, then $(n_1, n_2) \in R_1$, and we have that $h(n_1) = h(n_2)$. Therefore $\overline{h}$ is well defined.

For all $\{[n_i] \mid i \in I\} \subseteq N/R$, $a, b \in Q$, $[n] \in N/R$, we have that

$\overline{h}(\bigvee\limits_{i\in I}[n_i]) = \overline{h}([\bigvee\limits_{i\in I} n_i]) = h(\bigvee\limits_{i\in I} n_i) = \bigvee\limits_{i\in I} h(n_i) = \bigvee\limits_{i\in I} \overline{h}([n_i])$;

$\overline{h}(T_{N/R}(a, [n], b)) = \overline{h}([T(a, n, b)]) = h(T(a, n, b)) = T_{E'}(a, h(n), b) = T_{E'}(a, \overline{h}([n]), b)$.

Thus, $\overline{h}$ is a Q-P quantale module, and $\overline{h}$ is the unique homomorphism satisfy $\overline{h} \circ \pi = h$. Therefore (π, E) is the coequalizer of f and g. □

From now until the end of Section 3, we suppose Q be a unital quantale with unit e. Let X be a nonempty set, we consider the complete lattice $(Q^X, \bigvee^X)$, where Q^X is the set of all the function from X to Q and $(\bigvee\limits_{i\in I}^X f_i)(x) = \bigvee\limits_{i\in I} f_i(x)$ for all $x \in X$.

Theorem 3.8. Let X be a nonempty set, and Q is idempotent and unital quantale with unit e, define $T_X : Q \times Q^X \times Q \longrightarrow Q^X$ such that $T_X(a, f, b)(x) = a \& f(x) \& b$, for all $a, b \in Q, f \in Q^X$, $x \in X$. Then (Q^X, T_X) is the free double quantale module generated by X, equipped with the map $\varphi : x \in X \longmapsto \varphi_x \in Q^X$, where φ_x is defined by $\varphi_x(y) = \begin{cases} 0, & y \neq x, \\ e\,, & y = x. \end{cases}$ for all $y \in X$.

Proof. It's easy to prove that (Q^X, T_X) is a double quantale module. Let (M, T_M) be any double quantale module and $g : X \longrightarrow M$ be an arbitrary map. First observe that for all $f \in Q^X$, Q be a unital quantale with unit e, hence $f = T_X(e, f, e)$ by definition 2.2. So every elments of Q^X could denote by $T_X(c, f, d)$ for some $c, d \in Q, f \in Q^X$. Define map $h_g : Q^X \longrightarrow M$ such that $h_g(T_X(c, f, d)) = \bigvee\limits_{x\in X} T_M(c, T_M(f(x), g(x), f(x)), d)$, for all $T_X(c, f, d) \in Q^X$, $c, d \in Q$.

For all $x' \in Z$, $(h_g \circ \varphi)(x') = h_g(\varphi_{x'}) = \bigvee_{x\in X} T_M(\varphi_{x'}(x), g(x), \varphi_{x'}(x)) = T_M(e, g(x), e) = f(x)$, hence $h_g \circ \varphi = f$. This implies that the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "g"'] & Q^X \arrow[d, "h_g"] \\ & M \end{tikzcd}$$

We will prove that h_g is a Q-P quantale module homomorphism.

For all $\{f_i\}_{i\in I}$, $a, b \in Q$, $f \in Q^X$, we have

(i) $h_g(\bigvee_{i\in I} f_i) = h_g(T_X(e, \bigvee_{i\in I} f_i, e) = \bigvee_{x\in X} T_M(e, T_M(\bigvee_{i\in I} f_i, g(x), \bigvee_{i\in I} f_i), e)$

$$= \bigvee_{x\in X} T_M(\bigvee_{i\in I} f_i, g(x), \bigvee_{i\in I} f_i)$$
$$= \bigvee_{i\in I} \bigvee_{x\in X} T_M(f_i, g(x), f_i)$$
$$= \bigvee_{i\in I} h_g(f_i);$$

(ii) $h_g(T_X(a, f, b)) = \bigvee_{x\in X} T_M(a, T_M(f(x), g(x), f(x)), b)$

$$= T_M(a, \bigvee_{x\in X} T_M(f(x), g(x), f(x)), b)$$
$$= T_M(a, h_g(f), b).$$

Therefore, h_g is a Q-P quantale module homomorphism.

Next, we will prove that h_g is an unique Q-P quantale module homomorphism satisfying the above conditions.

Now, let $h'_g : Q^X \longrightarrow M$ be another unique Q-P quantale module homomorphism such that $h'_g \circ \varphi = g$. For all $T_X(c, f, d) \in Q^X$, we have

$h_g(T_X(c, f, d)) = \bigvee_{x\in X} T_M(c, T_M(f(x), g(x), f(x)), d)$

$$= \bigvee_{x\in X} T_M(c, T_M(f(x), (h'_g \circ \varphi)(x), f(x)), d)$$
$$= T_M(c, h'_g(\bigvee_{x\in X} T_X(f(x), \varphi_x, f(x))), d)$$
$$= T_M(c, h'_g(f), d) \qquad (\bigvee_{x\in X} T_X(f(x), \varphi_x, f(x)) = f)$$
$$= h'_g(T_X(c, f, d)).$$

Therefore, (Q^X, T_X) is the free Q-P quantale module generated by X, equipped with the map φ.

Definition 3.9. Let X be a nonempty set, Q, P is unital quantale , (Q^X, T_X) is called *free Q-P quantale module* generated by X.

Theorem 3.10. The forgetfull functor $U : {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}} \longrightarrow \mathbf{Set}$ have a left adjoint.

Proof. Let X and Y be nonempty sets, (Q^X, T_X) and (Q^Y, T_Y) be the free Q-p quantale module generated by X and Y respectively.

Corresponding map $f : X \longrightarrow Y$ defines $M(f) : Q^X \longrightarrow Q^Y$ such that $M(f)(g)(y) = \bigvee\{g(x) \mid f(x) = y, x \in X\}$, for all g in Q^X, $y \in Y$. Obiviously, $M(f)$ is well defined.

We check $M(f)$ is a Q-p quantale module homomorphism.

For all $g_i, g \in Q^X, a \in Q, b \in P, y \in Y$ we have

(i) $M(f)(\bigvee_{i\in I} g_i) = \bigvee\{\bigvee_{i\in I} g_i(x) \mid f(x) = y, x \in X\}$

$= \bigvee_{i\in I}(\bigvee\{g_i(x) \mid f(x) = y, x \in X\})$

$= \bigvee_{i\in I} M(f)(g_i)(y).$

Thus $M(f)$ preserves arbitrary joins.

(ii) $M(f)(T_X(a, g, b))(y) = \bigvee\{T_X(a, g, b)(x) \mid f(x) = y, x \in X\}$

$= \bigvee\{a\&g(x)\&b \mid f(x) = y, x \in X\}$

$= a\&(\bigvee\{g(x) \mid f(x) = y, x \in X\})\&b$

$= a\&(M(f)(g)(y))\&b$

$= T_Y(a, M(f)(g), b)(y).$

Thus $M(f)(T_X(a, g, b))(y) = T_Y(a, M(f)(g), b)(y)$.
It is readily verified that $M(f)$ is a Q-P quantale module homomorphism.

Next, we will check that $M : \mathbf{Set} \longrightarrow {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ is a functor.
Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, id_X is the identity function on X. For all $h \in Q^X$, $x \in X$, $z \in Z$, we have

(i) $M(id_X)(h)(x) = \bigvee\{h(x) \mid id_X(x) = x\} = h(x) = id_{Q^X}(h)(x)$, it shows that M preserves identity function.

(ii) $(M(g) \circ M(f))(h)(z) = \bigvee\{M(f)(h)(y) \mid g(y) = z, y \in Y\}$

$= \bigvee\{\bigvee\{h(x) \mid f(x) = y, x \in X\} \mid g(y) = z, y \in Y\}$

$= \bigvee\{h(x) \mid f(x) = y, g(y) = z, x \in X, y \in Y\}$

$= \bigvee\{h(x) \mid g(f(x)) = z, x \in X\}$

$= M(g \circ f)(h)(z),$

then M preserves composition.

Finally, we will prove that M is the left adjoint of U.

By theorem 3.8, we have (Q^X, T_X) is the free Q-P quantale module generated by X, equipped with the map φ, therefore, M is the left adjoint of U. □

Theorem 3.11. The forgetful functor $U : {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}} \longrightarrow \mathbf{Set}$ preserves and reflects regular epimorphisms.

Proof. It is easy to be verified that the forgetful functor U preserves regular epimorphisms. We will check the forgetful functor U reflects regular epimorphisms.

At first, every regular epimorphisms is a surjective homomorphism in ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ by Theorem 3.7.

Next, we prove that every surjective homomorphism is a regular epimorphisms in ${}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$.

Let $h : M_1 \longrightarrow M_2$ be a surjective Q-P quantale module homomorphism. Since the surjective morphism is an regular epimorphism in **Set**. Then h is a regular epimorphism in **Set**, there exists a set X and maps f, g such that (h, M_2) is a coequalizer of f and g.

Let (Q^X, T_X) be a Q-P quantale module generated by X. Since Q be a unital quantale with unit e, hence $s = T_X(e, s, e)$ for all $s \in Q^X$.

Define map $h_f, h_g : Q^X \longrightarrow M$ such that $h_f(T_X(a, s, b)) = \bigvee_{x\in X} T_{M_1}(a, T_{M_1}(s(x), f(x), s(x)), b)$.

$h_g(T_X(a, s, b)) = \bigvee_{x\in X} T_{M_1}(a, T_{M_1}(s(x), g(x), s(x)), b)$, for all $T_X(a, s, b) \in Q^X$, $s \in Q^X$, $a, b \in Q$.

We know that h_f and h_g are Q-P quantale module homomorphisms by theorem 3.8.

Since h_f is a Q-P quantale module homomorphism, and $h \circ f = h \circ g$, then $h \circ h_f = h \circ h_g$. Suppose there is a Q-P quantale module homomorphism $h' : M_1 \longrightarrow M_2$ with $h' \circ h_f = h' \circ h_g$, then we have $h' \circ f = h' \circ g$.

Because (h, M_2) is the coequalizer of f and g, there is a unique Q-P quantale module homomorphism $\overline{h} : M_2 \longrightarrow M_3$ such that $h' = \overline{h} \circ h$. Since h is a surjective of Q-P quantale module homomorphism, then there exists $x', y' \in M_1$ and $\{x'_i\}_{i\in I} \subseteq M_1$ such that $h(x_1) = x, h(y_1) = y, h(x'_i) = x_i$.

We check that $\overline{h}$ be a Q-P quantale module homomorphism in the following.

(i) $\overline{h}(\bigvee\limits_{i\in I} x_i) = \overline{h}(\bigvee\limits_{i\in I} h(x'_i)) = \overline{h}h(\bigvee\limits_{i\in I} x'_i) = h'(\bigvee\limits_{i\in I} x'_i) = \bigvee\limits_{i\in I} h(x'_i) = \bigvee\limits_{i\in I} \overline{h}h(x'_i) = \bigvee\limits_{i\in I} \overline{h}(x_i)$,

(ii) For any $a \in Q, b \in P,\ m \in M_2$, since h is a surjective of double quantale module homomorphism, there exists m' in M such that $h(m') = m$.

So we have $T_3(a, \overline{h}(m), b) = T_3(a, \overline{h}(h(m')), b) = T_3(a, h'(m'), b) = h'(T_1(a, m', b))$
$= \overline{h}h(T_1(a, m', b)) = \overline{h}(T_2(a, h(m'), b) = \overline{h}(T_2(a, m, b))$.

Hence, (h, M_2) is an coequalizer of h_f and h_g in $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$, so h is a regular epimorphism in $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$. Therefore, the regular epimorphisms are precisely surjective homomorphisms in $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$. Since the forgetfull functor $U : {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}} \longrightarrow \mathbf{Set}$ reflects surjective homomorphisms, hence $U : {}_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}} \longrightarrow \mathbf{Set}$ reflects regular epimorphisms.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & M_1 \arrow[r, "h'"] \arrow[d, "h"] & M_3 \\ & M_2 \arrow[ur, "\overline{h}"'] & \end{tikzcd} \qquad \begin{tikzcd} Q^X \arrow[r, shift left, "h_f"] \arrow[r, shift right, "h_g"'] & M_1 \arrow[r, "h'"] \arrow[d, "h"'] & M_3 \\ & M_2 \arrow[ur, "\overline{h}"'] & \end{tikzcd}$$

□

The combination of theorem 3.7, theorem 3.10 and theorem 3.11, we can obtain the main result of this paper.

Theorem 3.12. The category $_{\mathbf{Q}}\mathbf{Mod}_{\mathbf{P}}$ is algeraic.

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