



# NCEES

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## Reference Handbook

9.0 Version for Computer-Based Testing

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# PREFACE

## **About the *Handbook***

The Fundamentals of Engineering (FE) exam is computer-based, and the *FE Reference Handbook* is the only resource material you may use during the exam. Reviewing it before exam day will help you become familiar with the charts, formulas, tables, and other reference information provided. You won't be allowed to bring your personal copy of the *Handbook* into the exam room. Instead, the computer-based exam will include a PDF version of the *Handbook* for your use. No printed copies of the *Handbook* will be allowed in the exam room.

The PDF version of the *FE Reference Handbook* that you use on exam day will be very similar to the printed version. Pages not needed to solve exam questions—such as the cover, introductory material, and exam specifications—will not be included in the PDF version. In addition, NCEES will periodically revise and update the *Handbook*, and each FE exam will be administered using the updated version.

The *FE Reference Handbook* does not contain all the information required to answer every question on the exam. Basic theories, conversions, formulas, and definitions examinees are expected to know have not been included. Special material required for the solution of a particular exam question will be included in the question itself.

## **Updates on exam content and procedures**

NCEES.org is our home on the Web. Visit us there for updates on everything exam-related, including specifications, exam-day policies, scoring, and practice tests. A PDF version of the *FE Reference Handbook* similar to the one you will use on exam day is also available there.

## **Errata**

To report errata in this book, email your correction using our feedback form on NCEES.org. Examinees are not penalized for any errors in the *Handbook* that affect an exam question.





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## UNITS

The FE exam and this handbook use both the metric system of units and the U.S. Customary System (USCS). In the USCS system of units, both force and mass are called pounds. Therefore, one must distinguish the pound-force (lbf) from the pound-mass (lbm).

The pound-force is that force which accelerates one pound-mass at  $32.174 \text{ ft/sec}^2$ . Thus,  $1 \text{ lbf} = 32.174 \text{ lbm-ft/sec}^2$ . The expression  $32.174 \text{ lbm-ft/(lbf-sec}^2)$  is designated as  $g_c$  and is used to resolve expressions involving both mass and force expressed as pounds. For instance, in writing Newton's second law, the equation would be written as  $F = ma/g_c$ , where  $F$  is in lbf,  $m$  in lbm, and  $a$  is in  $\text{ft/sec}^2$ .

Similar expressions exist for other quantities. Kinetic Energy,  $KE = mv^2/2g_c$ , with  $KE$  in (ft-lbf); Potential Energy,  $PE = mgh/g_c$ , with  $PE$  in (ft-lbf); Fluid Pressure,  $p = \rho gh/g_c$ , with  $p$  in (lbf/ft<sup>2</sup>); Specific Weight,  $SW = \rho g/g_c$ , in (lbf/ft<sup>3</sup>); Shear Stress,  $\tau = (\mu/g_c)(dv/dy)$ , with shear stress in (lbf/ft<sup>2</sup>). In all these examples,  $g_c$  should be regarded as a unit conversion factor. It is frequently not written explicitly in engineering equations. However, its use is required to produce a consistent set of units.

Note that the conversion factor  $g_c$  [ $\text{lbm-ft/(lbf-sec}^2)$ ] should not be confused with the local acceleration of gravity  $g$ , which has different units ( $\text{m/s}^2$  or  $\text{ft/sec}^2$ ) and may be either its standard value ( $9.807 \text{ m/s}^2$  or  $32.174 \text{ ft/sec}^2$ ) or some other local value.

If the problem is presented in USCS units, it may be necessary to use the constant  $g_c$  in the equation to have a consistent set of units.

METRIC PREFIXES			COMMONLY USED EQUIVALENTS	
Multiple	Prefix	Symbol		
$10^{-18}$	atto	a	1 gallon of water weighs 8.34 lbf 1 cubic foot of water weighs 62.4 lbf 1 cubic inch of mercury weighs 0.491 lbf The mass of 1 cubic meter of water is 1,000 kilograms 1 mg/L is 8.34 lbf/Mgal	
$10^{-15}$	femto	f		
$10^{-12}$	pico	p		
$10^{-9}$	nano	n		
$10^{-6}$	micro	$\mu$		
$10^{-3}$	milli	m		
$10^{-2}$	centi	c		
$10^{-1}$	deci	d		
$10^1$	deka	da		
$10^2$	hecto	h		
$10^3$	kilo	k	TEMPERATURE CONVERSIONS	
$10^6$	mega	M		
$10^9$	giga	G		
$10^{12}$	tera	T		
$10^{15}$	peta	P		
$10^{18}$	exa	E		
			$^{\circ}\text{F} = 1.8 (^{\circ}\text{C}) + 32$	
			$^{\circ}\text{C} = (^{\circ}\text{F} - 32)/1.8$	
			$^{\circ}\text{R} = ^{\circ}\text{F} + 459.69$	
			$\text{K} = ^{\circ}\text{C} + 273.15$	

### IDEAL GAS CONSTANTS

The universal gas constant, designated as  $\bar{R}$  in the table below, relates pressure, volume, temperature, and number of moles of an ideal gas. When that universal constant,  $\bar{R}$ , is divided by the molecular weight of the gas, the result, often designated as  $R$ , has units of energy per degree per unit mass [ $\text{kJ/(kg}\cdot\text{K)}$  or  $\text{ft-lbf/(lbm}\cdot^{\circ}\text{R)}$ ] and becomes characteristic of the particular gas. Some disciplines, notably chemical engineering, often use the symbol  $R$  to refer to the universal gas constant  $\bar{R}$ .

### FUNDAMENTAL CONSTANTS

Quantity		Symbol	Value	Units
electron charge		$e$	$1.6022 \times 10^{-19}$	C (coulombs)
Faraday constant		$F$	96,485	coulombs/(mol)
gas constant	metric	$\bar{R}$	8,314	J/(kmol $\cdot$ K)
gas constant	metric	$\bar{R}$	8.314	kPa $\cdot$ m <sup>3</sup> /(kmol $\cdot$ K)
gas constant	USCS	$\bar{R}$	1,545	ft-lbf/(lb mole $\cdot$ $^{\circ}$ R)
		$\bar{R}$	0.08206	L-atm/(mole $\cdot$ K)
gravitation–Newtonian constant		$G$	$6.673 \times 10^{-11}$	m <sup>3</sup> /(kg $\cdot$ s <sup>2</sup> )
gravitation–Newtonian constant		$G$	$6.673 \times 10^{-11}$	N $\cdot$ m <sup>2</sup> /kg <sup>2</sup>
gravity acceleration (standard)	metric	$g$	9.807	m/s <sup>2</sup>
gravity acceleration (standard)	USCS	$g$	32.174	ft/sec <sup>2</sup>
molar volume (ideal gas), $T = 273.15\text{K}$ , $p = 101.3 \text{ kPa}$		$V_m$	22,414	L/kmol
speed of light in vacuum		$c$	299,792,000	m/s
Stefan-Boltzmann constant		$\sigma$	$5.67 \times 10^{-8}$	W/(m <sup>2</sup> $\cdot$ K <sup>4</sup> )

# CONVERSION FACTORS

Multiply	By	To Obtain	Multiply	By	To Obtain
acre	43,560	square feet (ft <sup>2</sup> )	joule (J)	$9.478 \times 10^{-4}$	Btu
ampere-hr (A-hr)	3,600	coulomb (C)	J	0.7376	ft-lbf
ångström (Å)	$1 \times 10^{-10}$	meter (m)	J	1	newton•m (N•m)
atmosphere (atm)	76.0	cm, mercury (Hg)	J/s	1	watt (W)
atm, std	29.92	in., mercury (Hg)			
atm, std	14.70	lb/in <sup>2</sup> abs (psia)	kilogram (kg)	2.205	pound (lbm)
atm, std	33.90	ft, water	kgf	9.8066	newton (N)
atm, std	$1.013 \times 10^5$	pascal (Pa)	kilometer (km)	3,281	feet (ft)
			km/hr	0.621	mph
bar	$1 \times 10^5$	Pa	kilopascal (kPa)	0.145	lb/in <sup>2</sup> (psi)
bar	0.987	atm	kilowatt (kW)	1.341	horsepower (hp)
barrels—oil	42	gallons—oil	kW	3,413	Btu/hr
Btu	1,055	joule (J)	kW	737.6	(ft-lbf )/sec
Btu	$2.928 \times 10^{-4}$	kilowatt-hr (kWh)	kW-hour (kWh)	3,413	Btu
Btu	778	ft-lbf	kWh	1.341	hp-hr
Btu/hr	$3.930 \times 10^{-4}$	horsepower (hp)	kWh	$3.6 \times 10^6$	joule (J)
Btu/hr	0.293	watt (W)	kip (K)	1,000	lbf
Btu/hr	0.216	ft-lbf/sec	K	4,448	newton (N)
calorie (g-cal)	$3.968 \times 10^{-3}$	Btu	liter (L)	61.02	in <sup>3</sup>
cal	$1.560 \times 10^{-6}$	hp-hr	L	0.264	gal (U.S. Liq)
cal	4.186	joule (J)	L	$10^{-3}$	m <sup>3</sup>
cal/sec	4.184	watt (W)	L/second (L/s)	2.119	ft <sup>3</sup> /min (cfm)
centimeter (cm)	$3.281 \times 10^{-2}$	foot (ft)	L/s	15.85	gal (U.S.)/min (gpm)
cm	0.394	inch (in)			
centipoise (cP)	0.001	pascal•sec (Pa•s)	meter (m)	3.281	feet (ft)
centipoise (cP)	1	g/(m•s)	m	1.094	yard
centipoise (cP)	2.419	lbm/hr-ft	m/second (m/s)	196.8	feet/min (ft/min)
centistoke (cSt)	$1 \times 10^{-6}$	m <sup>2</sup> /sec (m <sup>2</sup> /s)	mile (statute)	5,280	feet (ft)
cubic feet/second (cfs)	0.646317	million gallons/day (MGD)	mile (statute)	1.609	kilometer (km)
cubic foot (ft <sup>3</sup> )	7.481	gallon	mile/hour (mph)	88.0	ft/min (fpm)
cubic meters (m <sup>3</sup> )	1,000	liters	mph	1.609	km/h
electronvolt (eV)	$1.602 \times 10^{-19}$	joule (J)	mm of Hg	$1.316 \times 10^{-3}$	atm
			mm of H <sub>2</sub> O	$9.678 \times 10^{-5}$	atm
foot (ft)	30.48	cm	newton (N)	0.225	lbf
ft	0.3048	meter (m)	newton (N)	1	kg•m/s <sup>2</sup>
ft-pound (ft-lbf)	$1.285 \times 10^{-3}$	Btu	N•m	0.7376	ft-lbf
ft-lbf	$3.766 \times 10^{-7}$	kilowatt-hr (kWh)	N•m	1	joule (J)
ft-lbf	0.324	calorie (g-cal)			
ft-lbf	1.356	joule (J)	pascal (Pa)	$9.869 \times 10^{-6}$	atmosphere (atm)
			Pa	1	newton/m <sup>2</sup> (N/m <sup>2</sup> )
ft-lbf/sec	$1.818 \times 10^{-3}$	horsepower (hp)	Pa•sec (Pa•s)	10	poise (P)
			pound (lbm, avdp)	0.454	kilogram (kg)
gallon (U.S. Liq)	3.785	liter (L)	lbf	4.448	N
gallon (U.S. Liq)	0.134	ft <sup>3</sup>	lbf-ft	1.356	N•m
gallons of water	8.3453	pounds of water	lbf/in <sup>2</sup> (psi)	0.068	atm
gamma (γ, Γ)	$1 \times 10^{-9}$	tesla (T)	psi	2.307	ft of H <sub>2</sub> O
gauss	$1 \times 10^{-4}$	T	psi	2.036	in. of Hg
gram (g)	$2.205 \times 10^{-3}$	pound (lbm)	psi	6,895	Pa
hectare	$1 \times 10^4$	square meters (m <sup>2</sup> )	radian	$180/\pi$	degree
hectare	2.47104	acres			
horsepower (hp)	42.4	Btu/min	stokes	$1 \times 10^{-4}$	m <sup>2</sup> /s
hp	745.7	watt (W)			
hp	33,000	(ft-lbf)/min	therm	$1 \times 10^5$	Btu
hp	550	(ft-lbf)/sec	ton (metric)	1,000	kilogram (kg)
hp-hr	2,545	Btu	ton (short)	2,000	pound (lb)
hp-hr	$1.98 \times 10^6$	ft-lbf			
hp-hr	$2.68 \times 10^6$	joule (J)	watt (W)	3.413	Btu/hr
hp-hr	0.746	kWh	W	$1.341 \times 10^{-3}$	horsepower (hp)
			W	1	joule/s (J/s)
inch (in.)	2.540	centimeter (cm)	weber/m <sup>2</sup> (Wb/m <sup>2</sup> )	10,000	gauss
in. of Hg	0.0334	atm			
in. of Hg	13.60	in. of H <sub>2</sub> O			
in. of H <sub>2</sub> O	0.0361	lb/in <sup>2</sup> (psi)			
in. of H <sub>2</sub> O	0.002458	atm			

# ETHICS

Engineering is considered to be a “profession” rather than an “occupation” because of several important characteristics shared with other recognized learned professions, law, medicine, and theology: special knowledge, special privileges, and special responsibilities. Professions are based on a large knowledge base requiring extensive training. Professional skills are important to the well-being of society. Professions are self-regulating, in that they control the training and evaluation processes that admit new persons to the field. Professionals have autonomy in the workplace; they are expected to utilize their independent judgment in carrying out their professional responsibilities. Finally, professions are regulated by ethical standards.<sup>1</sup>

The expertise possessed by engineers is vitally important to public welfare. In order to serve the public effectively, engineers must maintain a high level of technical competence. However, a high level of technical expertise without adherence to ethical guidelines is as much a threat to public welfare as is professional incompetence. Therefore, engineers must also be guided by ethical principles.

The ethical principles governing the engineering profession are embodied in codes of ethics. Such codes have been adopted by state boards of registration, professional engineering societies, and even by some private industries. An example of one such code is the NCEES Rules of Professional Conduct, found in Section 240 of the *Model Rules* and presented here. As part of his/her responsibility to the public, an engineer is responsible for knowing and abiding by the code. Additional rules of conduct are also included in the *Model Rules*.

The three major sections of the *Model Rules* address (1) Licensee's Obligation to Society, (2) Licensee's Obligation to Employers and Clients, and (3) Licensee's Obligation to Other Licensees. The principles amplified in these sections are important guides to appropriate behavior of professional engineers.

Application of the code in many situations is not controversial. However, there may be situations in which applying the code may raise more difficult issues. In particular, there may be circumstances in which terminology in the code is not clearly defined, or in which two sections of the code may be in conflict. For example, what constitutes “valuable consideration” or “adequate” knowledge may be interpreted differently by qualified professionals. These types of questions are called *conceptual issues*, in which definitions of terms may be in dispute. In other situations, *factual issues* may also affect ethical dilemmas. Many decisions regarding engineering design may be based upon interpretation of disputed or incomplete information. In addition, *tradeoffs* revolving around competing issues of risk vs. benefit, or safety vs. economics may require judgments that are not fully addressed simply by application of the code.

No code can give immediate and mechanical answers to all ethical and professional problems that an engineer may face. Creative problem solving is often called for in ethics, just as it is in other areas of engineering.

## *Model Rules, Section 240.15, Rules of Professional Conduct*

### **A. LICENSEE'S OBLIGATION TO SOCIETY**

1. Licensees, in the performance of their services for clients, employers, and customers, shall be cognizant that their first and foremost responsibility is to the public welfare.
2. Licensees shall approve and seal only those design documents and surveys that conform to accepted engineering and surveying standards and safeguard the life, health, property, and welfare of the public.
3. Licensees shall notify their employer or client and such other authority as may be appropriate when their professional judgment is overruled under circumstances where the life, health, property, or welfare of the public is endangered.
4. Licensees shall be objective and truthful in professional reports, statements, or testimony. They shall include all relevant and pertinent information in such reports, statements, or testimony.
5. Licensees shall express a professional opinion publicly only when it is founded upon an adequate knowledge of the facts and a competent evaluation of the subject matter.
6. Licensees shall issue no statements, criticisms, or arguments on technical matters which are inspired or paid for by interested parties, unless they explicitly identify the interested parties on whose behalf they are speaking and reveal any interest they have in the matters.
7. Licensees shall not permit the use of their name or firm name by, nor associate in the business ventures with, any person or firm which is engaging in fraudulent or dishonest business or professional practices.
8. Licensees having knowledge of possible violations of any of these Rules of Professional Conduct shall provide the board with the information and assistance necessary to make the final determination of such violation.

<sup>1</sup> Harris, C.E., M.S. Pritchard, & M.J. Rabins, *Engineering Ethics: Concepts and Cases*, Wadsworth Publishing Company, pages 27–28, 1995.

## **B. LICENSEE'S OBLIGATION TO EMPLOYER AND CLIENTS**

1. Licensees shall undertake assignments only when qualified by education or experience in the specific technical fields of engineering or surveying involved.
2. Licensees shall not affix their signatures or seals to any plans or documents dealing with subject matter in which they lack competence, nor to any such plan or document not prepared under their direct control and personal supervision.
3. Licensees may accept assignments for coordination of an entire project, provided that each design segment is signed and sealed by the licensee responsible for preparation of that design segment.
4. Licensees shall not reveal facts, data, or information obtained in a professional capacity without the prior consent of the client or employer except as authorized or required by law. Licensees shall not solicit or accept gratuities, directly or indirectly, from contractors, their agents, or other parties in connection with work for employers or clients.
5. Licensees shall make full prior disclosures to their employers or clients of potential conflicts of interest or other circumstances which could influence or appear to influence their judgment or the quality of their service.
6. Licensees shall not accept compensation, financial or otherwise, from more than one party for services pertaining to the same project, unless the circumstances are fully disclosed and agreed to by all interested parties.
7. Licensees shall not solicit or accept a professional contract from a governmental body on which a principal or officer of their organization serves as a member. Conversely, licensees serving as members, advisors, or employees of a government body or department, who are the principals or employees of a private concern, shall not participate in decisions with respect to professional services offered or provided by said concern to the governmental body which they serve.

## **C. LICENSEE'S OBLIGATION TO OTHER LICENSEES**

1. Licensees shall not falsify or permit misrepresentation of their, or their associates', academic or professional qualifications. They shall not misrepresent or exaggerate their degree of responsibility in prior assignments nor the complexity of said assignments. Presentations incident to the solicitation of employment or business shall not misrepresent pertinent facts concerning employers, employees, associates, joint ventures, or past accomplishments.
2. Licensees shall not offer, give, solicit, or receive, either directly or indirectly, any commission, or gift, or other valuable consideration in order to secure work, and shall not make any political contribution with the intent to influence the award of a contract by public authority.
3. Licensees shall not attempt to injure, maliciously or falsely, directly or indirectly, the professional reputation, prospects, practice, or employment of other licensees, nor indiscriminately criticize other licensees' work.

## **SUSTAINABILITY**

The codes of ethics of a number of professional societies emphasize the need to develop sustainably. Sustainable development is the challenge of meeting human needs for natural resources, industrial products, energy, food, transportation, shelter, and effective waste management while conserving and protecting environmental quality and the natural resource base essential for future development.

# SAFETY

## DEFINITION OF SAFETY

Safety is the condition of protecting people from threats or failures that could harm their physical, emotional, occupational, psychological, or financial well-being. Safety is also the control of known threats to attain an acceptable level of risk.

The United States relies on public codes and standards, engineering designs, and corporate policies to ensure that a structure or place does what it should do to maintain a steady state of safety—that is, long-term stability and reliability. Some *Safety/Regulatory Agencies* that develop codes and standards commonly used in the United States are shown in the table.

Acronym	Name	Jurisdiction
CSA	Canadian Standards Association	Nonprofit standards organization
FAA	Federal Aviation Administration	Federal regulatory agency
IEC	International Electrotechnical Commission	Nonprofit standards organization
ITSNA	Intertek Testing Services NA (formerly Edison Testing Labs)	Nationally recognized testing laboratory
MSHA	Mine Safety and Health Administration	Federal regulatory agency
NFPA	National Fire Protection Association	Nonprofit trade association
OSHA	Occupational Health and Safety Administration	Federal regulatory agency
UL	Underwriters Laboratories	Nationally recognized testing laboratory
USCG	United States Coast Guard	Federal regulatory agency
USDOT	United States Department of Transportation	Federal regulatory agency
EPA	United States Environmental Protection Agency	Federal regulatory agency

## SAFETY PREVENTION

A traditional preventive approach to both accidents and occupational illness involves recognizing, evaluating, and controlling hazards and work conditions that may cause physical or other injuries.

*Hazard* is the capacity to cause harm. It is an inherent quality of a material or a condition. For example, a rotating saw blade or an uncontrolled high-pressure jet of water has the capability (hazard) to slice through flesh. A toxic chemical or a pathogen has the capability (hazard) to cause illness.

*Risk* is the chance or probability that a person will experience harm and is not the same as a hazard. Risk always involves both probability and severity elements. The hazard associated with a rotating saw blade or the water jet continues to exist, but the probability of causing harm, and thus the risk, can be reduced by installing a guard or by controlling the jet's path. Risk is expressed by the equation:

$$\text{Risk} = \text{Hazard} \times \text{Probability}$$

When people discuss the hazards of disease-causing agents, the term *exposure* is typically used more than *probability*. If a certain type of chemical has a toxicity hazard, the risk of illness rises with the degree to which that chemical contacts your body or enters your lungs. In that case, the equation becomes:

$$\text{Risk} = \text{Hazard} \times \text{Exposure}$$

Organizations evaluate hazards using multiple techniques and data sources.

### Job Safety Analysis

Job safety analysis (JSA) is known by many names, including activity hazard analysis (AHA), or job hazard analysis (JHA). Hazard analysis helps integrate accepted safety and health principles and practices into a specific task. In a JSA, each basic step of the job is reviewed, potential hazards identified, and recommendations documented as to the safest way to do the job. JSA techniques work well when used on a task that the analysts understand well. JSA analysts look for specific types of potential accidents and ask basic questions about each step, such as these:

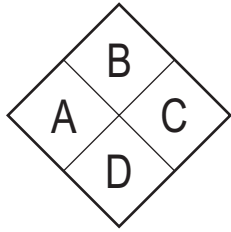
- Can the employee strike against or otherwise make injurious contact with the object?
- Can the employee be caught in, on, or between objects?
- Can the employee strain muscles by pushing, pulling, or lifting?
- Is exposure to toxic gases, vapors, dust, heat, electrical currents, or radiation possible?



## HAZARD ASSESSMENTS

### Hazard Assessment

The fire/hazard diamond below summarizes common hazard data available on the MSDS and is frequently shown on chemical labels.



#### Position A – Health Hazard (Blue)

- 0 = normal material
- 1 = slightly hazardous
- 2 = hazardous
- 3 = extreme danger
- 4 = deadly

#### Position B – Flammability (Red)

- 0 = will not burn
- 1 = will ignite if preheated
- 2 = will ignite if moderately heated
- 3 = will ignite at most ambient temperature
- 4 = burns readily at ambient conditions

#### Position C – Reactivity (Yellow)

- 0 = stable and not reactive with water
- 1 = unstable if heated
- 2 = violent chemical change
- 3 = shock short may detonate
- 4 = may detonate

#### Position D – (White)

- ALKALI = alkali
- OXY = oxidizer
- ACID = acid
- Cor = corrosive
- W = use no water
- = radiation hazard

### Flammability

*Flammable* describes any solid, liquid, vapor, or gas that will ignite easily and burn rapidly. A flammable liquid is defined by NFPA and USDOT as a liquid with a flash point below 100°F (38°C). Flammability is further defined with lower and upper limits:

LFL = lower flammability limit (volume % in air)

UFL = upper flammability limit (volume % in air)

A vapor-air mixture will only ignite and burn over the range of concentrations between LFL and UFL. Examples are:

Compound	LFL	UFL
Ethyl alcohol	3.3	19
Ethyl ether	1.9	36
Ethylene	2.7	36
Methane	5	15
Propane	2.1	9.5

#### ♦ Predicting Lower Flammable Limits of Mixtures of Flammable Gases (Le Chatelier's Rule)

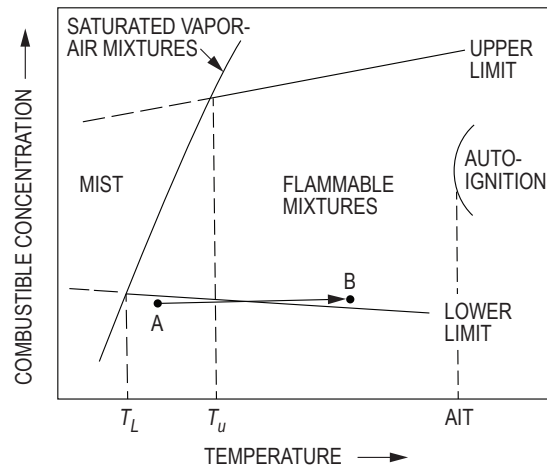
Based on an empirical rule developed by Le Chatelier, the lower flammable limit of mixtures of multiple flammable gases in air can be determined. A generalization of Le Chatelier's rule is

$$\sum_{i=1}^n (C_i / \text{LFL}_i) \geq 1$$

where  $C_i$  is the volume percent of fuel gas,  $i$ , in the fuel/air mixture and  $\text{LFL}_i$  is the volume percent of fuel gas,  $i$ , at its lower flammable limit in air alone. If the indicated sum is greater than unity, the mixture is above the lower flammable limit. This can be restated in terms of the lower flammable limit concentration of the fuel mixture,  $\text{LFL}_m$ , as follows:

$$\text{LFL}_m = \frac{100}{\sum_{i=1}^n (C_{fi} / \text{LFL}_i)}$$

where  $C_{fi}$  is the volume percent of fuel gas  $i$  in the fuel gas mixture.



♦ *The SFPE Handbook of Fire Protection Engineering*, National Fire Protection Association, 1988. With permission from the Society of Fire Protection Engineers.

## Material Safety Data Sheets (MSDS)

An MSDS contains technical information on the product, including chemical source, composition, hazards and health effects, first aid, firefighting precautions, accidental-release measures, handling and storage, exposure controls and personal protection, physical and chemical properties, stability and reactivity, toxicological information, ecological hazards, disposal, transport, and other regulatory information.

The MSDS forms for all chemical compounds stored, handled, or used on-site should be filed by a designated site safety officer.

The MSDS form is provided by the supplier or must be developed when new chemicals are synthesized.

**Signal Words.** The signal word found on every product's label is based on test results from various oral, dermal, and inhalation toxicity tests, as well skin and eye corrosion assays in some cases. Signal words are placed on labels to convey a level of care that should be taken (especially personal protection) when handling and using a product, from purchase to disposal of the empty container, as demonstrated by the Pesticide Toxicity Table.

**Pesticide Toxicity Categories**

Signal Word on Label	Toxicity Category	Acute-Oral LD <sub>50</sub> for Rats	Amount Needed to Kill an Average Size Adult	Notes
Danger–Poison	Highly Toxic	50 or less	Taste to a teaspoon	Skull and crossbones; Keep Out of Reach of Children
Warning	Moderately Toxic	50 to 500	One to six teaspoons	Keep Out of Reach of Children
Caution	Slightly Toxic	500 to 5,000	One ounce to a pint	Keep Out of Reach of Children
Caution	Relatively Non-Toxic	>5,000	More than a pint	Keep Out of Reach of Children

LD<sub>50</sub> - See Risk Assessment/Toxicology section on page 9.

From *Regulating Pesticides*, U.S. Environmental Protection Agency.

## Granular Storage and Process Safety

Some materials that are not inherently hazardous can become hazardous during storage or processing. An example is the handling of grain in grain bins. Grain bins should not be entered when the grain is being removed since grains flow to the center of the emptying bin and create suffocation hazards. Bridging may occur at the top surface due to condensation and resulting spoilage creating a crust.

Organic vapors and dusts associated with grain handling often contain toxic yeasts or molds and have low oxygen contents. These organic vapors and dusts may also be explosive.

## Confined Space Safety

Many workplaces contain spaces that are considered “confined” because their configurations hinder the activities of employees who must enter, work in, and exit them. A confined space has limited or restricted means for entry or exit and is not designed for continuous employee occupancy. Confined spaces include, but are not limited to, underground vaults, tanks, storage bins, manholes, pits, silos, process vessels, and pipelines. OSHA uses the term “permit-required confined spaces” (permit space) to describe a confined space that has one or more of the following characteristics: contains or has the potential to contain a hazardous atmosphere; contains a material that has the potential to engulf an entrant; has walls that converge inward or floors that slope downward and taper into a smaller area that could trap or asphyxiate an entrant; or contains any other recognized safety or health hazard such as unguarded machinery, exposed live wires or heat stress.

OSHA has developed OSHA standards, directives (instructions for compliance officers), standard interpretations (official letters of interpretation of the standards), and national consensus standards related to confined spaces. The following gases are often present in confined spaces:

Ammonia – irritating at 50 ppm and deadly above 1,000 ppm; sharp, cutting odor

Hydrogen sulfide – irritating at 10 ppm and deadly at 500 ppm; accumulates at lower levels and in corners where circulation is minimal; rotten egg odor

Methane – explosive at levels above 50,000 ppm, lighter than air, odorless

Carbon dioxide – heavier than air, accumulates at lower levels and in corners where circulation is minimal, displaces air leading to asphyxiation

## Electrical Safety

Current Level (Milliamperes)	Probable Effect on Human Body
1 mA	Perception level. Slight tingling sensation. Still dangerous under certain conditions.
5 mA	Slight shock felt; not painful but disturbing. Average individual can let go. However, strong involuntary reactions to shocks in this range may lead to injuries.
6 mA–16 mA	Painful shock, begin to lose muscular control. Commonly referred to as the freezing current or "let-go" range.
17 mA–99 mA	Extreme pain, respiratory arrest, severe muscular contractions. Individual cannot let go. Death is possible.
100 mA–2,000 mA	Ventricular fibrillation (uneven, uncoordinated pumping of the heart.) Muscular contraction and nerve damage begins to occur. Death is likely.
> 2,000 mA	Cardiac arrest, internal organ damage, and severe burns. Death is probable.

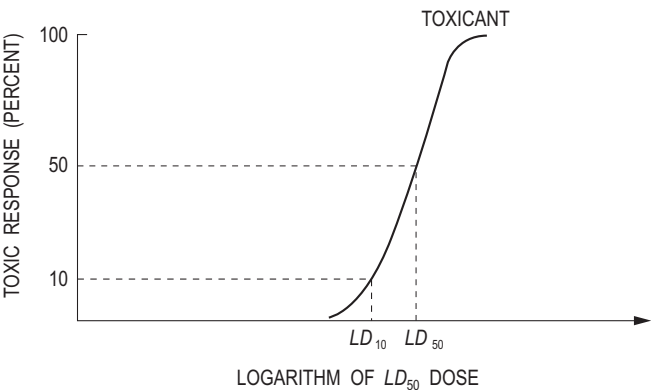
NIOSH [1998]. Worker Deaths by Electrocution; A Summary of NIOSH Surveillance and Investigative Findings. Ohio: U.S. Health and Human Services.

Greenwald E.K. [1991]. Electrical Hazards and Accidents—Their Cause and Prevention. New York: Van Nostrand Reinhold.

RISK ASSESSMENT/TOXICOLOGY

Dose-Response Curves

The dose-response curve relates toxic response (i.e., percentage of test population exhibiting a specified symptom or dying) to the logarithm of the dosage [i.e., mg/(kg·day) ingested]. A typical dose-response curve is shown below.



**LC<sub>50</sub>**  
Median lethal concentration in air that, based on laboratory tests, is expected to kill 50% of a group of test animals when administered as a single exposure over one or four hours.

**LD<sub>50</sub>**  
Median lethal single dose, based on laboratory tests, expected to kill 50% of a group of test animals, usually by oral or skin exposure.

Similar definitions exist for **LC<sub>10</sub>** and **LD<sub>10</sub>**, where the corresponding percentages are 10%.

The following table lists the probable effect on the human body of different current levels.

Comparative Acutely Lethal Doses		
Actual Ranking No.	LD <sub>50</sub> (mg/kg)	Toxic Chemical
1	15,000	PCBs
2	10,000	Alcohol (ethanol)
3	4,000	Table salt—sodium chloride
4	1,500	Ferrous sulfate—an iron supplement
5	1,375	Malathion—pesticide
6	900	Morphine
7	150	Phenobarbital—a sedative
8	142	Tylenol (acetaminophen)
9	2	Strychnine—a rat poison
10	1	Nicotine
11	0.5	Curare—an arrow poison
12	0.001	2,3,7,8-TCDD (dioxin)
13	0.00001	Botulinum toxin (food poison)

Selected Chemical Interaction Effects

Effect	Relative toxicity (hypothetical)	Example
Additive	2 + 3 = 5	Organophosphate pesticides
Synergistic	2 + 3 = 20	Cigarette smoking + asbestos
Antagonistic	6 + 6 = 8	Toluene + benzene or caffeine + alcohol

Exposure Limits for Selected Compounds		
N	Allowable Workplace Exposure Level (mg/m <sup>3</sup> )	Chemical (use)
1	0.1	Iodine
2	5	Aspirin
3	10	Vegetable oil mists (cooking oil)
4	55	1,1,2-Trichloroethane (solvent/degreaser)
5	188	Perchloroethylene (dry-cleaning fluid)
6	170	Toluene (organic solvent)
7	269	Trichloroethylene (solvent/degreaser)
8	590	Tetrahydrofuran (organic solvent)
9	890	Gasoline (fuel)
10	1,590	Naphtha (rubber solvent)
11	1,910	1,1,1-Trichloroethane (solvent/degreaser)

◆ Adapted from *Loomis's Essentials of Toxicology*, 4th ed., Loomis, T.A., and A.W. Hayes, Academic Press, San Diego, 1996.  
● Adapted from Williams, P.L., R.C. James, and S.M. Roberts, *Principles of Toxicology: Environmental and Industrial Applications*, 2nd ed., Wiley, 2000.  
■ American Conference of Government Industrial Hygienists (ACGIH) 1996.

## Carcinogens

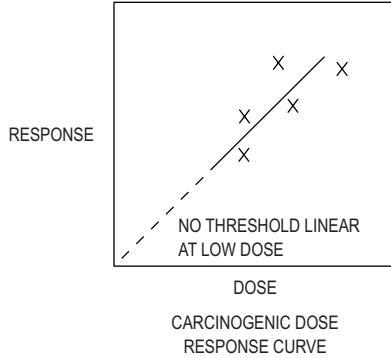
For carcinogens, the added risk of cancer is calculated as follows:

$$\text{Risk} = \text{dose} \times \text{toxicity} = CDI \times CSF$$

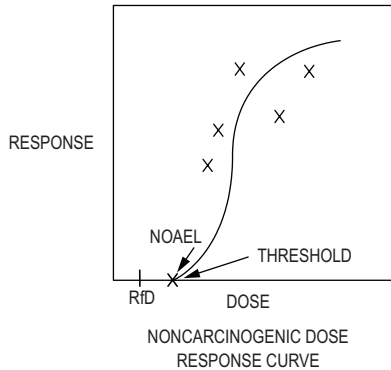
where

$CDI$  = Chronic Daily Intake

$CSF$  = Cancer Slope Factor. Slope of the dose-response curve for carcinogenic materials.



## Noncarcinogens



Dose is expressed

$$\left( \frac{\text{mass of chemical}}{\text{body weight} \cdot \text{exposure time}} \right)$$

$NOAEL$  = No Observable Adverse Effect Level. The dose below which there are no harmful effects

For noncarcinogens, a hazard index ( $HI$ ) is calculated as follows:

$$HI = \text{chronic daily intake} / RfD$$

## Reference Dose

Reference dose ( $RfD$ ) is determined from the Noncarcinogenic Dose-Response Curve Using  $NOAEL$ .

$$RfD = \frac{NOAEL}{UF}$$

and

$$SHD = RfD \times W = \frac{NOAEL \times W}{UF}$$

where

$SHD$  = safe human dose (mg/day)

$NOAEL$  = threshold dose per kg test animal [mg/(kg•day)] from the dose-response curve

$UF$  = the total uncertainty factor, depending on nature and reliability of the animal test data

$W$  = the weight of the adult male (typically 70 kg)

## Threshold Limit Value (TLV)

TLV is the highest dose (ppm by volume in the atmosphere) the body is able to detoxify without any detectable effects.

Examples are:

Compound	TLV
Ammonia	25
Chlorine	0.5
Ethyl Chloride	1,000
Ethyl Ether	400

## HAZARDOUS WASTE COMPATIBILITY CHART

Reactivity Group		Reactivity															
No.	Name	Reactivity															
1	Acid, Minerals, Non-Oxidizing	1															
2	Acids, Minerals, Oxidizing	2															
3	Acids, Organic	3															
4	Alcohols & Glycols	4															
5	Aldehydes	5															
6	Amides	6															
7	Amines, Aliphatic & Aromatic	7															
8	Azo Compounds, Diazo Comp, Hydrazines	8															
9	Carbamates	9															
10	Caustics	10															
11	Cyanides	11															
12	Dithiocarbamates	12															
13	Esters	13															
14	Ethers	14															
15	Fluorides, Inorganic	15															
16	Hydrocarbons, Aromatic	16															
17	Halogenated Organics	17															
18	Isocyanates	18															
19	Ketones	19															
20	Mercaptans & Other Organic Sulfides	20															
21	Metal, Alkali & Alkaline Earth, Elemental	21															
104	Oxidizing Agents, Strong	104															
105	Reducing Agents, Strong	105															
106	Water & Mixtures Containing Water	106															
107	Water Reactive Substances	107															

## Exposure

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### Residential Exposure Equations for Various Pathways

Ingestion in drinking water

$$CDI = \frac{(CW)(IR)(EF)(ED)}{(BW)(AT)}$$

Ingestion while swimming

$$CDI = \frac{(CW)(CR)(ET)(EF)(ED)}{(BW)(AT)}$$

Dermal contact with water

$$AD = \frac{(CW)(SA)(PC)(ET)(EF)(ED)(CF)}{(BW)(AT)}$$

Ingestion of chemicals in soil

$$CDI = \frac{(CS)(IR)(CF)(FI)(EF)(ED)}{(BW)(AT)}$$

Dermal contact with soil

$$AD = \frac{(CS)(CF)(SA)(AF)(ABS)(EF)(ED)}{(BW)(AT)}$$

Inhalation of airborne (vapor phase) chemicals

$$CDI = \frac{(CA)(IR)(ET)(EF)(ED)}{(BW)(AT)}$$

Ingestion of contaminated fruits, vegetables, fish and shellfish

$$CDI = \frac{(CF)(IR)(FI)(EF)(ED)}{(BW)(AT)}$$

where ABS = absorption factor for soil contaminant (unitless)

AD = absorbed dose (mg/[kg•day])

AF = soil-to-skin adherence factor (mg/cm<sup>2</sup>)

AT = averaging time (days)

BW = body weight (kg)

CA = contaminant concentration in air (mg/m<sup>3</sup>)

CDI = chronic daily intake (mg/[kg•day])

CF = volumetric conversion factor for water  
= 1 L/1,000 cm<sup>3</sup>

= conversion factor for soil = 10<sup>-6</sup> kg/mg

CR = contact rate (L/hr)

CS = chemical concentration in soil (mg/kg)

CW = chemical concentration in water (mg/L)

ED = exposure duration (years)

EF = exposure frequency (days/yr or events/year)

ET = exposure time (hr/day or hr/event)

FI = fraction ingested (unitless)

IR = ingestion rate (L/day or mg soil/day or kg/meal)  
= inhalation rate (m<sup>3</sup>/hr)

PC = chemical-specific dermal permeability constant  
(cm/hr)

SA = skin surface area available for contact (cm<sup>2</sup>)

*Risk Assessment Guidance for Superfund. Volume 1, Human Health Evaluation Manual (part A). U.S. Environmental Protection Agency, EPA/540/1-89/002, 1989.*



## Intake Rates

EPA Recommended Values for Estimating Intake

Parameter	Standard Value
Average body weight, female adult	65.4 kg
Average body weight, male adult	78 kg
Average body weight, child <sup>a</sup>	
6–11 months	9 kg
1–5 years	16 kg
6–12 years	33 kg
Amount of water ingested, adult	2.3 L/day
Amount of water ingested, child	1.5 L/day
Amount of air breathed, female adult	11.3 m <sup>3</sup> /day
Amount of air breathed, male adult	15.2 m <sup>3</sup> /day
Amount of air breathed, child (3–5 years)	8.3 m <sup>3</sup> /day
Amount of fish consumed, adult	6 g/day
Water swallowing rate, while swimming	50 mL/hr
Inhalation rates	
adult (6-hr day)	0.98 m <sup>3</sup> /hr
adult (2-hr day)	1.47 m <sup>3</sup> /hr
child	0.46 m <sup>3</sup> /hr
Skin surface available, adult male	1.94 m <sup>2</sup>
Skin surface available, adult female	1.69 m <sup>2</sup>
Skin surface available, child	
3–6 years (average for male and female)	0.720 m <sup>2</sup>
6–9 years (average for male and female)	0.925 m <sup>2</sup>
9–12 years (average for male and female)	1.16 m <sup>2</sup>
12–15 years (average for male and female)	1.49 m <sup>2</sup>
15–18 years (female)	1.60 m <sup>2</sup>
15–18 years (male)	1.75 m <sup>2</sup>
Soil ingestion rate, child 1–6 years	>100 mg/day
Soil ingestion rate, persons > 6 years	50 mg/day
Skin adherence factor, gardener's hands	0.07 mg/cm <sup>2</sup>
Skin adherence factor, wet soil	0.2 mg/cm <sup>2</sup>
Exposure duration	
Lifetime (carcinogens, for noncarcinogens use actual exposure duration)	75 years
At one residence, 90th percentile	30 years
National median	5 years
Averaging time	(ED)(365 days/year)
Exposure frequency (EF)	
Swimming	7 days/year
Eating fish and shellfish	48 days/year
Exposure time (ET)	
Shower, 90th percentile	12 min
Shower, 50th percentile	7 min

<sup>a</sup>Data in this category taken from: Copeland, T., A. M. Holbrow, J. M. Otan, et al., "Use of probabilistic methods to understand the conservatism in California's approach to assessing health risks posed by air contaminants," *Journal of the Air and Waste Management Association*, vol. 44, pp. 1399-1413, 1994.

*Risk Assessment Guidance for Superfund*. Volume 1, *Human Health Evaluation Manual* (part A). U.S. Environmental Protection Agency, EPA/540/I-89/002, 1989.

## Concentrations of Vaporized Liquids

### Vaporization Rate ( $Q_m$ , mass/time) from a Liquid Surface

$$Q_m = [MK A_S P^{\text{sat}} / (R_g T_L)]$$

$M$  = molecular weight of volatile substance

$K$  = mass transfer coefficient

$A_S$  = area of liquid surface

$P^{\text{sat}}$  = saturation vapor pressure of the pure liquid at  $T_L$

$R_g$  = ideal gas constant

$T_L$  = absolute temperature of the liquid

### Mass Flow Rate of Liquid from a Hole in the Wall of a Process Unit

$$Q_m = A_H C_0 (2\rho g_c P_g)^{1/2}$$

$A_H$  = area of hole

$C_0$  = discharge coefficient

$\rho$  = density of the liquid

$g_c$  = gravitational constant

$P_g$  = gauge pressure within the process unit

### Concentration ( $C_{\text{ppm}}$ ) of Vaporized Liquid in Ventilated Space

$$C_{\text{ppm}} = [Q_m R_g T \times 10^6 / (k Q_V P M)]$$

$T$  = absolute ambient temperature

$k$  = non-ideal mixing factor

$Q_V$  = ventilation rate

$P$  = absolute ambient pressure

## Sweep-Through Concentration Change in a Vessel

$$Q_V t = V \ln[(C_1 - C_0) / (C_2 - C_0)]$$

$Q_V$  = volumetric flow rate

$t$  = time

$V$  = vessel volume

$C_0$  = inlet concentration

$C_1$  = initial concentration

$C_2$  = final concentration

## ERGONOMICS

### NIOSH Formula

#### Recommended Weight Limit (RWL)

$$RWL = 51(10/H)(1 - 0.0075|V - 30|)(0.82 + 1.8/D)(1 - 0.0032A)(FM)(CM)$$

where

RWL= recommended weight limit, in pounds

H = horizontal distance of the hand from the midpoint of the line joining the inner ankle bones to a point projected on the floor directly below the load center, in inches

V = vertical distance of the hands from the floor, in inches

D = vertical travel distance of the hands between the origin and destination of the lift, in inches

A = asymmetry angle, in degrees

FM = frequency multiplier (see table)

CM = coupling multiplier (see table)

**Frequency Multiplier Table**

	≤ 8 hr/day		≤ 2 hr/day		≤ 1 hr/day	
F, min <sup>-1</sup>	V < 30 in.	V ≥ 30 in.	V < 30 in.	V ≥ 30 in.	V < 30 in.	V ≥ 30 in.
0.2	0.85		0.95		1.00	
0.5	0.81		0.92		0.97	
1	0.75		0.88		0.94	
2	0.65		0.84		0.91	
3	0.55		0.79		0.88	
4	0.45		0.72		0.84	
5	0.35		0.60		0.80	
6	0.27		0.50		0.75	
7	0.22		0.42		0.70	
8	0.18		0.35		0.60	
9		0.15	0.30		0.52	
10		0.13	0.26		0.45	
11				0.23	0.41	
12				0.21	0.37	
13			0.00			0.34
14						0.31
15						0.28

Waters, Thomas R., Ph.D., et al, *Applications Manual for the Revised NIOSH Lifting Equation*, Table 5, U.S. Department of Health and Human Services (NIOSH), January 1994.

**Coupling Multiplier (CM) Table**  
(Function of Coupling of Hands to Load)

Container		Loose Part / Irreg. Object		
Optimal Design		Not	Comfort Grip	Not
Opt. Handles or Cut-outs	Not	POOR	GOOD	
GOOD	Flex Fingers 90 Degrees	Not		
	FAIR	POOR		

Coupling	V < 30 in. or 75 cm	V ≥ 30 in. or 75 cm
GOOD	1.00	
FAIR	0.95	
POOR	0.90	

Waters, Thomas R., Ph.D., et al, *Applications Manual for the Revised NIOSH Lifting Equation*, Table 7, U.S. Department of Health and Human Services (NIOSH), January 1994.

## Biomechanics of the Human Body

### Basic Equations

$$H_x + F_x = 0$$

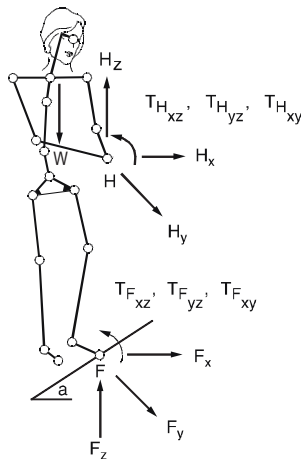
$$H_y + F_y = 0$$

$$H_z + W + F_z = 0$$

$$T_{Hxz} + T_{Wxz} + T_{Fxz} = 0$$

$$T_{Hyz} + T_{Wyz} + T_{Fyz} = 0$$

$$T_{Hxy} + T_{Fxy} = 0$$



The coefficient of friction  $\mu$  and the angle  $\alpha$  at which the floor is inclined determine the equations at the foot.

$$F_x = \mu F_z$$

With the slope angle  $\alpha$

$$F_x = \alpha F_z \cos \alpha$$

Of course, when motion must be considered, dynamic conditions come into play according to Newton's Second Law. Force transmitted with the hands is counteracted at the foot. Further, the body must also react with internal forces at all points between the hand and the foot.

### Incidence Rates

Two concepts can be important when completing OSHA forms. These concepts are *incidence rates* and *severity rates*. On occasion it is necessary to calculate the total injury/illness incident rate of an organization in order to complete OSHA forms. This calculation must include fatalities and all injuries requiring medical treatment beyond mere first aid. The formula for determining the total injury/illness incident rate is as follows:

$$IR = N \times 200,000 \div T$$

$IR$  = Total injury/illness incidence rate

$N$  = Number of injuries, illnesses, and fatalities

$T$  = Total hours worked by all employees during the period in question

The number 200,000 in the formula represents the number of hours 100 employees work in a year (40 hours per week  $\times$  50 weeks = 2,000 hours per year per employee). Using the same basic formula with only minor substitutions, safety managers can calculate the following types of incidence rates:

1. Injury rate
2. Illness rate
3. Fatality rate
4. Lost workday cases rate
5. Number of lost workdays rate
6. Specific hazard rate
7. Lost workday injuries rate

### NOISE POLLUTION

$$\text{SPL (dB)} = 10 \log_{10} (P^2 / P_0^2)$$

$$\text{SPL}_{\text{total}} = 10 \log_{10} \Sigma 10^{\text{SPL}/10}$$

Point Source Attenuation

$$\Delta \text{SPL (dB)} = 10 \log_{10} (r_1 / r_2)^2$$

Line Source Attenuation

$$\Delta \text{SPL (dB)} = 10 \log_{10} (r_1 / r_2)$$

where

$\text{SPL (dB)}$  = sound pressure level, measured in decibels

$P$  = sound pressure (Pa)

$P_0$  = reference sound pressure ( $2 \times 10^{-5}$  Pa)

$\text{SPL}_{\text{total}}$  = sum of multiple sources

$\Delta \text{SPL (dB)}$  = change in sound pressure level with distance, measured in decibels

$r_1$  = distance from source to receptor at point 1

$r_2$  = distance from source to receptor at point 2

### PERMISSIBLE NOISE EXPOSURE (OSHA)

Noise dose  $D$  should not exceed 100%.

$$D = 100\% \times \Sigma \frac{C_i}{T_i}$$

where  $C_i$  = time spent at specified sound pressure level, SPL, (hours)

$T_i$  = time permitted at SPL (hours)

$\Sigma C_i$  = 8 (hours)

Noise Level (dBA)	Permissible Time (hr)
80	32
85	16
90	8
95	4
100	2
105	1
110	0.5
115	0.25
120	0.125
125	0.063
130	0.031

If  $D > 100\%$ , noise abatement required.

If  $50\% \leq D \leq 100\%$ , hearing conservation program required.

Note:  $D = 100\%$  is equivalent to 90 dBA time-weighted average (TWA).  $D = 50\%$  equivalent to TWA of 85 dBA.

Hearing conservation program requires: (1) testing employee hearing, (2) providing hearing protection at employee's request, and (3) monitoring noise exposure.

Exposure to impulsive or impact noise should not exceed 140 dB sound pressure level (SPL).

# MATHEMATICS

## DISCRETE MATH

### Symbols

$x \in X$	$x$ is a member of $X$
$\{ \}, \phi$	The empty (or null) set
$S \subseteq T$	$S$ is a subset of $T$
$S \subset T$	$S$ is a proper subset of $T$
$(a, b)$	Ordered pair
$P^{(s)}$	Power set of $S$
$(a_1, a_2, \dots, a_n)$	$n$ -tuple
$A \times B$	Cartesian product of $A$ and $B$
$A \cup B$	Union of $A$ and $B$
$A \cap B$	Intersection of $A$ and $B$
$\forall x$	Universal qualification for all $x$ ; for any $x$ ; for each $x$
$\exists y$	Uniqueness qualification there exists $y$

A binary relation from  $A$  to  $B$  is a subset of  $A \times B$ .

### Matrix of Relation

If  $A = \{a_1, a_2, \dots, a_m\}$  and  $B = \{b_1, b_2, \dots, b_n\}$  are finite sets containing  $m$  and  $n$  elements, respectively, then a relation  $R$  from  $A$  to  $B$  can be represented by the  $m \times n$  matrix  $M_R = [m_{ij}]$ , which is defined by:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

### Directed Graphs or Digraphs of Relation

A directed graph or digraph, consists of a set  $V$  of vertices (or nodes) together with a set  $E$  of ordered pairs of elements of  $V$  called edges (or arcs). For edge  $(a, b)$ , the vertex  $a$  is called the initial vertex and vertex  $b$  is called the terminal vertex. An edge of form  $(a, a)$  is called a loop.

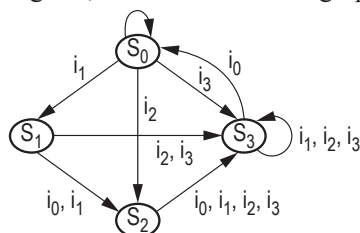
### Finite State Machine

A finite state machine consists of a finite set of states  $S_i = \{s_0, s_1, \dots, s_n\}$  and a finite set of inputs  $I$ ; and a transition function  $f$  that assigns to each state and input pair a new state.

A state (or truth) table can be used to represent the finite state machine.

State	Input			
	$i_0$	$i_1$	$i_2$	$i_3$
$S_0$	$S_0$	$S_1$	$S_2$	$S_3$
$S_1$	$S_2$	$S_2$	$S_3$	$S_3$
$S_2$	$S_3$	$S_3$	$S_3$	$S_3$
$S_3$	$S_0$	$S_3$	$S_3$	$S_3$

Another way to represent a finite state machine is to use a state diagram, which is a directed graph with labeled edges.



The characteristic of how a function maps one set ( $X$ ) to another set ( $Y$ ) may be described in terms of being either injective, surjective, or bijective.

An injective (one-to-one) relationship exists if, and only if,  $\forall x_1, x_2 \in X$ , if  $f(x_1) = f(x_2)$ , then  $x_1 = x_2$

A surjective (onto) relationship exists when  $\forall y \in Y, \exists x \in X$  such that  $f(x) = y$

A bijective relationship is both injective (one-to-one) and surjective (onto).

## STRAIGHT LINE

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is  $y - y_1 = m(x - x_1)$

Given two points: slope,  $m = (y_2 - y_1)/(x_2 - x_1)$

The angle between lines with slopes  $m_1$  and  $m_2$  is

$$\alpha = \arctan [(m_2 - m_1)/(1 + m_2 \cdot m_1)]$$

Two lines are perpendicular if  $m_1 = -1/m_2$

The distance between two points is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

## QUADRATIC EQUATION

$$ax^2 + bx + c = 0$$

$$x = \text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## QUADRIC SURFACE (SPHERE)

The standard form of the equation is

$$(x - h)^2 + (y - k)^2 + (z - m)^2 = r^2$$

with center at  $(h, k, m)$ .

In a three-dimensional space, the distance between two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

## LOGARITHMS

The logarithm of  $x$  to the Base  $b$  is defined by

$$\log_b(x) = c, \text{ where } b^c = x$$

Special definitions for  $b = e$  or  $b = 10$  are:

$$\ln x, \text{ Base} = e$$

$$\log x, \text{ Base} = 10$$

To change from one Base to another:

$$\log_b x = (\log_a x) / (\log_a b)$$

$$\text{e.g., } \ln x = (\log_{10} x) / (\log_{10} e) = 2.302585 (\log_{10} x)$$

### Identities

$$\log_b b^n = n$$

$$\log x^c = c \log x; x^c = \text{antilog } (c \log x)$$

$$\log xy = \log x + \log y$$

$$\log_b b = 1; \log 1 = 0$$

$$\log x/y = \log x - \log y$$

## ALGEBRA OF COMPLEX NUMBERS

Complex numbers may be designated in rectangular form or polar form. In rectangular form, a complex number is written in terms of its real and imaginary components.

$$z = a + jb, \text{ where}$$

$a$  = the real component,

$b$  = the imaginary component, and

$$j = \sqrt{-1} \text{ (some disciplines use } i = \sqrt{-1} \text{)}$$

In polar form  $z = c \angle \theta$  where

$$c = \sqrt{a^2 + b^2},$$

$$\theta = \tan^{-1}(b/a),$$

$$a = c \cos \theta, \text{ and}$$

$$b = c \sin \theta.$$

Complex numbers can be added and subtracted in rectangular form. If

$$z_1 = a_1 + jb_1 = c_1 (\cos \theta_1 + j \sin \theta_1)$$

$$= c_1 \angle \theta_1 \text{ and}$$

$$z_2 = a_2 + jb_2 = c_2 (\cos \theta_2 + j \sin \theta_2)$$

$$= c_2 \angle \theta_2, \text{ then}$$

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \text{ and}$$

$$z_1 - z_2 = (a_1 - a_2) + j(b_1 - b_2)$$

While complex numbers can be multiplied or divided in rectangular form, it is more convenient to perform these operations in polar form.

$$z_1 \times z_2 = (c_1 \times c_2) \angle (\theta_1 + \theta_2)$$

$$z_1/z_2 = (c_1/c_2) \angle (\theta_1 - \theta_2)$$

The complex conjugate of a complex number  $z_1 = (a_1 + jb_1)$  is defined as  $z_1^* = (a_1 - jb_1)$ . The product of a complex number and its complex conjugate is  $z_1 z_1^* = a_1^2 + b_1^2$ .

## Polar Coordinate System

$$x = r \cos \theta; y = r \sin \theta; \theta = \arctan (y/x)$$

$$r = |x + jy| = \sqrt{x^2 + y^2}$$

$$x + jy = r (\cos \theta + j \sin \theta) = r e^{j\theta}$$

$$[r_1 (\cos \theta_1 + j \sin \theta_1)] [r_2 (\cos \theta_2 + j \sin \theta_2)] =$$

$$r_1 r_2 [\cos (\theta_1 + \theta_2) + j \sin (\theta_1 + \theta_2)]$$

$$(x + jy)^n = [r (\cos \theta + j \sin \theta)]^n$$

$$= r^n (\cos n\theta + j \sin n\theta)$$

$$\frac{r_1 (\cos \theta_1 + j \sin \theta_1)}{r_2 (\cos \theta_2 + j \sin \theta_2)} = \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + j \sin (\theta_1 - \theta_2)]$$

### Euler's Identity

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

### Roots

If  $k$  is any positive integer, any complex number (other than zero) has  $k$  distinct roots. The  $k$  roots of  $r (\cos \theta + j \sin \theta)$  can be found by substituting successively  $n = 0, 1, 2, \dots, (k-1)$  in the formula

$$w = \sqrt[k]{r} \left[ \cos \left( \frac{\theta}{k} + n \frac{360^\circ}{k} \right) + j \sin \left( \frac{\theta}{k} + n \frac{360^\circ}{k} \right) \right]$$

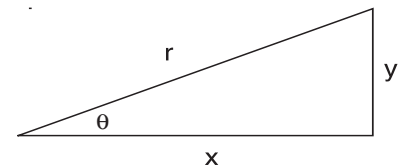
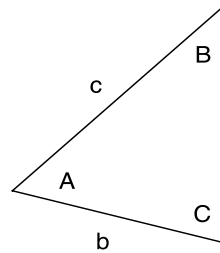
## TRIGONOMETRY

Trigonometric functions are defined using a right triangle.

$$\sin \theta = y/r, \cos \theta = x/r$$

$$\tan \theta = y/x, \cot \theta = x/y$$

$$\csc \theta = r/y, \sec \theta = r/x$$



### Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

### Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Brink, R.W., *A First Year of College Mathematics*, D. Appleton-Century Co., Inc., Englewood Cliffs, NJ, 1937.



## Identities

$$\cos \theta = \sin (\theta + \pi/2) = -\sin (\theta - \pi/2)$$

$$\sin \theta = \cos (\theta - \pi/2) = -\cos (\theta + \pi/2)$$

$$\csc \theta = 1/\sin \theta$$

$$\sec \theta = 1/\cos \theta$$

$$\tan \theta = \sin \theta / \cos \theta$$

$$\cot \theta = 1/\tan \theta$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$$

$$\tan 2\alpha = (2 \tan \alpha) / (1 - \tan^2 \alpha)$$

$$\cot 2\alpha = (\cot^2 \alpha - 1) / (2 \cot \alpha)$$

$$\tan (\alpha + \beta) = (\tan \alpha + \tan \beta) / (1 - \tan \alpha \tan \beta)$$

$$\cot (\alpha + \beta) = (\cot \alpha \cot \beta - 1) / (\cot \alpha + \cot \beta)$$

$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan (\alpha - \beta) = (\tan \alpha - \tan \beta) / (1 + \tan \alpha \tan \beta)$$

$$\cot (\alpha - \beta) = (\cot \alpha \cot \beta + 1) / (\cot \beta - \cot \alpha)$$

$$\sin (\alpha/2) = \pm \sqrt{(1 - \cos \alpha) / 2}$$

$$\cos (\alpha/2) = \pm \sqrt{(1 + \cos \alpha) / 2}$$

$$\tan (\alpha/2) = \pm \sqrt{(1 - \cos \alpha) / (1 + \cos \alpha)}$$

$$\cot (\alpha/2) = \pm \sqrt{(1 + \cos \alpha) / (1 - \cos \alpha)}$$

$$\sin \alpha \sin \beta = (1/2)[\cos (\alpha - \beta) - \cos (\alpha + \beta)]$$

$$\cos \alpha \cos \beta = (1/2)[\cos (\alpha - \beta) + \cos (\alpha + \beta)]$$

$$\sin \alpha \cos \beta = (1/2)[\sin (\alpha + \beta) + \sin (\alpha - \beta)]$$

$$\sin \alpha + \sin \beta = 2 \sin [(1/2)(\alpha + \beta)] \cos [(1/2)(\alpha - \beta)]$$

$$\sin \alpha - \sin \beta = 2 \cos [(1/2)(\alpha + \beta)] \sin [(1/2)(\alpha - \beta)]$$

$$\cos \alpha + \cos \beta = 2 \cos [(1/2)(\alpha + \beta)] \cos [(1/2)(\alpha - \beta)]$$

$$\cos \alpha - \cos \beta = -2 \sin [(1/2)(\alpha + \beta)] \sin [(1/2)(\alpha - \beta)]$$

## MENSURATION OF AREAS AND VOLUMES

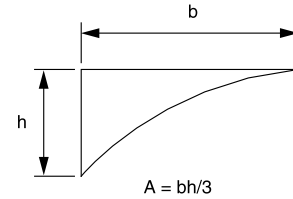
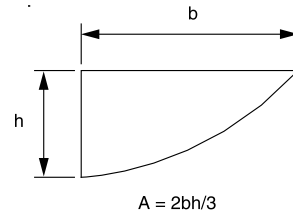
### Nomenclature

$A$  = total surface area

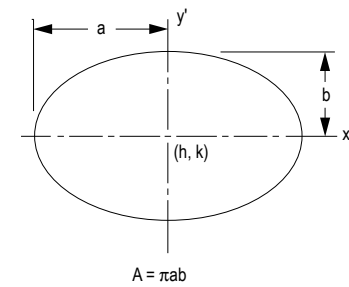
$P$  = perimeter

$V$  = volume

### Parabola



### Ellipse



$$P_{approx} = 2\pi \sqrt{(a^2 + b^2)/2}$$

$$P = \pi(a+b) \left[ 1 + \left(\frac{1}{2}\right)^2 \lambda^2 + \left(\frac{1}{2} \times \frac{1}{4}\right)^2 \lambda^4 + \left(\frac{1}{2} \times \frac{1}{4} \times \frac{3}{6}\right)^2 \lambda^6 + \left(\frac{1}{2} \times \frac{1}{4} \times \frac{3}{6} \times \frac{5}{8}\right)^2 \lambda^8 + \left(\frac{1}{2} \times \frac{1}{4} \times \frac{3}{6} \times \frac{5}{8} \times \frac{7}{10}\right)^2 \lambda^{10} + \dots \right]$$

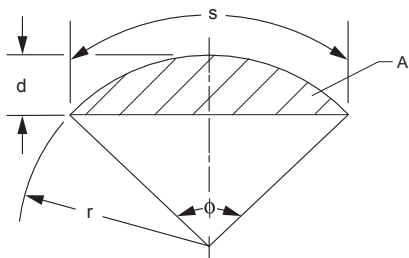
where

$$\lambda = (a - b) / (a + b)$$

♦ Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

## MENSURATION OF AREAS AND VOLUMES (continued)

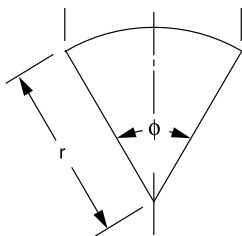
### Circular Segment



$$A = \left[ r^2 (\phi - \sin \phi) \right] / 2$$

$$\phi = s/r = 2 \left\{ \arccos[(r - d)/r] \right\}$$

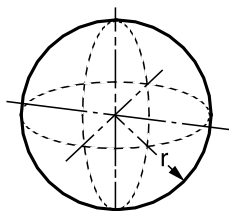
### Circular Sector



$$A = \phi r^2 / 2 = sr / 2$$

$$\phi = s/r$$

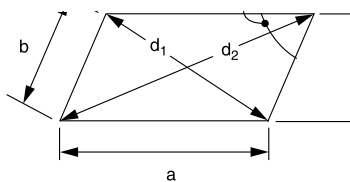
### Sphere



$$V = 4\pi r^3 / 3 = \pi d^3 / 6$$

$$A = 4\pi r^2 = \pi d^2$$

### Parallelogram



$$P = 2(a + b)$$

$$d_1 = \sqrt{a^2 + b^2 - 2ab(\cos \phi)}$$

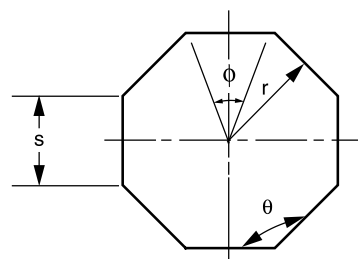
$$d_2 = \sqrt{a^2 + b^2 + 2ab(\cos \phi)}$$

$$d_1^2 + d_2^2 = 2(a^2 + b^2)$$

$$A = ah = ab(\sin \phi)$$

If  $a = b$ , the parallelogram is a rhombus.

### Regular Polygon ( $n$ equal sides)



$$\phi = 2\pi/n$$

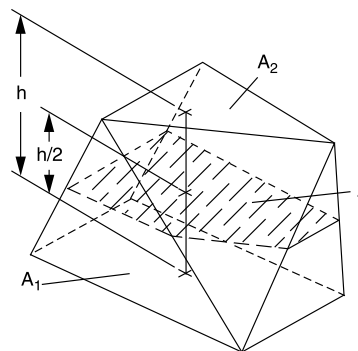
$$\theta = \left[ \frac{\pi(n-2)}{n} \right] = \pi \left( 1 - \frac{2}{n} \right)$$

$$P = ns$$

$$s = 2r \left[ \tan(\phi/2) \right]$$

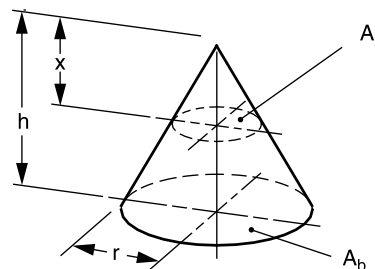
$$A = (nsr)/2$$

### Prismoid



$$V = (h/6)(A_1 + A_2 + 4A)$$

### Right Circular Cone



$$V = (\pi r^2 h) / 3$$

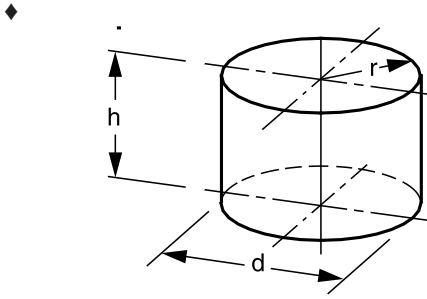
$$A = \text{side area} + \text{base area}$$

$$= \pi r(r + \sqrt{r^2 + h^2})$$

$$A_x : A_b = x^2 : h^2$$

♦ Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

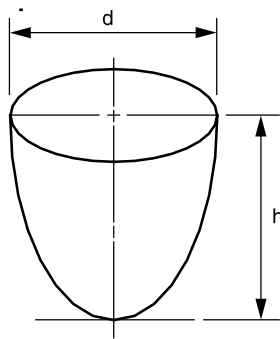
## Right Circular Cylinder



$$V = \pi r^2 h = \frac{\pi d^2 h}{4}$$

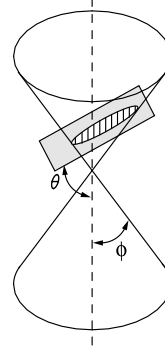
$$A = \text{side area} + \text{end areas} = 2\pi r(h + r)$$

## Paraboloid of Revolution



$$V = \frac{\pi d^2 h}{8}$$

## CONIC SECTIONS

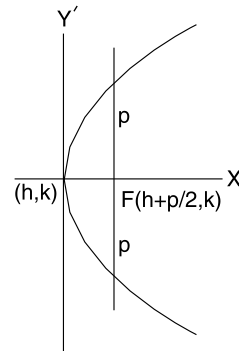


$$e = \text{eccentricity} = \cos \theta / (\cos \phi)$$

[Note:  $X'$  and  $Y'$ , in the following cases, are translated axes.]

### Case 1. Parabola $e = 1$ :

•

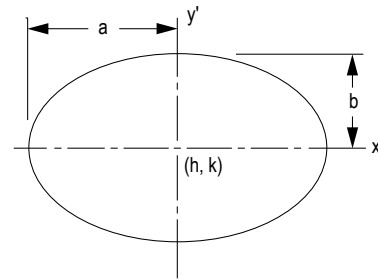


$$(y - k)^2 = 2p(x - h); \text{ Center at } (h, k)$$

is the standard form of the equation. When  $h = k = 0$ ,

Focus:  $(p/2, 0)$ ; Directrix:  $x = -p/2$

### Case 2. Ellipse $e < 1$ :



$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1; \text{ Center at } (h, k)$$

is the standard form of the equation. When  $h = k = 0$ ,

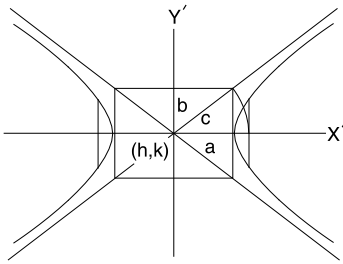
$$\text{Eccentricity: } e = \sqrt{1 - (b^2/a^2)} = c/a$$

$$b = a\sqrt{1 - e^2};$$

Focus:  $(\pm ae, 0)$ ; Directrix:  $x = \pm a/e$

- ♦ Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.
- Brink, R.W., *A First Year of College Mathematics*, D. Appleton-Century Co., Inc., 1937.

### Case 3. Hyperbola $e > 1$ :



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1; \text{ Center at } (h, k)$$

is the standard form of the equation. When  $h = k = 0$ ,

$$\text{Eccentricity: } e = \sqrt{1 + (b^2/a^2)} = c/a$$

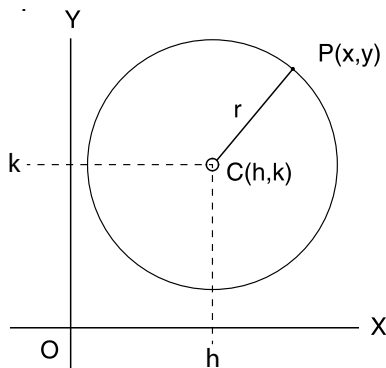
$$b = a\sqrt{e^2 - 1};$$

$$\text{Focus: } (\pm ae, 0); \text{ Directrix: } x = \pm a/e$$

### Case 4. Circle $e = 0$ :

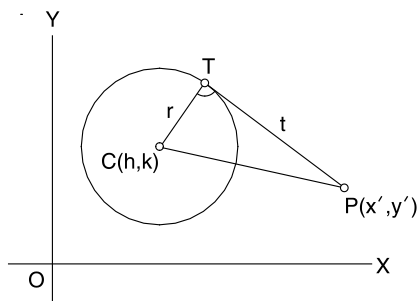
$(x-h)^2 + (y-k)^2 = r^2$ ; Center at  $(h, k)$  is the standard form of the equation with radius

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$



Length of the tangent line from a point on a circle to a point  $(x', y')$ :

$$t^2 = (x' - h)^2 + (y' - k)^2 - r^2$$



### Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where not both  $A$  and  $C$  are zero.

If  $B^2 - 4AC < 0$ , an *ellipse* is defined.

If  $B^2 - 4AC > 0$ , a *hyperbola* is defined.

If  $B^2 - 4AC = 0$ , the conic is a *parabola*.

If  $A = C$  and  $B = 0$ , a *circle* is defined.

If  $A = B = C = 0$ , a *straight line* is defined.

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$$h = -a; k = -b$$

$$r = \sqrt{a^2 + b^2 - c}$$

If  $a^2 + b^2 - c$  is positive, a *circle*, center  $(-a, -b)$ .

If  $a^2 + b^2 - c$  equals zero, a *point* at  $(-a, -b)$ .

If  $a^2 + b^2 - c$  is negative, locus is *imaginary*.

### DIFFERENTIAL CALCULUS

#### The Derivative

For any function  $y = f(x)$ ,

the derivative  $= D_x y = dy/dx = y'$

$$y' = \lim_{\Delta x \rightarrow 0} [(\Delta y)/(\Delta x)]$$

$$= \lim_{\Delta x \rightarrow 0} \{[f(x + \Delta x) - f(x)]/(\Delta x)\}$$

$y' =$  the slope of the curve  $f(x)$ .

#### Test for a Maximum

$y = f(x)$  is a maximum for

$x = a$ , if  $f'(a) = 0$  and  $f''(a) < 0$ .

#### Test for a Minimum

$y = f(x)$  is a minimum for

$x = a$ , if  $f'(a) = 0$  and  $f''(a) > 0$ .

#### Test for a Point of Inflection

$y = f(x)$  has a point of inflection at  $x = a$ ,

if  $f''(a) = 0$ , and

if  $f''(x)$  changes sign as  $x$  increases through  $x = a$ .

#### The Partial Derivative

In a function of two independent variables  $x$  and  $y$ , a derivative with respect to one of the variables may be found if the other variable is *assumed* to remain constant. If  $y$  is kept *fixed*, the function

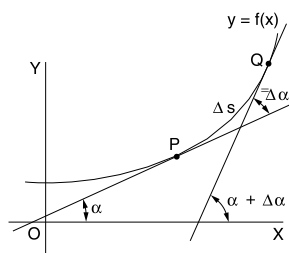
$$z = f(x, y)$$

becomes a function of the *single variable*  $x$ , and its derivative (if it exists) can be found. This derivative is called the *partial derivative of  $z$  with respect to  $x$* . The partial derivative with respect to  $x$  is denoted as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

## The Curvature of Any Curve

◆



The curvature  $K$  of a curve at  $P$  is the limit of its average curvature for the arc  $PQ$  as  $Q$  approaches  $P$ . This is also expressed as: the curvature of a curve at a given point is the rate-of-change of its inclination with respect to its arc length.

$$K = \lim_{\Delta s \rightarrow 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}$$

### Curvature in Rectangular Coordinates

$$K = \frac{y''}{[1 + (y')^2]^{3/2}}$$

When it may be easier to differentiate the function with respect to  $y$  rather than  $x$ , the notation  $x'$  will be used for the derivative.

$$x' = dx/dy$$

$$K = \frac{-x''}{[1 + (x')^2]^{3/2}}$$

### The Radius of Curvature

The *radius of curvature*  $R$  at any point on a curve is defined as the absolute value of the reciprocal of the curvature  $K$  at that point.

$$R = \frac{1}{|K|} \quad (K \neq 0)$$

$$R = \left| \frac{[1 + (y')^2]^{3/2}}{y''} \right| \quad (y'' \neq 0)$$

### L'Hospital's Rule (L'Hôpital's Rule)

If the fractional function  $f(x)/g(x)$  assumes one of the indeterminate forms  $0/0$  or  $\infty/\infty$  (where  $\alpha$  is finite or infinite), then

$$\lim_{x \rightarrow \alpha} f(x)/g(x)$$

is equal to the first of the expressions

$$\lim_{x \rightarrow \alpha} \frac{f'(x)}{g'(x)}, \lim_{x \rightarrow \alpha} \frac{f''(x)}{g''(x)}, \lim_{x \rightarrow \alpha} \frac{f'''(x)}{g'''(x)}$$

which is not indeterminate, provided such first indicated limit exists.

## INTEGRAL CALCULUS

The definite integral is defined as:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i = \int_a^b f(x) dx$$

Also,  $\Delta x_i \rightarrow 0$  for all  $i$ .

A table of derivatives and integrals is available in the Derivatives and Indefinite Integrals section. The integral equations can be used along with the following methods of integration:

- Integration by Parts (integral equation #6),
- Integration by Substitution, and
- Separation of Rational Fractions into Partial Fractions.

◆ Wade, Thomas L., *Calculus*, Ginn & Company/Simon & Schuster Publishers, 1953.

## DERIVATIVES AND INDEFINITE INTEGRALS

In these formulas,  $u$ ,  $v$ , and  $w$  represent functions of  $x$ . Also,  $a$ ,  $c$ , and  $n$  represent constants. All arguments of the trigonometric functions are in radians. A constant of integration should be added to the integrals. To avoid terminology difficulty, the following definitions are followed:  $\arcsin u = \sin^{-1} u$ ,  $(\sin u)^{-1} = 1/\sin u$ .

1.  $dc/dx = 0$
2.  $dx/dx = 1$
3.  $d(cu)/dx = c du/dx$
4.  $d(u + v - w)/dx = du/dx + dv/dx - dw/dx$
5.  $d(uv)/dx = u dv/dx + v du/dx$
6.  $d(uvw)/dx = uv dw/dx + uw dv/dx + vw du/dx$
7.  $\frac{d(u/v)}{dx} = \frac{v du/dx - u dv/dx}{v^2}$
8.  $d(u^n)/dx = nu^{n-1} du/dx$
9.  $d[f(u)]/dx = \{d[f(u)]/du\} du/dx$
10.  $du/dx = 1/(dx/du)$
11.  $\frac{d(\log_a u)}{dx} = (\log_a e) \frac{1}{u} \frac{du}{dx}$
12.  $\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx}$
13.  $\frac{d(a^u)}{dx} = (\ln a) a^u \frac{du}{dx}$
14.  $d(e^u)/dx = e^u du/dx$
15.  $d(u^v)/dx = vu^{v-1} du/dx + (\ln u) u^v dv/dx$
16.  $d(\sin u)/dx = \cos u du/dx$
17.  $d(\cos u)/dx = -\sin u du/dx$
18.  $d(\tan u)/dx = \sec^2 u du/dx$
19.  $d(\cot u)/dx = -\csc^2 u du/dx$
20.  $d(\sec u)/dx = \sec u \tan u du/dx$
21.  $d(\csc u)/dx = -\csc u \cot u du/dx$
22.  $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \quad (-\pi/2 \leq \sin^{-1} u \leq \pi/2)$
23.  $\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \quad (0 \leq \cos^{-1} u \leq \pi)$
24.  $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx} \quad (-\pi/2 < \tan^{-1} u < \pi/2)$
25.  $\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2} \frac{du}{dx} \quad (0 < \cot^{-1} u < \pi)$
26.  $\frac{d(\sec^{-1} u)}{dx} = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$
27.  $\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{u\sqrt{u^2-1}} \frac{du}{dx} \quad \begin{matrix} (0 < \sec^{-1} u < \pi/2) & (-\pi \leq \sec^{-1} u < -\pi/2) \\ (0 < \csc^{-1} u \leq \pi/2) & (-\pi < \csc^{-1} u \leq -\pi/2) \end{matrix}$
1.  $\int df(x) = f(x)$
2.  $\int dx = x$
3.  $\int a f(x) dx = a \int f(x) dx$
4.  $\int [u(x) \pm v(x)] dx = \int u(x) dx \pm \int v(x) dx$
5.  $\int x^m dx = \frac{x^{m+1}}{m+1} \quad (m \neq -1)$
6.  $\int u(x) dv(x) = u(x) v(x) - \int v(x) du(x)$
7.  $\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b|$
8.  $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$
9.  $\int a^x dx = \frac{a^x}{\ln a}$
10.  $\int \sin x dx = -\cos x$
11.  $\int \cos x dx = \sin x$
12.  $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}$
13.  $\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4}$
14.  $\int x \sin x dx = \sin x - x \cos x$
15.  $\int x \cos x dx = \cos x + x \sin x$
16.  $\int \sin x \cos x dx = (\sin^2 x)/2$
17.  $\int \sin ax \cos bx dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} \quad (a^2 \neq b^2)$
18.  $\int \tan x dx = -\ln|\cos x| = \ln|\sec x|$
19.  $\int \cot x dx = -\ln|\csc x| = \ln|\sin x|$
20.  $\int \tan^2 x dx = \tan x - x$
21.  $\int \cot^2 x dx = -\cot x - x$
22.  $\int e^{ax} dx = (1/a) e^{ax}$
23.  $\int x e^{ax} dx = (e^{ax}/a^2)(ax-1)$
24.  $\int \ln x dx = x[\ln(x)-1] \quad (x > 0)$
25.  $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad (a \neq 0)$
26.  $\int \frac{dx}{ax^2+c} = \frac{1}{\sqrt{ac}} \tan^{-1}\left(x\sqrt{\frac{a}{c}}\right), \quad (a > 0, c > 0)$
- 27a.  $\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (4ac-b^2 > 0)$
- 27b.  $\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}} \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| \quad (b^2-4ac > 0)$
- 27c.  $\int \frac{dx}{ax^2+bx+c} = -\frac{2}{2ax+b}, \quad (b^2-4ac = 0)$

## CENTROIDS AND MOMENTS OF INERTIA

The *location of the centroid of an area*, bounded by the axes and the function  $y = f(x)$ , can be found by integration.

$$x_c = \frac{\int x dA}{A}$$

$$y_c = \frac{\int y dA}{A}$$

$$A = \int f(x) dx$$

$$dA = f(x) dx = g(y) dy$$

The *first moment of area* with respect to the  $y$ -axis and the  $x$ -axis, respectively, are:

$$M_y = \int x dA = x_c A$$

$$M_x = \int y dA = y_c A$$

The *moment of inertia (second moment of area)* with respect to the  $y$ -axis and the  $x$ -axis, respectively, are:

$$I_y = \int x^2 dA$$

$$I_x = \int y^2 dA$$

The moment of inertia taken with respect to an axis passing through the area's centroid is the *centroidal moment of inertia*. The *parallel axis theorem* for the moment of inertia with respect to another axis parallel with and located  $d$  units from the centroidal axis is expressed by

$$I_{\text{parallel axis}} = I_c + Ad^2$$

In a plane,  $J = \int r^2 dA = I_x + I_y$

## PROGRESSIONS AND SERIES

### Arithmetic Progression

To determine whether a given finite sequence of numbers is an arithmetic progression, subtract each number from the following number. If the differences are equal, the series is arithmetic.

1. The first term is  $a$ .
2. The common difference is  $d$ .
3. The number of terms is  $n$ .
4. The last or  $n$ th term is  $l$ .
5. The sum of  $n$  terms is  $S$ .

$$l = a + (n - 1)d$$

$$S = n(a + l)/2 = n[2a + (n - 1)d]/2$$

### Geometric Progression

To determine whether a given finite sequence is a geometric progression (G.P.), divide each number after the first by the preceding number. If the quotients are equal, the series is geometric:

1. The first term is  $a$ .
2. The common ratio is  $r$ .
3. The number of terms is  $n$ .
4. The last or  $n$ th term is  $l$ .
5. The sum of  $n$  terms is  $S$ .

$$l = ar^{n-1}$$

$$S = a(1 - r^n)/(1 - r); r \neq 1$$

$$S = (a - rl)/(1 - r); r \neq 1$$

$$\lim_{n \rightarrow \infty} S_n = a/(1 - r); r < 1$$

A G.P. converges if  $|r| < 1$  and it diverges if  $|r| > 1$ .

### Properties of Series

$$\sum_{i=1}^n c = nc; \quad c = \text{constant}$$

$$\sum_{i=1}^n cx_i = c \sum_{i=1}^n x_i$$

$$\sum_{i=1}^n (x_i + y_i - z_i) = \sum_{i=1}^n x_i + \sum_{i=1}^n y_i - \sum_{i=1}^n z_i$$

$$\sum_{x=1}^n x = (n + n^2)/2$$

### Power Series

$$\sum_{i=0}^{\infty} a_i (x - a)^i$$

1. A power series, which is convergent in the interval  $-R < x < R$ , defines a function of  $x$  that is continuous for all values of  $x$  within the interval and is said to represent the function in that interval.
2. A power series may be differentiated term by term within its interval of convergence. The resulting series has the same interval of convergence as the original series (except possibly at the end points of the series).
3. A power series may be integrated term by term provided the limits of integration are within the interval of convergence of the series.
4. Two power series may be added, subtracted, or multiplied, and the resulting series in each case is convergent, at least, in the interval common to the two series.
5. Using the process of long division (as for polynomials), two power series may be divided one by the other within their common interval of convergence.

### Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \dots$$

is called *Taylor's series*, and the function  $f(x)$  is said to be expanded about the point  $a$  in a Taylor's series.

If  $a = 0$ , the Taylor's series equation becomes a *Maclaurin's series*.



## DIFFERENTIAL EQUATIONS

A common class of ordinary linear differential equations is

$$b_n \frac{d^n y(x)}{dx^n} + \dots + b_1 \frac{dy(x)}{dx} + b_0 y(x) = f(x)$$

where  $b_n, \dots, b_1, b_0$  are constants.

When the equation is a homogeneous differential equation,  $f(x) = 0$ , the solution is

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \dots + C_i e^{r_i x} + \dots + C_n e^{r_n x}$$

where  $r_n$  is the  $n$ th distinct root of the characteristic polynomial  $P(r)$  with

$$P(r) = b_n r^n + b_{n-1} r^{n-1} + \dots + b_1 r + b_0$$

If the root  $r_1 = r_2$ , then  $C_2 e^{r_2 x}$  is replaced with  $C_2 x e^{r_1 x}$ .

Higher orders of multiplicity imply higher powers of  $x$ . The complete solution for the differential equation is

$$y(x) = y_h(x) + y_p(x),$$

where  $y_p(x)$  is any particular solution with  $f(x)$  present. If  $f(x)$  has  $e^{r_n x}$  terms, then resonance is manifested. Furthermore, specific  $f(x)$  forms result in specific  $y_p(x)$  forms, some of which are:

$f(x)$	$y_p(x)$
$A$	$B$
$Ae^{\alpha x}$	$Be^{\alpha x}, \alpha \neq r_n$
$A_1 \sin \omega x + A_2 \cos \omega x$	$B_1 \sin \omega x + B_2 \cos \omega x$

If the independent variable is time  $t$ , then transient dynamic solutions are implied.

### First-Order Linear Homogeneous Differential Equations with Constant Coefficients

$y' + ay = 0$ , where  $a$  is a real constant:

Solution,  $y = Ce^{-at}$

where  $C$  = a constant that satisfies the initial conditions.

### First-Order Linear Nonhomogeneous Differential Equations

$$\tau \frac{dy}{dt} + y = Kx(t) \quad x(t) = \begin{cases} A & t < 0 \\ B & t > 0 \end{cases}$$

$$y(0) = KA$$

$\tau$  is the time constant

$K$  is the gain

The solution is

$$y(t) = KA + (KB - KA) \left( 1 - \exp\left(-\frac{t}{\tau}\right) \right) \text{ or}$$

$$\frac{t}{\tau} = \ln \left[ \frac{KB - KA}{KB - y} \right]$$

## Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

An equation of the form

$$y'' + ay' + by = 0$$

can be solved by the method of undetermined coefficients

where a solution of the form  $y = Ce^{rx}$  is sought. Substitution of this solution gives

$$(r^2 + ar + b) Ce^{rx} = 0$$

and since  $Ce^{rx}$  cannot be zero, the characteristic equation must vanish or

$$r^2 + ar + b = 0$$

The roots of the characteristic equation are

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

and can be real and distinct for  $a^2 > 4b$ , real and equal for  $a^2 = 4b$ , and complex for  $a^2 < 4b$ .

If  $a^2 > 4b$ , the solution is of the form (overdamped)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If  $a^2 = 4b$ , the solution is of the form (critically damped)

$$y = (C_1 + C_2 x) e^{r_1 x}$$

If  $a^2 < 4b$ , the solution is of the form (underdamped)

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x), \text{ where}$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

## FOURIER TRANSFORM

The Fourier transform pair, one form of which is

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = [1/(2\pi)] \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

can be used to characterize a broad class of signal models in terms of their frequency or spectral content. Some useful transform pairs are:

$f(t)$	$F(\omega)$
$\delta(t)$	1
$u(t)$	$\pi \delta(\omega) + 1/j\omega$
$u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) = r_{\text{rect}} \frac{t}{\tau}$	$\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2}$
$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0)$

Some mathematical liberties are required to obtain the second and fourth form. Other Fourier transforms are derivable from the Laplace transform by replacing  $s$  with  $j\omega$  provided

$$f(t) = 0, t < 0$$

$$\int_0^{\infty} |f(t)| dt < \infty$$

## FOURIER SERIES

Every periodic function  $f(t)$  which has the period  $T = 2\pi/\omega_0$  and has certain continuity conditions can be represented by a series plus a constant

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

The above holds if  $f(t)$  has a continuous derivative  $f'(t)$  for all  $t$ . It should be noted that the various sinusoids present in the series are orthogonal on the interval 0 to  $T$  and as a result the coefficients are given by

$$a_0 = (1/T) \int_0^T f(t) dt$$

$$a_n = (2/T) \int_0^T f(t) \cos(n\omega_0 t) dt \quad n = 1, 2, \dots$$

$$b_n = (2/T) \int_0^T f(t) \sin(n\omega_0 t) dt \quad n = 1, 2, \dots$$

The constants  $a_n$  and  $b_n$  are the *Fourier coefficients* of  $f(t)$  for the interval 0 to  $T$  and the corresponding series is called the *Fourier series of  $f(t)$*  over the same interval.

The integrals have the same value when evaluated over any interval of length  $T$ .

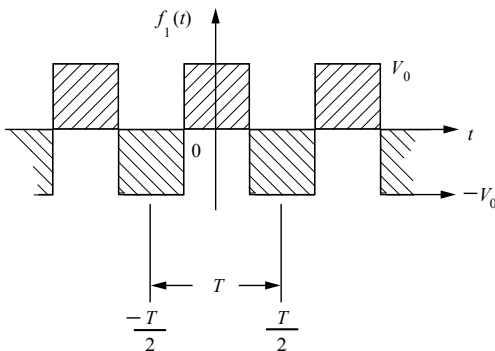
If a Fourier series representing a periodic function is truncated after term  $n = N$  the mean square value  $F_N^2$  of the truncated series is given by Parseval's relation. This relation says that the mean-square value is the sum of the mean-square values of the Fourier components, or

$$F_N^2 = a_0^2 + (1/2) \sum_{n=1}^N (a_n^2 + b_n^2)$$

and the RMS value is then defined to be the square root of this quantity or  $F_N$ .

Three useful and common Fourier series forms are defined in terms of the following graphs (with  $\omega_0 = 2\pi/T$ ).

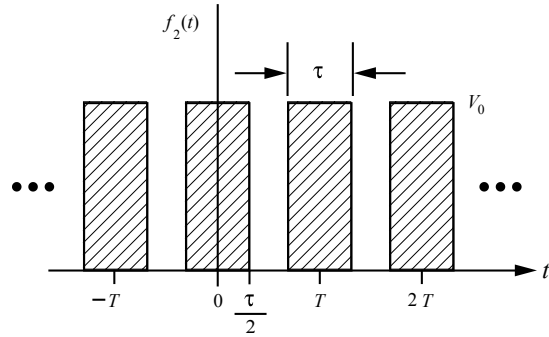
Given:



then

$$f_1(t) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} (-1)^{(n-1)/2} (4V_0/n\pi) \cos(n\omega_0 t)$$

Given:

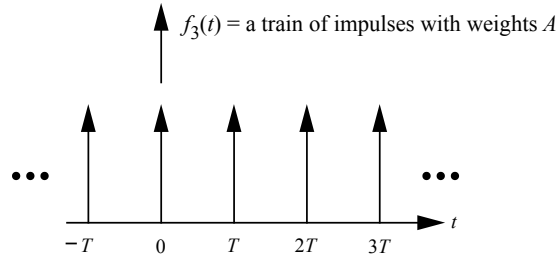


then

$$f_2(t) = \frac{V_0 \tau}{T} + \frac{2V_0 \tau}{T} \sum_{n=1}^{\infty} \frac{\sin(n\pi\tau/T)}{(n\pi\tau/T)} \cos(n\omega_0 t)$$

$$f_2(t) = \frac{V_0 \tau}{T} \sum_{n=-\infty}^{\infty} \frac{\sin(n\pi\tau/T)}{(n\pi\tau/T)} e^{jn\omega_0 t}$$

Given:



then

$$f_3(t) = \sum_{n=-\infty}^{\infty} A \delta(t - nT)$$

$$f_3(t) = (A/T) + (2A/T) \sum_{n=1}^{\infty} \cos(n\omega_0 t)$$

$$f_3(t) = (A/T) \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

## The Fourier Transform and its Inverse

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df$$

We say that  $x(t)$  and  $X(f)$  form a *Fourier transform pair*:

$$x(t) \leftrightarrow X(f)$$

### Fourier Transform Pairs

$x(t)$	$X(f)$
1	$\delta(f)$
$\delta(t)$	1
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\Pi(t/\tau)$	$\tau \operatorname{sinc}(\tau f)$
$\operatorname{sinc}(Bt)$	$\frac{1}{B}\Pi(f/B)$
$\Lambda(t/\tau)$	$\tau \operatorname{sinc}^2(\tau f)$
$e^{-at}u(t)$	$\frac{1}{a + j2\pi f} \quad a > 0$
$te^{-at}u(t)$	$\frac{1}{(a + j2\pi f)^2} \quad a > 0$
$e^{-a t }$	$\frac{2a}{a^2 + (2\pi f)^2} \quad a > 0$
$e^{-(at)^2}$	$\frac{\sqrt{\pi}}{a} e^{-(\pi f/a)^2}$
$\cos(2\pi f_0 t + \theta)$	$\frac{1}{2}[e^{j\theta}\delta(f - f_0) + e^{-j\theta}\delta(f + f_0)]$
$\sin(2\pi f_0 t + \theta)$	$\frac{1}{2j}[e^{j\theta}\delta(f - f_0) - e^{-j\theta}\delta(f + f_0)]$
$\sum_{n=-\infty}^{n=+\infty} \delta(t - nT_s)$	$f_s \sum_{k=-\infty}^{k=+\infty} \delta(f - kf_s) \quad f_s = \frac{1}{T_s}$

## Fourier Transform Theorems

Linearity	$ax(t) + by(t)$	$aX(f) + bY(f)$
Scale change	$x(at)$	$\frac{1}{ a }X\left(\frac{f}{a}\right)$
Time reversal	$x(-t)$	$X(-f)$
Duality	$X(t)$	$x(-f)$
Time shift	$x(t - t_0)$	$X(f)e^{-j2\pi ft_0}$
Frequency shift	$x(t)e^{j2\pi f_0 t}$	$X(f - f_0)$
Modulation	$x(t)\cos 2\pi f_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
Multiplication	$x(t)y(t)$	$X(f) * Y(f)$
Convolution	$x(t) * y(t)$	$X(f)Y(f)$
Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda$	$\frac{1}{j2\pi f} X(f) + \frac{1}{2}X(0)\delta(f)$

## LAPLACE TRANSFORMS

The unilateral Laplace transform pair

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st} dt$$

where  $s = \sigma + j\omega$

represents a powerful tool for the transient and frequency response of linear time invariant systems. Some useful Laplace transform pairs are:

$f(t)$	$F(s)$
$\delta(t)$ , Impulse at $t = 0$	1
$u(t)$ , Step at $t = 0$	$1/s$
$t[u(t)]$ , Ramp at $t = 0$	$1/s^2$
$e^{-\alpha t}$	$1/(s + \alpha)$
$te^{-\alpha t}$	$1/(s + \alpha)^2$
$e^{-\alpha t} \sin \beta t$	$\beta/[(s + \alpha)^2 + \beta^2]$
$e^{-\alpha t} \cos \beta t$	$(s + \alpha)/[(s + \alpha)^2 + \beta^2]$
$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{m=0}^{n-1} s^{n-m-1} \frac{d^m f(0)}{dt^m}$
$\int_0^t f(\tau) d\tau$	$(1/s)F(s)$
$\int_0^t x(t - \tau)h(\tau)d\tau$	$H(s)X(s)$
$f(t - \tau) u(t - \tau)$	$e^{-\tau s} F(s)$
$\lim_{t \rightarrow \infty} f(t)$	$\lim_{s \rightarrow 0} sF(s)$
$\lim_{t \rightarrow 0} f(t)$	$\lim_{s \rightarrow \infty} sF(s)$

The last two transforms represent the Final Value Theorem (F.V.T.) and Initial Value Theorem (I.V.T.), respectively. It is assumed that the limits exist.

## MATRICES

A matrix is an ordered rectangular array of numbers with  $m$  rows and  $n$  columns. The element  $a_{ij}$  refers to row  $i$  and column  $j$ .

### Multiplication of Two Matrices

$$A = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \quad A_{2,3} \text{ is a 3-row, 2-column matrix}$$

$$B = \begin{bmatrix} H & I \\ J & K \end{bmatrix} \quad B_{2,2} \text{ is a 2-row, 2-column matrix}$$

In order for multiplication to be possible, the number of columns in A must equal the number of rows in B.

Multiplying matrix B by matrix A occurs as follows:

$$C = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \cdot \begin{bmatrix} H & I \\ J & K \end{bmatrix}$$

$$C = \begin{bmatrix} (A \cdot H + B \cdot J) & (A \cdot I + B \cdot K) \\ (C \cdot H + D \cdot J) & (C \cdot I + D \cdot K) \\ (E \cdot H + F \cdot J) & (E \cdot I + F \cdot K) \end{bmatrix}$$

Matrix multiplication is not commutative.

### Addition

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} + \begin{bmatrix} G & H & I \\ J & K & L \end{bmatrix} = \begin{bmatrix} A+G & B+H & C+I \\ D+J & E+K & F+L \end{bmatrix}$$

### Identity Matrix

The matrix  $\mathbf{I} = (a_{ij})$  is a square  $n \times n$  matrix with 1's on the diagonal and 0's everywhere else.

### Matrix Transpose

Rows become columns. Columns become rows.

$$A = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \quad A^T = \begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

### Inverse $[\ ]^{-1}$

The inverse  $\mathbf{B}$  of a square  $n \times n$  matrix  $\mathbf{A}$  is

$$\mathbf{B} = \mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{|\mathbf{A}|}, \text{ where}$$

$\text{adj}(\mathbf{A})$  = adjoint of  $\mathbf{A}$  (obtained by replacing  $\mathbf{A}^T$  elements with their cofactors) and  $|\mathbf{A}|$  = determinant of  $\mathbf{A}$ .

$$[\mathbf{A}][\mathbf{A}]^{-1} = [\mathbf{A}]^{-1}[\mathbf{A}] = [\mathbf{I}] \text{ where } \mathbf{I} \text{ is the identity matrix.}$$

## DETERMINANTS

A *determinant of order  $n$*  consists of  $n^2$  numbers, called the *elements* of the determinant, arranged in  $n$  rows and  $n$  columns and enclosed by two vertical lines.

In any determinant, the *minor* of a given element is the determinant that remains after all of the elements are struck out that lie in the same row and in the same column as the given element. Consider an element which lies in the  $j$ th column and the  $i$ th row. The *cofactor* of this element is the value of the minor of the element (if  $i + j$  is even), and it is the negative of the value of the minor of the element (if  $i + j$  is odd).

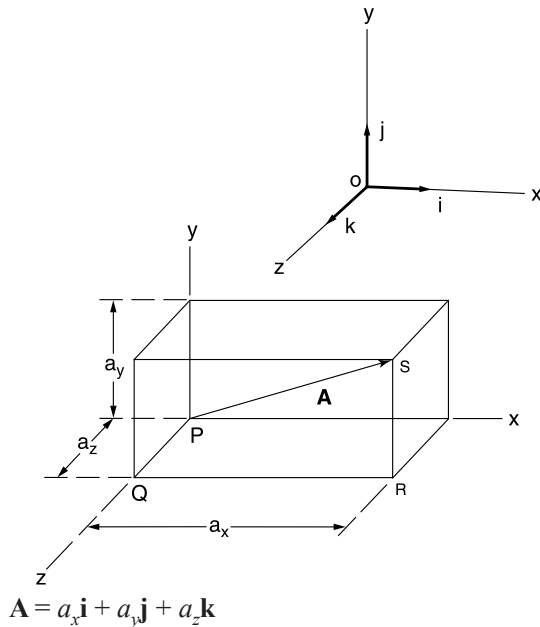
If  $n$  is greater than 1, the *value* of a determinant of order  $n$  is the sum of the  $n$  products formed by multiplying each element of some specified row (or column) by its cofactor. This sum is called the *expansion of the determinant* [according to the elements of the specified row (or column)]. For a second-order determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

For a third-order determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

## VECTORS



*Addition and subtraction:*

$$\mathbf{A} + \mathbf{B} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$$

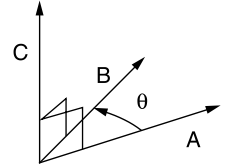
$$\mathbf{A} - \mathbf{B} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k}$$

The *dot product* is a *scalar product* and represents the projection of  $\mathbf{B}$  onto  $\mathbf{A}$  times  $|\mathbf{A}|$ . It is given by

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= a_x b_x + a_y b_y + a_z b_z \\ &= |\mathbf{A}| |\mathbf{B}| \cos \theta = \mathbf{B} \cdot \mathbf{A} \end{aligned}$$

The *cross product* is a *vector product* of magnitude  $|\mathbf{B}| |\mathbf{A}| \sin \theta$  which is perpendicular to the plane containing  $\mathbf{A}$  and  $\mathbf{B}$ . The product is

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{B} \times \mathbf{A}$$



The sense of  $\mathbf{A} \times \mathbf{B}$  is determined by the right-hand rule.

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \mathbf{n} \sin \theta, \text{ where}$$

$\mathbf{n}$  = unit vector perpendicular to the plane of  $\mathbf{A}$  and  $\mathbf{B}$ .

## Gradient, Divergence, and Curl

$$\nabla \phi = \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \phi$$

$$\nabla \cdot \mathbf{V} = \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot (V_1 \mathbf{i} + V_2 \mathbf{j} + V_3 \mathbf{k})$$

$$\nabla \times \mathbf{V} = \left( \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \times (V_1 \mathbf{i} + V_2 \mathbf{j} + V_3 \mathbf{k})$$

The Laplacian of a scalar function  $\phi$  is

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

## Identities

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}; \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$$

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

If  $\mathbf{A} \cdot \mathbf{B} = 0$ , then either  $\mathbf{A} = 0$ ,  $\mathbf{B} = 0$ , or  $\mathbf{A}$  is perpendicular to  $\mathbf{B}$ .

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$

$$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = (\mathbf{B} \times \mathbf{A}) + (\mathbf{C} \times \mathbf{A})$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}; \mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}$$

If  $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ , then either  $\mathbf{A} = \mathbf{0}$ ,  $\mathbf{B} = \mathbf{0}$ , or  $\mathbf{A}$  is parallel to  $\mathbf{B}$ .

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi) = (\nabla \cdot \nabla) \phi$$

$$\nabla \times \nabla \phi = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = \mathbf{0}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

## DIFFERENCE EQUATIONS

Any system whose input  $v(t)$  and output  $y(t)$  are defined only at the equally spaced intervals

$$f(t) = y' = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$$

can be described by a difference equation.

### First-Order Linear Difference Equation

$$\Delta t = t_{i+1} - t_i$$

$$y_{i+1} = y_i + y'(\Delta t)$$

## NUMERICAL METHODS

### Newton's Method for Root Extraction

Given a function  $f(x)$  which has a simple root of  $f(x) = 0$  at  $x = a$  an important computational task would be to find that root. If  $f(x)$  has a continuous first derivative then the  $(j+1)$ st estimate of the root is

$$a^{j+1} = a^j - \left. \frac{f(x)}{\frac{df(x)}{dx}} \right|_{x=a^j}$$

The initial estimate of the root  $a^0$  must be near enough to the actual root to cause the algorithm to converge to the root.

### Newton's Method of Minimization

Given a scalar value function

$$h(x) = h(x_1, x_2, \dots, x_n)$$

find a vector  $x^* \in R_n$  such that

$$h(x^*) \leq h(x) \text{ for all } x$$

Newton's algorithm is

$$x_{k+1} = x_k - \left( \left. \frac{\partial^2 h}{\partial x^2} \right|_{x=x_k} \right)^{-1} \left. \frac{\partial h}{\partial x} \right|_{x=x_k}, \text{ where}$$

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \\ \dots \\ \frac{\partial h}{\partial x_n} \end{bmatrix}$$

and

$$\frac{\partial^2 h}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 h}{\partial x_1 \partial x_2} & \dots & \dots & \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 h}{\partial x_2^2} & \dots & \dots & \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial^2 h}{\partial x_1 \partial x_n} & \frac{\partial^2 h}{\partial x_2 \partial x_n} & \dots & \dots & \frac{\partial^2 h}{\partial x_n^2} \end{bmatrix}$$

## Numerical Integration

Three of the more common numerical integration algorithms used to evaluate the integral

$$\int_a^b f(x) dx$$

are:

*Euler's or Forward Rectangular Rule*

$$\int_a^b f(x) dx \approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x)$$

*Trapezoidal Rule*

for  $n = 1$

$$\int_a^b f(x) dx \approx \Delta x \left[ \frac{f(a) + f(b)}{2} \right]$$

for  $n > 1$

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(a + k\Delta x) + f(b) \right]$$

*Simpson's Rule/Parabolic Rule* ( $n$  must be an even integer)

for  $n = 2$

$$\int_a^b f(x) dx \approx \left( \frac{b-a}{6} \right) \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

for  $n \geq 4$

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3} \left[ f(a) + 2 \sum_{k=2,4,6,\dots}^{n-2} f(a + k\Delta x) + 4 \sum_{k=1,3,5,\dots}^{n-1} f(a + k\Delta x) + f(b) \right]$$

with  $\Delta x = (b-a)/n$

$n$  = number of intervals between data points

## Numerical Solution of Ordinary Differential Equations

### Euler's Approximation

Given a differential equation

$$dx/dt = f(x, t) \text{ with } x(0) = x_0$$

At some general time  $k\Delta t$

$$x[(k+1)\Delta t] \cong x(k\Delta t) + \Delta t f[x(k\Delta t), k\Delta t]$$

which can be used with starting condition  $x_0$  to solve recursively for  $x(\Delta t)$ ,  $x(2\Delta t)$ , ...,  $x(n\Delta t)$ .

The method can be extended to  $n$ th order differential equations by recasting them as  $n$  first-order equations.

In particular, when  $dx/dt = f(x)$

$$x[(k+1)\Delta t] \cong x(k\Delta t) + \Delta t f[x(k\Delta t)]$$

which can be expressed as the recursive equation

$$x_{k+1} = x_k + \Delta t (dx_k/dt)$$

$$x_{k+1} = x + \Delta t [f(x(k), t(k))]$$

# ENGINEERING PROBABILITY AND STATISTICS

## DISPERSION, MEAN, MEDIAN, AND MODE VALUES

If  $X_1, X_2, \dots, X_n$  represent the values of a random sample of  $n$  items or observations, the *arithmetic mean* of these items or observations, denoted  $\bar{X}$ , is defined as

$$\bar{X} = (1/n)(X_1 + X_2 + \dots + X_n) = (1/n) \sum_{i=1}^n X_i$$

$\bar{X} \rightarrow \mu$  for sufficiently large values of  $n$ .

The *weighted arithmetic mean* is

$$\bar{X}_w = \frac{\sum w_i X_i}{\sum w_i}, \text{ where}$$

$X_i$  = the value of the  $i$ th observation, and

$w_i$  = the weight applied to  $X_i$ .

The *variance* of the population is the *arithmetic mean* of the *squared deviations from the population mean*. If  $\mu$  is the arithmetic mean of a discrete population of size  $N$ , the *population variance* is defined by

$$\begin{aligned}\sigma^2 &= (1/N) [(X_1 - \mu)^2 + (X_2 - \mu)^2 + \dots + (X_N - \mu)^2] \\ &= (1/N) \sum_{i=1}^N (X_i - \mu)^2\end{aligned}$$

*Standard deviation* formulas are

$$\sigma_{\text{population}} = \sqrt{(1/N) \sum (X_i - \mu)^2}$$

$$\sigma_{\text{sum}} = \sqrt{\sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2}$$

$$\sigma_{\text{series}} = \sigma \sqrt{n}$$

$$\sigma_{\text{mean}} = \frac{\sigma}{\sqrt{n}}$$

$$\sigma_{\text{product}} = \sqrt{A^2 \sigma_b^2 + B^2 \sigma_a^2}$$

The *sample variance* is

$$s^2 = [1/(n-1)] \sum_{i=1}^n (X_i - \bar{X})^2$$

The *sample standard deviation* is

$$s = \sqrt{[1/(n-1)] \sum_{i=1}^n (X_i - \bar{X})^2}$$

The *sample coefficient of variation* =  $CV = s/\bar{X}$

The *sample geometric mean* =  $\sqrt[n]{X_1 X_2 X_3 \dots X_n}$

The *sample root-mean-square value* =  $\sqrt{(1/n) \sum X_i^2}$

When the discrete data are rearranged in increasing order and  $n$  is odd, the median is the value of the  $\left(\frac{n+1}{2}\right)^{\text{th}}$  item

When  $n$  is even, the median is the average of the  $\left(\frac{n}{2}\right)^{\text{th}}$  and  $\left(\frac{n}{2} + 1\right)^{\text{th}}$  items.

The *mode* of a set of data is the value that occurs with greatest frequency.

The *sample range*  $R$  is the largest sample value minus the smallest sample value.

## PERMUTATIONS AND COMBINATIONS

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of  $n$  distinct objects *taken  $r$  at a time* is

$$P(n, r) = \frac{n!}{(n-r)!}$$

$nPr$  is an alternative notation for  $P(n, r)$

2. The number of different *combinations* of  $n$  distinct objects *taken  $r$  at a time* is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

$nCr$  and  $\binom{n}{r}$  are alternative notations for  $C(n, r)$

3. The number of different *permutations* of  $n$  objects *taken  $n$  at a time*, given that  $n_i$  are of type  $i$ , where  $i = 1, 2, \dots, k$  and  $\sum n_i = n$ , is

$$P(n; n_1, n_2, \dots, n_k) = \frac{n!}{n_1! n_2! \dots n_k!}$$

## SETS

### De Morgan's Law

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

### Associative Law

$$A \cup (B \cap C) = (A \cup B) \cap C$$

$$A \cap (B \cup C) = (A \cap B) \cup C$$

### Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

## LAWS OF PROBABILITY

### Property 1. General Character of Probability

The probability  $P(E)$  of an event  $E$  is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

### Property 2. Law of Total Probability

$$P(A + B) = P(A) + P(B) - P(A, B), \text{ where}$$

$P(A + B)$  = the probability that either  $A$  or  $B$  occur alone or that both occur together

$P(A)$  = the probability that  $A$  occurs

$P(B)$  = the probability that  $B$  occurs

$P(A, B)$  = the probability that both  $A$  and  $B$  occur simultaneously



**Property 3. Law of Compound or Joint Probability**

If neither  $P(A)$  nor  $P(B)$  is zero,

$$P(A, B) = P(A)P(B | A) = P(B)P(A | B), \text{ where}$$

$P(B | A)$  = the probability that  $B$  occurs given the fact that  $A$  has occurred

$P(A | B)$  = the probability that  $A$  occurs given the fact that  $B$  has occurred

If either  $P(A)$  or  $P(B)$  is zero, then  $P(A, B) = 0$ .

**Bayes' Theorem**

$$P(B_j | A) = \frac{P(B_j)P(A | B_j)}{\sum_{i=1}^n P(A | B_i)P(B_i)}$$

where  $P(A_j)$  is the probability of event  $A_j$  within the population of  $A$

$P(B_j)$  is the probability of event  $B_j$  within the population of  $B$

**PROBABILITY FUNCTIONS, DISTRIBUTIONS, AND EXPECTED VALUES**

A random variable  $X$  has a probability associated with each of its possible values. The probability is termed a discrete probability if  $X$  can assume only discrete values, or

$$X = x_1, x_2, x_3, \dots, x_n$$

The *discrete probability* of any single event,  $X = x_i$ , occurring is defined as  $P(x_i)$  while the *probability mass function* of the random variable  $X$  is defined by

$$f(x_k) = P(X = x_k), k = 1, 2, \dots, n$$

**Probability Density Function**

If  $X$  is continuous, the *probability density function*,  $f$ , is defined such that

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

**Cumulative Distribution Functions**

The *cumulative distribution function*,  $F$ , of a discrete random variable  $X$  that has a probability distribution described by  $P(x_i)$  is defined as

$$F(x_m) = \sum_{k=1}^m P(x_k) = P(X \leq x_m), m = 1, 2, \dots, n$$

If  $X$  is continuous, the *cumulative distribution function*,  $F$ , is defined by

$$F(x) = \int_{-\infty}^x f(t) dt$$

which implies that  $F(a)$  is the probability that  $X \leq a$ .

**Expected Values**

Let  $X$  be a discrete random variable having a probability mass function

$$f(x_k), k = 1, 2, \dots, n$$

The expected value of  $X$  is defined as

$$\mu = E[X] = \sum_{k=1}^n x_k f(x_k)$$

The variance of  $X$  is defined as

$$\sigma^2 = V[X] = \sum_{k=1}^n (x_k - \mu)^2 f(x_k)$$

Let  $X$  be a continuous random variable having a density function  $f(X)$  and let  $Y = g(X)$  be some general function. The expected value of  $Y$  is:

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) dx$$

The mean or expected value of the random variable  $X$  is now defined as

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x) dx$$

while the variance is given by

$$\sigma^2 = V[X] = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

The standard deviation is given by

$$\sigma = \sqrt{V[X]}$$

The coefficient of variation is defined as  $\sigma/\mu$ .

**Sums of Random Variables**

$$Y = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

The expected value of  $Y$  is:

$$\mu_y = E(Y) = a_1 E(X_1) + a_2 E(X_2) + \dots + a_n E(X_n)$$

If the random variables are statistically *independent*, then the variance of  $Y$  is:

$$\begin{aligned} \sigma_y^2 &= V(Y) = a_1^2 V(X_1) + a_2^2 V(X_2) + \dots + a_n^2 V(X_n) \\ &= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \dots + a_n^2 \sigma_n^2 \end{aligned}$$

Also, the standard deviation of  $Y$  is:

$$\sigma_y = \sqrt{\sigma_y^2}$$



### Binomial Distribution

$P(x)$  is the probability that  $x$  successes will occur in  $n$  trials. If  $p$  = probability of success and  $q$  = probability of failure =  $1 - p$ , then

$$P_n(x) = C(n, x) p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x},$$

where

$x = 0, 1, 2, \dots, n$

$C(n, x)$  = the number of combinations

$n, p$  = parameters

The variance is given by the form:

$$\sigma^2 = p(1 - q)/n$$

### Normal Distribution (Gaussian Distribution)

This is a unimodal distribution, the mode being  $x = \mu$ , with two points of inflection (each located at a distance  $\sigma$  to either side of the mode). The averages of  $n$  observations tend to become normally distributed as  $n$  increases. The variate  $x$  is said to be normally distributed if its density function  $f(x)$  is given by an expression of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ where}$$

$\mu$  = the population mean

$\sigma$  = the standard deviation of the population

$-\infty \leq x \leq \infty$

When  $\mu = 0$  and  $\sigma^2 = \sigma = 1$ , the distribution is called a *standardized* or *unit normal* distribution. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ where } -\infty \leq x \leq \infty.$$

It is noted that  $Z = \frac{x - \mu}{\sigma}$  follows a standardized normal distribution function.

A unit normal distribution table is included at the end of this section. In the table, the following notations are utilized:

$F(x)$  = the area under the curve from  $-\infty$  to  $x$

$R(x)$  = the area under the curve from  $x$  to  $\infty$

$W(x)$  = the area under the curve between  $-x$  and  $x$

$F(-x) = 1 - F(x)$

### The Central Limit Theorem

Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed random variables each having mean  $\mu$  and variance  $\sigma^2$ . Then for large  $n$ , the Central Limit Theorem asserts that the sum

$Y = X_1 + X_2 + \dots + X_n$  is approximately normal.

$$\mu_{\bar{y}} = \mu$$

and the standard deviation

$$\sigma_{\bar{y}} = \frac{\sigma}{\sqrt{n}}$$

### t-Distribution

Student's  $t$ -distribution has the probability density function given by:

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where

$\nu$  = number of *degrees of freedom*

$n$  = sample size

$\nu = n - 1$

$\Gamma$  = gamma function

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$-\infty \leq t \leq \infty$

A table later in this section gives the values of  $t_{\alpha, \nu}$  for values of  $\alpha$  and  $\nu$ . Note that, in view of the symmetry of the  $t$ -distribution,  $t_{1-\alpha, \nu} = -t_{\alpha, \nu}$ .

The function for  $\alpha$  follows:

$$\alpha = \int_{t_{\alpha, \nu}}^{\infty} f(t) dt$$

### $\chi^2$ - Distribution

If  $Z_1, Z_2, \dots, Z_n$  are independent unit normal random variables, then

$$\chi^2 = Z_1^2 + Z_2^2 + \dots + Z_n^2$$

is said to have a chi-square distribution with  $n$  degrees of freedom.

A table at the end of this section gives values of  $\chi_{\alpha, n}^2$  for selected values of  $\alpha$  and  $n$ .

### Gamma Function

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt, \quad n > 0$$

## LINEAR REGRESSION AND GOODNESS OF FIT

### Least Squares

$y = \hat{a} + \hat{b}x$ , where

$y$ -intercept,  $\hat{a} = \bar{y} - \hat{b}\bar{x}$

and slope,  $\hat{b} = S_{xy}/S_{xx}$

$$S_{xy} = \sum_{i=1}^n x_i y_i - (1/n) \left( \sum_{i=1}^n x_i \right) \left( \sum_{i=1}^n y_i \right)$$

$$S_{xx} = \sum_{i=1}^n x_i^2 - (1/n) \left( \sum_{i=1}^n x_i \right)^2$$

$$\bar{y} = (1/n) \left( \sum_{i=1}^n y_i \right)$$

$$\bar{x} = (1/n) \left( \sum_{i=1}^n x_i \right)$$

where

$n$  = sample size

$S_{xx}$  = sum of squares of  $x$

$S_{yy}$  = sum of squares of  $y$

$S_{xy}$  = sum of  $x$ - $y$  products

**Standard Error of Estimate ( $S_e^2$ ):**

$$S_e^2 = \frac{S_{xx}S_{yy} - S_{xy}^2}{S_{xx}(n - 2)} = MSE, \text{ where}$$

$$S_{yy} = \sum_{i=1}^n y_i^2 - (1/n) \left( \sum_{i=1}^n y_i \right)^2$$

**Confidence Interval for Intercept ( $\hat{a}$ ):**

$$\hat{a} \pm t_{\alpha/2, n-2} \sqrt{\left( \frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right) MSE}$$

**Confidence Interval for Slope ( $\hat{b}$ ):**

$$\hat{b} \pm t_{\alpha/2, n-2} \sqrt{\frac{MSE}{S_{xx}}}$$

**Sample Correlation Coefficient ( $R$ ) and Coefficient of Determination ( $R^2$ ):**

$$R = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$R^2 = \frac{S_{xy}^2}{S_{xx}S_{yy}}$$

## HYPOTHESIS TESTING

### One-Way Analysis of Variance (ANOVA)

Given independent random samples of size  $n_i$  from  $k$  populations, then:

$$\begin{aligned} & \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x})^2 \\ &= \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 + \sum_{i=1}^k n_i (\bar{x}_i - \bar{x})^2 \text{ or} \end{aligned}$$

$$SS_{\text{total}} = SS_{\text{error}} + SS_{\text{treatments}}$$

Let  $T$  be the grand total of all  $N = \sum_i n_i$  observations and  $T_i$  be the total of the  $n_i$  observations of the  $i$ th sample.

$$C = T^2/N$$

$$SS_{\text{total}} = \sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}^2 - C$$

$$SS_{\text{treatments}} = \sum_{i=1}^k (T_i^2/n_i) - C$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}}$$

**One-Way ANOVA Table**

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	<i>F</i>
Between Treatments	$k - 1$	$SS_{\text{treatments}}$	$MST = \frac{SS_{\text{treatments}}}{k - 1}$	$\frac{MST}{MSE}$
Error	$N - k$	$SS_{\text{error}}$	$MSE = \frac{SS_{\text{error}}}{N - k}$	
Total	$N - 1$	$SS_{\text{total}}$		

**Two-Way ANOVA Table**

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	<i>F</i>
Between Treatments	$k - 1$	$SS_{\text{treatments}}$	$MST = \frac{SS_{\text{treatments}}}{k - 1}$	$\frac{MST}{MSE}$
Between Blocks	$n - 1$	$SS_{\text{blocks}}$	$MSB = \frac{SS_{\text{blocks}}}{n - 1}$	$\frac{MSB}{MSE}$
Error	$(k - 1)(n - 1)$	$SS_{\text{error}}$	$MSE = \frac{SS_{\text{error}}}{(k - 1)(n - 1)}$	
Total	$N - 1$	$SS_{\text{total}}$		

Consider an unknown parameter  $\theta$  of a statistical distribution. Let the null hypothesis be

$$H_0: \mu = \mu_0$$

and let the alternative hypothesis be

$$H_1: \mu = \mu_1$$

Rejecting  $H_0$  when it is true is known as a type I error, while accepting  $H_0$  when it is wrong is known as a type II error.

Furthermore, the probabilities of type I and type II errors are usually represented by the symbols  $\alpha$  and  $\beta$ , respectively:

$\alpha$  = probability (type I error)

$\beta$  = probability (type II error)

The probability of a type I error is known as the level of significance of the test.

**Table A. Tests on Means of Normal Distribution—Variance Known**

<i>Hypothesis</i>	<i>Test Statistic</i>	<i>Criteria for Rejection</i>
$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$		$ Z_0  > Z_{\alpha/2}$
$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$	$Z_0 \equiv \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$	$Z_0 < -Z_\alpha$
$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$		$Z_0 > Z_\alpha$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 \neq \gamma$		$ Z_0  > Z_{\alpha/2}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 < \gamma$	$Z_0 \equiv \frac{\bar{X}_1 - \bar{X}_2 - \gamma}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$	$Z_0 < -Z_\alpha$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 > \gamma$		$Z_0 > Z_\alpha$

**Table B. Tests on Means of Normal Distribution—Variance Unknown**

<i>Hypothesis</i>	<i>Test Statistic</i>	<i>Criteria for Rejection</i>
$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$		$ t_0  > t_{\alpha/2, n-1}$
$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$	$t_0 = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$	$t_0 < -t_{\alpha, n-1}$
$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$		$t_0 > t_{\alpha, n-1}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 \neq \gamma$	$\left\{ \begin{array}{l} \text{Variances} \\ \text{equal} \end{array} \right. \quad \left\{ \begin{array}{l} t_0 = \frac{\bar{X}_1 - \bar{X}_2 - \gamma}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\ v = n_1 + n_2 - 2 \end{array} \right.$	$ t_0  > t_{\alpha/2, v}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 < \gamma$	$\left\{ \begin{array}{l} \text{Variances} \\ \text{unequal} \end{array} \right. \quad \left\{ \begin{array}{l} t_0 = \frac{\bar{X}_1 - \bar{X}_2 - \gamma}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \\ v = \frac{\left( \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2} \right)^2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}} \end{array} \right.$	$t_0 < -t_{\alpha, v}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 > \gamma$		$t_0 > t_{\alpha, v}$

In Table B,  $S_p^2 = [(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2]/v$

**Table C. Tests on Variances of Normal Distribution with Unknown Mean**

<i>Hypothesis</i>	<i>Test Statistic</i>	<i>Criteria for Rejection</i>
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 \neq \sigma_0^2$		$\chi_0^2 > \chi_{\alpha/2, n-1}^2$ or $\chi_0^2 < \chi_{1-\alpha/2, n-1}^2$
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 < \sigma_0^2$	$\chi_0^2 = \frac{(n-1)S^2}{\sigma_0^2}$	$\chi_0^2 < \chi_{1-\alpha/2, n-1}^2$
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 > \sigma_0^2$		$\chi_0^2 > \chi_{\alpha, n-1}^2$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 \neq \sigma_2^2$	$F_0 = \frac{S_1^2}{S_2^2}$	$F_0 > F_{\alpha/2, n_1-1, n_2-1}$ $F_0 < F_{1-\alpha/2, n_1-1, n_2-1}$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 < \sigma_2^2$	$F_0 = \frac{S_2^2}{S_1^2}$	$F_0 > F_{\alpha, n_2-1, n_1-1}$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 > \sigma_2^2$	$F_0 = \frac{S_1^2}{S_2^2}$	$F_0 > F_{\alpha, n_1-1, n_2-1}$

Assume that the values of  $\alpha$  and  $\beta$  are given. The sample size can be obtained from the following relationships. In (A) and (B),  $\mu_1$  is the value assumed to be the true mean.

(A)  $H_0: \mu = \mu_0; H_1: \mu \neq \mu_0$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_{\alpha/2}\right) - \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - Z_{\alpha/2}\right)$$

An approximate result is

$$n \simeq \frac{(Z_{\alpha/2} + Z_{\beta})^2 \sigma^2}{(\mu_1 - \mu_0)^2}$$

(B)  $H_0: \mu = \mu_0; H_1: \mu > \mu_0$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_{\alpha}\right)$$

$$n = \frac{(Z_{\alpha} + Z_{\beta})^2 \sigma^2}{(\mu_1 - \mu_0)^2}$$

## CONFIDENCE INTERVALS, SAMPLE DISTRIBUTIONS AND SAMPLE SIZE

### Confidence Interval for the Mean $\mu$ of a Normal Distribution

(A) Standard deviation  $\sigma$  is known

$$\bar{X} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

(B) Standard deviation  $\sigma$  is not known

$$\bar{X} - t_{\alpha/2} \frac{s}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where  $t_{\alpha/2}$  corresponds to  $n - 1$  degrees of freedom.

### Confidence Interval for the Difference Between Two Means $\mu_1$ and $\mu_2$

(A) Standard deviations  $\sigma_1$  and  $\sigma_2$  known

$$\bar{X}_1 - \bar{X}_2 - Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

(B) Standard deviations  $\sigma_1$  and  $\sigma_2$  are not known

$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2} \sqrt{\frac{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) [(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2]}{n_1 + n_2 - 2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + t_{\alpha/2} \sqrt{\frac{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) [(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2]}{n_1 + n_2 - 2}}$$

where  $t_{\alpha/2}$  corresponds to  $n_1 + n_2 - 2$  degrees of freedom.

### Confidence Intervals for the Variance $\sigma^2$ of a Normal Distribution

$$\frac{(n-1)s^2}{x_{\alpha/2, n-1}^2} \leq \sigma^2 \leq \frac{(n-1)s^2}{x_{1-\alpha/2, n-1}^2}$$

### Sample Size

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \quad n = \left[ \frac{z_{\alpha/2} \sigma}{\bar{x} - \mu} \right]^2$$

## TEST STATISTICS

The following definitions apply.

$$Z_{\text{var}} = \frac{\bar{X} - \mu_o}{\frac{\sigma}{\sqrt{n}}}$$

$$t_{\text{var}} = \frac{\bar{X} - \mu_o}{\frac{s}{\sqrt{n}}}$$

where

$Z_{\text{var}}$  is the standard normal  $Z$  score

$t_{\text{var}}$  is the sample distribution test statistic

$\sigma$  is known standard deviation

$\mu_o$  is population mean

$\bar{X}$  is hypothesized mean or sample mean

$n$  is sample size

$s$  is computed sample standard deviation

The  $Z$  score is applicable when the standard deviation(s) is known. The test statistic is applicable when the standard deviation(s) is computed at time of sampling.

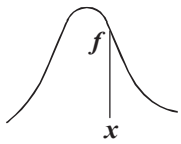
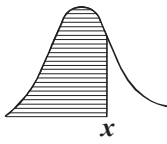
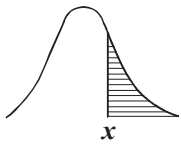
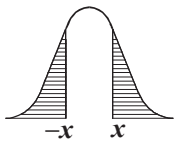
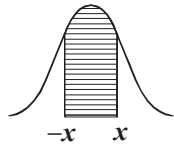
$Z_{\alpha}$  corresponds to the appropriate probability under the normal probability curve for a given  $Z_{\text{var}}$ .

$t_{\alpha, n-1}$  corresponds to the appropriate probability under the  $t$  distribution with  $n - 1$  degrees of freedom for a given  $t_{\text{var}}$ .

Values of  $Z_{\alpha/2}$

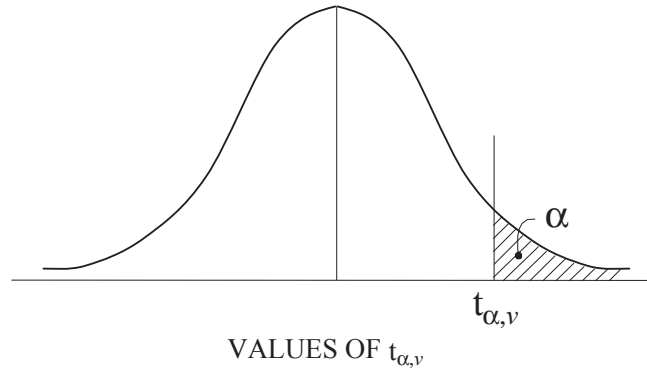
Confidence Interval	$Z_{\alpha/2}$
80%	1.2816
90%	1.6449
95%	1.9600
96%	2.0537
98%	2.3263
99%	2.5758

### Unit Normal Distribution

					
$x$	$f(x)$	$F(x)$	$R(x)$	$2R(x)$	$W(x)$
0.0	0.3989	0.5000	0.5000	1.0000	0.0000
0.1	0.3970	0.5398	0.4602	0.9203	0.0797
0.2	0.3910	0.5793	0.4207	0.8415	0.1585
0.3	0.3814	0.6179	0.3821	0.7642	0.2358
0.4	0.3683	0.6554	0.3446	0.6892	0.3108
0.5	0.3521	0.6915	0.3085	0.6171	0.3829
0.6	0.3332	0.7257	0.2743	0.5485	0.4515
0.7	0.3123	0.7580	0.2420	0.4839	0.5161
0.8	0.2897	0.7881	0.2119	0.4237	0.5763
0.9	0.2661	0.8159	0.1841	0.3681	0.6319
1.0	0.2420	0.8413	0.1587	0.3173	0.6827
1.1	0.2179	0.8643	0.1357	0.2713	0.7287
1.2	0.1942	0.8849	0.1151	0.2301	0.7699
1.3	0.1714	0.9032	0.0968	0.1936	0.8064
1.4	0.1497	0.9192	0.0808	0.1615	0.8385
1.5	0.1295	0.9332	0.0668	0.1336	0.8664
1.6	0.1109	0.9452	0.0548	0.1096	0.8904
1.7	0.0940	0.9554	0.0446	0.0891	0.9109
1.8	0.0790	0.9641	0.0359	0.0719	0.9281
1.9	0.0656	0.9713	0.0287	0.0574	0.9426
2.0	0.0540	0.9772	0.0228	0.0455	0.9545
2.1	0.0440	0.9821	0.0179	0.0357	0.9643
2.2	0.0355	0.9861	0.0139	0.0278	0.9722
2.3	0.0283	0.9893	0.0107	0.0214	0.9786
2.4	0.0224	0.9918	0.0082	0.0164	0.9836
2.5	0.0175	0.9938	0.0062	0.0124	0.9876
2.6	0.0136	0.9953	0.0047	0.0093	0.9907
2.7	0.0104	0.9965	0.0035	0.0069	0.9931
2.8	0.0079	0.9974	0.0026	0.0051	0.9949
2.9	0.0060	0.9981	0.0019	0.0037	0.9963
3.0	0.0044	0.9987	0.0013	0.0027	0.9973
Fractiles					
1.2816	0.1755	0.9000	0.1000	0.2000	0.8000
1.6449	0.1031	0.9500	0.0500	0.1000	0.9000
1.9600	0.0584	0.9750	0.0250	0.0500	0.9500
2.0537	0.0484	0.9800	0.0200	0.0400	0.9600
2.3263	0.0267	0.9900	0.0100	0.0200	0.9800
2.5758	0.0145	0.9950	0.0050	0.0100	0.9900



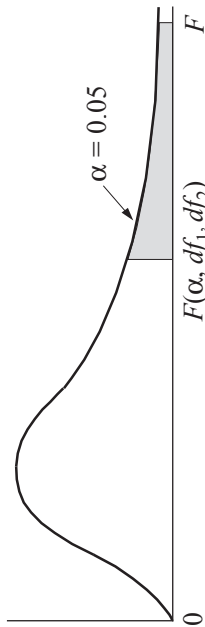
### Student's $t$ -Distribution



$\nu$	$\alpha$								$\nu$
	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	1
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	2
3	0.765	0.978	1.350	1.638	2.353	3.182	4.541	5.841	3
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	4
<b>5</b>	<b>0.727</b>	<b>0.920</b>	<b>1.156</b>	<b>1.476</b>	<b>2.015</b>	<b>2.571</b>	<b>3.365</b>	<b>4.032</b>	<b>5</b>
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	6
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	7
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	8
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	9
<b>10</b>	<b>0.700</b>	<b>0.879</b>	<b>1.093</b>	<b>1.372</b>	<b>1.812</b>	<b>2.228</b>	<b>2.764</b>	<b>3.169</b>	<b>10</b>
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	11
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	12
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	13
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	14
<b>15</b>	<b>0.691</b>	<b>0.866</b>	<b>1.074</b>	<b>1.341</b>	<b>1.753</b>	<b>2.131</b>	<b>2.602</b>	<b>2.947</b>	<b>15</b>
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	16
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	17
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	18
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	19
<b>20</b>	<b>0.687</b>	<b>0.860</b>	<b>1.064</b>	<b>1.325</b>	<b>1.725</b>	<b>2.086</b>	<b>2.528</b>	<b>2.845</b>	<b>20</b>
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	21
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	22
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	23
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	24
<b>25</b>	<b>0.684</b>	<b>0.856</b>	<b>1.058</b>	<b>1.316</b>	<b>1.708</b>	<b>2.060</b>	<b>2.485</b>	<b>2.787</b>	<b>25</b>
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	26
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	27
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	28
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	29
<b>30</b>	<b>0.683</b>	<b>0.854</b>	<b>1.055</b>	<b>1.310</b>	<b>1.697</b>	<b>2.042</b>	<b>2.457</b>	<b>2.750</b>	<b>30</b>
$\infty$	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	$\infty$

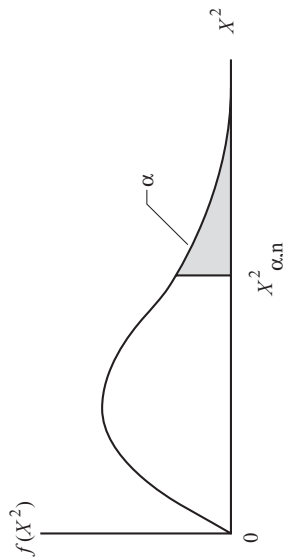
CRITICAL VALUES OF THE  $F$  DISTRIBUTION – TABLE

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of  $F$  corresponding to a specified upper tail area ( $\alpha$ ).



Denominator $df_2$	Numerator $df_1$																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	$\infty$
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
$\infty$	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

# CRITICAL VALUES OF $\chi^2$ DISTRIBUTION



Degrees of Freedom	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720	4.60517	5.99147	7.37776	9.21034	10.5966
3	0.0717212	0.114832	0.215795	0.351846	0.584375	6.25139	7.81473	9.34840	11.3449	12.8381
4	0.206990	0.297110	0.484419	0.710721	1.063623	7.77944	9.48773	11.1433	13.2767	14.8602
5	0.411740	0.554300	0.831211	1.145476	1.61031	9.23635	11.0705	12.8325	15.0863	16.7496
6	0.675727	0.872085	1.237347	1.63539	2.20413	10.6446	12.5916	14.4494	16.8119	18.5476
7	0.989265	1.239043	1.68987	2.16735	2.83311	12.0170	14.0671	16.0128	18.4753	20.2777
8	1.344419	1.646482	2.17973	2.73264	3.48954	13.3616	15.5073	17.5346	20.0902	21.9550
9	1.734926	2.087912	2.70039	3.32511	4.16816	14.6837	16.9190	19.0228	21.6660	23.5893
10	2.15585	2.55821	3.24697	3.94030	4.86518	15.9871	18.3070	20.4831	23.2093	25.1882
11	2.60321	3.05347	3.81575	4.57481	5.57779	17.2750	19.6751	21.9200	24.7250	26.7569
12	3.07382	3.57056	4.40379	5.22603	6.30380	18.5494	21.0261	23.3367	26.2170	28.2995
13	3.56503	4.10691	5.00874	5.89186	7.04150	19.8119	22.3621	24.7356	27.6883	29.8194
14	4.07468	4.66043	5.62872	6.57063	7.78953	21.0642	23.6848	26.1190	29.1413	31.3193
15	4.60094	5.22935	6.26214	7.26094	8.54675	22.3072	24.9958	27.4884	30.5779	32.8013
16	5.14224	5.81221	6.90766	7.96164	9.31223	23.5418	26.2962	28.8454	31.9999	34.2672
17	5.69724	6.40776	7.56418	8.67176	10.0852	24.7690	27.5871	30.1910	33.4087	35.7185
18	6.26481	7.01491	8.23075	9.39046	10.8649	25.9894	28.8693	31.5264	34.8053	37.1564
19	6.84398	7.63273	8.90655	10.1170	11.6509	27.2036	30.1435	32.8523	36.1908	38.5822
20	7.43386	8.26040	9.59083	10.8508	12.4426	28.4120	31.4104	34.1696	37.5662	39.9968
21	8.03366	8.89720	10.28293	11.5913	13.2396	29.6151	32.6705	35.4789	38.9321	41.4010
22	8.64272	9.54249	10.9823	12.3380	14.0415	30.8133	33.9244	36.7807	40.2894	42.7956
23	9.26042	10.19567	11.6885	13.0905	14.8479	32.0069	35.1725	38.0757	41.6384	44.1813
24	9.88623	10.8564	12.4011	13.8484	15.6587	33.1963	36.4151	39.3641	42.9798	45.5585
25	10.5197	11.5240	13.1197	14.6114	16.4734	34.3816	37.6525	40.6465	44.3141	46.9278
26	11.1603	12.1981	13.8439	15.3791	17.2919	35.5631	38.8852	41.9232	45.6417	48.2899
27	11.8076	12.8786	14.5733	16.1513	18.1138	36.7412	40.1133	43.1944	46.9630	49.6449
28	12.4613	13.5648	15.3079	16.9279	18.9392	37.9159	41.3372	44.4607	48.2782	50.9933
29	13.1211	14.2565	16.0471	17.7083	19.7677	39.0875	42.5569	45.7222	49.5879	52.3356
30	13.7867	14.9535	16.7908	18.4926	20.5992	40.2560	43.7729	46.9792	50.8922	53.6720
40	20.7065	22.1643	24.4331	26.5093	29.0505	51.8050	55.7585	59.3417	63.6907	66.7659
50	27.9907	29.7067	32.3574	34.7642	37.6886	63.1671	67.5048	71.4202	76.1539	79.4900
60	35.5346	37.4848	40.4817	43.1879	46.4589	74.3970	79.0819	83.2976	88.3794	91.9517
70	43.2752	45.4418	48.7576	51.7393	55.3290	85.5271	90.5312	95.0231	100.425	104.215
80	51.1720	53.5400	57.1532	60.3915	64.2778	96.5782	101.879	106.629	112.329	116.321
90	59.1963	61.7541	65.6466	69.1260	73.2912	107.565	113.145	118.136	124.116	128.299
100	67.3276	70.0648	74.2219	77.9295	82.3581	118.498	124.342	129.561	135.807	140.169

Source: Thompson, C. M. "Tables of the Percentage Points of the  $\chi^2$ -Distribution," *Biometrika*, ©1941, 32, 188-189. Reproduced by permission of Oxford University Press.

# Cumulative Binomial Probabilities $P(X \leq x)$

		$P$										
$n$	$x$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
1	0	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	0.0500	0.0100
2	0	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100	0.0025	0.0001
	1	0.9900	0.9600	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900	0.0975	0.0199
3	0	0.7290	0.5120	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010	0.0001	0.0000
	1	0.9720	0.8960	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280	0.0073	0.0003
	2	0.9990	0.9920	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710	0.1426	0.0297
4	0	0.6561	0.4096	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001	0.0000	0.0000
	1	0.9477	0.8192	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037	0.0005	0.0000
	2	0.9963	0.9728	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523	0.0140	0.0006
	3	0.9999	0.9984	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439	0.1855	0.0394
5	0	0.5905	0.3277	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000	0.0000	0.0000
	1	0.9185	0.7373	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005	0.0000	0.0000
	2	0.9914	0.9421	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086	0.0012	0.0000
	3	0.9995	0.9933	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815	0.0226	0.0010
	4	1.0000	0.9997	0.9976	0.9898	0.6988	0.9222	0.8319	0.6723	0.4095	0.2262	0.0490
6	0	0.5314	0.2621	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000	0.0000	0.0000
	1	0.8857	0.6554	0.4202	0.2333	0.1094	0.0410	0.0109	0.0016	0.0001	0.0000	0.0000
	2	0.9842	0.9011	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013	0.0001	0.0000
	3	0.9987	0.9830	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0159	0.0022	0.0000
	4	0.9999	0.9984	0.9891	0.9590	0.9806	0.7667	0.5798	0.3446	0.1143	0.0328	0.0015
	5	1.0000	0.9999	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686	0.2649	0.0585
7	0	0.4783	0.2097	0.0824	0.0280	0.0078	0.0106	0.0002	0.0000	0.0000	0.0000	0.0000
	1	0.8503	0.5767	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000	0.0000	0.0000
	2	0.9743	0.8520	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002	0.0000	0.0000
	3	0.9973	0.9667	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027	0.0002	0.0000
	4	0.9998	0.9953	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257	0.0038	0.0000
	5	1.0000	0.9996	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497	0.0444	0.0020
	6	1.0000	1.0000	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217	0.3017	0.0679
8	0	0.4305	0.1678	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.8131	0.5033	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	0.0000	0.0000	0.0000
	2	0.9619	0.7969	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000	0.0000	0.0000
	3	0.9950	0.9437	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004	0.0000	0.0000
	4	0.9996	0.9896	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050	0.0004	0.0000
	5	1.0000	0.9988	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381	0.0058	0.0001
	6	1.0000	0.9999	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869	0.0572	0.0027
	7	1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695	0.3366	0.0773
9	0	0.3874	0.1342	0.0404	0.0101	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7748	0.4362	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	0.0000	0.0000	0.0000
	2	0.9470	0.7382	0.4628	0.2318	0.0889	0.0250	0.0043	0.0003	0.0000	0.0000	0.0000
	3	0.9917	0.9144	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001	0.0000	0.0000
	4	0.9991	0.9804	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009	0.0000	0.0000
	5	0.9999	0.9969	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083	0.0006	0.0000
	6	1.0000	0.9997	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530	0.0084	0.0001
	7	1.0000	1.0000	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252	0.0712	0.0034
	8	1.0000	1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126	0.3698	0.0865

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4th ed. Reproduced by permission of John Wiley & Sons, 2007.

**Cumulative Binomial Probabilities  $P(X \leq x)$  (continued)**

		<i>P</i>										
<i>n</i>	<i>x</i>	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
10	0	0.3487	0.1074	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7361	0.3758	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	0.0000	0.0000
	2	0.9298	0.6778	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	0.0000	0.0000	0.0000
	3	0.9872	0.8791	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000	0.0000	0.0000
	4	0.9984	0.9672	0.8497	0.6331	0.3770	0.1662	0.0473	0.0064	0.0001	0.0000	0.0000
	5	0.9999	0.9936	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016	0.0001	0.0000
	6	1.0000	0.9991	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128	0.0010	0.0000
	7	1.0000	0.9999	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702	0.0115	0.0001
	8	1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639	0.0861	0.0043
15	9	1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513	0.4013	0.0956
	0	0.2059	0.0352	0.0047	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.4590	0.1671	0.0353	0.0052	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.8159	0.3980	0.1268	0.0271	0.0037	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.9444	0.6482	0.2969	0.0905	0.0176	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000
	4	0.9873	0.8358	0.5155	0.2173	0.0592	0.0093	0.0007	0.0000	0.0000	0.0000	0.0000
	5	0.9978	0.9389	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	0.0000	0.0000	0.0000
	6	0.9997	0.9819	0.8689	0.6098	0.3036	0.0950	0.0152	0.0008	0.0000	0.0000	0.0000
	7	1.0000	0.9958	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000	0.0000	0.0000
20	8	1.0000	0.9992	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003	0.0000	0.0000
	9	1.0000	0.9999	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022	0.0001	0.0000
	10	1.0000	1.0000	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127	0.0006	0.0000
	11	1.0000	1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556	0.0055	0.0000
	12	1.0000	1.0000	1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841	0.0362	0.0004
	13	1.0000	1.0000	1.0000	1.0000	0.9995	0.9948	0.9647	0.8329	0.4510	0.1710	0.0096
	14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9953	0.9648	0.7941	0.5367	0.1399
	0	0.1216	0.0115	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.3917	0.0692	0.0076	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
20	2	0.6769	0.2061	0.0355	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.8670	0.4114	0.1071	0.0160	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.9568	0.6296	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.9887	0.8042	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.9976	0.9133	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000
	7	0.9996	0.9679	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	0.0000	0.0000	0.0000
	8	0.9999	0.9900	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	0.0000	0.0000	0.0000
	9	1.0000	0.9974	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	0.0000	0.0000	0.0000
	10	1.0000	0.9994	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000	0.0000	0.0000
	11	1.0000	0.9999	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001	0.0000	0.0000
	12	1.0000	1.0000	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004	0.0000	0.0000
	13	1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024	0.0000	0.0000
	14	1.0000	1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113	0.0003	0.0000
	15	1.0000	1.0000	1.0000	0.9997	0.9941	0.9490	0.7625	0.3704	0.0432	0.0026	0.0000
	16	1.0000	1.0000	1.0000	1.0000	0.9987	0.9840	0.8929	0.5886	0.1330	0.0159	0.0000
	17	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9645	0.7939	0.3231	0.0755	0.0010
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9924	0.9308	0.6083	0.2642	0.0169
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9885	0.8784	0.6415	0.1821

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4th ed. Reproduced by permission of John Wiley & Sons, 2007.

## STATISTICAL QUALITY CONTROL

### Average and Range Charts

$n$	$A_2$	$D_3$	$D_4$
2	1.880	0	3.268
3	1.023	0	2.574
4	0.729	0	2.282
5	0.577	0	2.114
6	0.483	0	2.004
7	0.419	0.076	1.924
8	0.373	0.136	1.864
9	0.337	0.184	1.816
10	0.308	0.223	1.777

$X_i$  = an individual observation

$n$  = the sample size of a group

$k$  = the number of groups

$R$  = (range) the difference between the largest and smallest observations in a sample of size  $n$ .

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_k}{k}$$

$$\bar{R} = \frac{R_1 + R_2 + \dots + R_k}{k}$$

The  $R$  Chart formulas are:

$$CL_R = \bar{R}$$

$$UCL_R = D_4 \bar{R}$$

$$LCL_R = D_3 \bar{R}$$

The  $\bar{X}$  Chart formulas are:

$$CL_X = \bar{\bar{X}}$$

$$UCL_X = \bar{\bar{X}} + A_2 \bar{R}$$

$$LCL_X = \bar{\bar{X}} - A_2 \bar{R}$$

### Standard Deviation Charts

$n$	$A_3$	$B_3$	$B_4$
2	2.659	0	3.267
3	1.954	0	2.568
4	1.628	0	2.266
5	1.427	0	2.089
6	1.287	0.030	1.970
7	1.182	0.119	1.882
8	1.099	0.185	1.815
9	1.032	0.239	1.761
10	0.975	0.284	1.716

$$UCL_S = \bar{\bar{X}} + A_3 \bar{S}$$

$$CL_S = \bar{\bar{S}}$$

$$LCL_S = \bar{\bar{X}} - A_3 \bar{S}$$

$$UCL_S = B_4 \bar{S}$$

$$CL_S = \bar{S}$$

$$LCL_S = B_3 \bar{S}$$

### Approximations

The following table and equations may be used to generate initial approximations of the items indicated.

$n$	$c_4$	$d_2$	$d_3$
2	0.7979	1.128	0.853
3	0.8862	1.693	0.888
4	0.9213	2.059	0.880
5	0.9400	2.326	0.864
6	0.9515	2.534	0.848
7	0.9594	2.704	0.833
8	0.9650	2.847	0.820
9	0.9693	2.970	0.808
10	0.9727	3.078	0.797

$$\hat{\sigma} = \bar{R}/d_2$$

$$\hat{\sigma} = \bar{S}/c_4$$

$$\sigma_R = d_3 \hat{\sigma}$$

$$\sigma_S = \hat{\sigma} \sqrt{1 - c_4^2}, \text{ where}$$

$\hat{\sigma}$  = an estimate of  $\sigma$

$\sigma_R$  = an estimate of the standard deviation of the ranges of the samples

$\sigma_S$  = an estimate of the standard deviation of the standard deviations of the samples

### Tests for Out of Control

1. A single point falls outside the (three sigma) control limits.
2. Two out of three successive points fall on the same side of and more than two sigma units from the center line.
3. Four out of five successive points fall on the same side of and more than one sigma unit from the center line.
4. Eight successive points fall on the same side of the center line.

**Probability and Density Functions: Means and Variances**

Variable	Equation	Mean	Variance
Binomial Coefficient	$\binom{n}{x} = \frac{n!}{x!(n-x)!}$		
Binomial	$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$	$np$	$np(1-p)$
Hyper Geometric	$h(x; n, r, N) = \binom{r}{x} \frac{\binom{N-r}{n-x}}{\binom{N}{n}}$	$\frac{nr}{N}$	$\frac{r(N-r)n(N-n)}{N^2(N-1)}$
Poisson	$f(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$	$\lambda$	$\lambda$
Geometric	$g(x; p) = p(1-p)^{x-1}$	$1/p$	$(1-p)/p^2$
Negative Binomial	$f(y; r, p) = \binom{y+r-1}{r-1} p^r (1-p)^y$	$r/p$	$r(1-p)/p^2$
Multinomial	$f(x_1, \dots, x_k) = \frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k}$	$np_i$	$np_i(1-p_i)$
Uniform	$f(x) = 1/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Gamma	$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$ ; $\alpha > 0, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$
Exponential	$f(x) = \frac{1}{\beta} e^{-x/\beta}$	$\beta$	$\beta^2$
Weibull	$f(x) = \frac{\alpha}{\beta} x^{\alpha-1} e^{-x^\alpha/\beta}$	$\beta^{1/\alpha} \Gamma[(\alpha+1)/\alpha]$	$\beta^{2/\alpha} \left[ \Gamma\left(\frac{\alpha+1}{\alpha}\right) - \Gamma^2\left(\frac{\alpha+1}{\alpha}\right) \right]$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\mu$	$\sigma^2$
Triangular	$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(m-a)} & \text{if } a \leq x \leq m \\ \frac{2(b-x)}{(b-a)(b-m)} & \text{if } m < x \leq b \end{cases}$	$\frac{a+b+m}{3}$	$\frac{a^2 + b^2 + m^2 - ab - am - bm}{18}$

# CHEMISTRY

## DEFINITIONS

*Avogadro's Number* – The number of elementary particles in a mol of a substance.

$$1 \text{ mol} = 1 \text{ gram mole}$$

$$1 \text{ mol} = 6.02 \times 10^{23} \text{ particles}$$

*Mol* – The amount of a substance that contains as many particles as 12 grams of  $^{12}\text{C}$  (carbon 12).

*Molarity of Solutions* – The number of gram moles of a substance dissolved in a liter of solution.

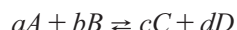
*Molality of Solutions* – The number of gram moles of a substance per 1,000 grams of solvent.

*Normality of Solutions* – The product of the molarity of a solution and the number of valence changes taking place in a reaction.

*Molar Volume of an Ideal Gas* [at  $0^\circ\text{C}$  ( $32^\circ\text{F}$ ) and 1 atm (14.7 psia)]; 22.4 L/(g mole) [359 ft<sup>3</sup>/(lb mole)].

*Mole Fraction of a Substance* – The ratio of the number of moles of a substance to the total moles present in a mixture of substances.

*Equilibrium Constant of a Chemical Reaction*



$$K_{\text{eq}} = \frac{[C]^c [D]^d}{[A]^a [B]^b}$$

*Le Chatelier's Principle for Chemical Equilibrium* – When a stress (such as a change in concentration, pressure, or temperature) is applied to a system in equilibrium, the equilibrium shifts in such a way that tends to relieve the stress.

*Heats of Reaction, Solution, Formation, and Combustion* –

Chemical processes generally involve the absorption or evolution of heat. In an endothermic process, heat is absorbed (enthalpy change is positive). In an exothermic process, heat is evolved (enthalpy change is negative).

*Solubility Product* of a slightly soluble substance  $AB$ :



$$\text{Solubility Product Constant} = K_{\text{sp}} = [A^+]^m [B^-]^n$$

*Faraday's Law* – One gram equivalent weight of matter is chemically altered at each electrode for 96,485 coulombs, or 1 Faraday, of electricity passed through the electrolyte.

A *catalyst* is a substance that alters the rate of a chemical reaction. The catalyst does not affect the position of equilibrium of a reversible reaction.

The *atomic number* is the number of protons in the atomic nucleus.

*Boiling Point Elevation* – The presence of a nonvolatile solute in a solvent raises the boiling point of the resulting solution.

*Freezing Point Depression* – The presence of a solute lowers the freezing point of the resulting solution.

**ACIDS, BASES, and pH** (aqueous solutions)

$$\text{pH} = \log_{10} \left( \frac{1}{[H^+]} \right), \text{ where}$$

$[H^+]$  = molar concentration of hydrogen ion, in gram moles per liter. *Acids* have  $\text{pH} < 7$ . *Bases* have  $\text{pH} > 7$ .



## VIII

51 CHEMISTRY

## SELECTED RULES OF NOMENCLATURE IN ORGANIC CHEMISTRY

### Alcohols

Three systems of nomenclature are in general use. In the first, the alkyl group attached to the hydroxyl group is named and the separate word *alcohol* is added. In the second system, the higher alcohols are considered as derivatives of the first member of the series, which is called *carbinol*. The third method is the modified Geneva system in which (1) the longest carbon chain containing the hydroxyl group determines the surname, (2) the ending *e* of the corresponding saturated hydrocarbon is replaced by *ol*, (3) the carbon chain is numbered from the end that gives the hydroxyl group the smaller number, and (4) the side chains are named and their positions indicated by the proper number. Alcohols in general are divided into three classes. In *primary* alcohols the hydroxyl group is united to a primary carbon atom, that is, a carbon atom united directly to only one other carbon atom. *Secondary* alcohols have the hydroxyl group united to a secondary carbon atom, that is, one united to two other carbon atoms. *Tertiary* alcohols have the hydroxyl group united to a tertiary carbon atom, that is, one united to three other carbon atoms.

### Ethers

Ethers are generally designated by naming the alkyl groups and adding the word *ether*. The group RO is known as an *alkoxyl group*. Ethers may also be named as alkoxy derivatives of hydrocarbons.

### Carboxylic Acids

The name of each linear carboxylic acid is unique to the number of carbon atoms it contains. 1: (one carbon atom) Formic. 2: Acetic. 3: Propionic. 4: Butyric. 5: Valeric. 6: Caproic. 7: Enanthic. 8: Caprylic. 9: Pelargonic. 10: Capric.

### Aldehydes

The common names of aldehydes are derived from the acids that would be formed on oxidation, that is, the acids having the same number of carbon atoms. In general the *ic acid* is dropped and *aldehyde* added.

### Ketones

The common names of ketones are derived from the acid which on pyrolysis would yield the ketone. A second method, especially useful for naming mixed ketones, simply names the alkyl groups and adds the word *ketone*. The name is written as three separate words.

### Unsaturated Acyclic Hydrocarbons



















The simplest compounds in this class of hydrocarbon chemicals are olefins or alkenes with a single carbon-carbon double bond, having the general formula of  $C_nH_{2n}$ . The simplest example in this category is ethylene,  $C_2H_4$ .

Dienes are acyclic hydrocarbons with two carbon-carbon double bonds, having the general formula of  $C_nH_{2n-2}$ ; butadiene ( $C_4H_6$ ) is an example of such.

Similarly, trienes have three carbon-carbon double bonds with the general formula of  $C_nH_{2n-4}$ ; hexatriene ( $C_6H_8$ ) is such an example.

The simplest alkynes have a single carbon-carbon triple bond with the general formula of  $C_nH_{2n-2}$ . This series of compounds begins with acetylene, or  $C_2H_2$ .

### Important Families of Organic Compounds

FAMILY												
	Alkane	Alkene	Alkyne	Arene	Haloalkane	Alcohol	Ether	Amine	Aldehyde	Ketone	Carboxylic Acid	Ester
Specific Example	CH <sub>3</sub> CH <sub>3</sub>	H <sub>2</sub> C = CH <sub>2</sub>	HC ≡ CH		CH <sub>3</sub> CH <sub>2</sub> Cl	CH <sub>3</sub> CH <sub>2</sub> OH	CH <sub>3</sub> OCH <sub>3</sub>	CH <sub>3</sub> NH <sub>2</sub>				
IUPAC Name	Ethane	Ethene or Ethylene	Ethyne or Acetylene	Benzene	Chloroethane	Ethanol	Methoxy-methane	Methan-amine	Ethanal	Acetone	Ethanoic Acid	Methyl ethanoate
Common Name	Ethane	Ethylene	Acetylene	Benzene	Ethyl chloride	Ethyl alcohol	Dimethyl ether	Methyl-amine	Acetal-dehyde	Dimethyl ketone	Acetic Acid	Methyl acetate
General Formula	RH	RCH = CH <sub>2</sub> RCH = CHR R <sub>2</sub> C = CHR R <sub>2</sub> C = CR <sub>2</sub>	RC ≡ CH RC ≡ CR	ArH	RX	ROH	ROR	RNH <sub>2</sub> R <sub>2</sub> NH R <sub>3</sub> N				
Functional Group	C-H and C-C bonds		- C ≡ C -	Aromatic Ring								

**Common Names and Molecular Formulas of Some Industrial  
(Inorganic and Organic) Chemicals**

Common Name	Chemical Name	Molecular Formula
Muriatic acid	Hydrochloric acid	HCl
Cumene	Isopropyl benzene	$\text{C}_6\text{H}_5\text{CH}(\text{CH}_3)_2$
Styrene	Vinyl benzene	$\text{C}_6\text{H}_5\text{CH}=\text{CH}_2$
—	Hypochlorite ion	$\text{OCl}^{-1}$
—	Chlorite ion	$\text{ClO}_2^{-1}$
—	Chlorate ion	$\text{ClO}_3^{-1}$
—	Perchlorate ion	$\text{ClO}_4^{-1}$
Gypsum	Calcium sulfate	$\text{CaSO}_4$
Limestone	Calcium carbonate	$\text{CaCO}_3$
Dolomite	Magnesium carbonate	$\text{MgCO}_3$
Bauxite	Aluminum oxide	$\text{Al}_2\text{O}_3$
Anatase	Titanium dioxide	$\text{TiO}_2$
Rutile	Titanium dioxide	$\text{TiO}_2$
—	Vinyl chloride	$\text{CH}_2=\text{CHCl}$
—	Ethylene oxide	$\text{C}_2\text{H}_4\text{O}$
Pyrite	Ferrous sulfide	FeS
Epsom salt	Magnesium sulfate	$\text{MgSO}_4$
Hydroquinone	p-Dihydroxy benzene	$\text{C}_6\text{H}_4(\text{OH})_2$
Soda ash	Sodium carbonate	$\text{Na}_2\text{CO}_3$
Salt	Sodium chloride	NaCl
Potash	Potassium carbonate	$\text{K}_2\text{CO}_3$
Baking soda	Sodium bicarbonate	$\text{NaHCO}_3$
Lye	Sodium hydroxide	NaOH
Caustic soda	Sodium hydroxide	NaOH
—	Vinyl alcohol	$\text{CH}_2=\text{CHOH}$
Carbolic acid	Phenol	$\text{C}_6\text{H}_5\text{OH}$
Aniline	Aminobenzene	$\text{C}_6\text{H}_5\text{NH}_2$
—	Urea	$(\text{NH}_2)_2\text{CO}$
Toluene	Methyl benzene	$\text{C}_6\text{H}_5\text{CH}_3$
Xylene	Dimethyl benzene	$\text{C}_6\text{H}_4(\text{CH}_3)_2$
—	Silane	$\text{SiH}_4$
—	Ozone	$\text{O}_3$
Neopentane	2,2-Dimethylpropane	$\text{CH}_3\text{C}(\text{CH}_3)_2\text{CH}_3$
Magnetite	Ferrous/ferric oxide	$\text{Fe}_3\text{O}_4$
Quicksilver	Mercury	Hg
Heavy water	Deuterium oxide	$(\text{H}^2)_2\text{O}$
—	Borane	$\text{BH}_3$
Eyewash	Boric acid (solution)	$\text{H}_3\text{BO}_3$
—	Deuterium	$\text{H}^2$
—	Tritium	$\text{H}^3$
Laughing gas	Nitrous oxide	$\text{N}_2\text{O}$
—	Phosgene	$\text{COCl}_2$
Wolfram	Tungsten	W
—	Permanganate ion	$\text{MnO}_4^{-1}$
—	Dichromate ion	$\text{Cr}_2\text{O}_7^{-2}$
—	Hydronium ion	$\text{H}_3\text{O}^{+1}$
Brine	Sodium chloride (solution)	NaCl
Battery acid	Sulfuric acid	$\text{H}_2\text{SO}_4$

## ELECTROCHEMISTRY

*Cathode* – The electrode at which reduction occurs.

*Anode* – The electrode at which oxidation occurs.

*Oxidation* – The loss of electrons.

*Reduction* – The gaining of electrons.

*Cation* – Positive ion

*Anion* – Negative ion

Standard Oxidation Potentials for Corrosion Reactions*	
Corrosion Reaction	Potential, $E_o$ , Volts vs. Normal Hydrogen Electrode
$\text{Au} \rightarrow \text{Au}^{3+} + 3\text{e}^-$	-1.498
$2\text{H}_2\text{O} \rightarrow \text{O}_2 + 4\text{H}^+ + 4\text{e}^-$	-1.229
$\text{Pt} \rightarrow \text{Pt}^{2+} + 2\text{e}^-$	-1.200
$\text{Pd} \rightarrow \text{Pd}^{2+} + 2\text{e}^-$	-0.987
$\text{Ag} \rightarrow \text{Ag}^+ + \text{e}^-$	-0.799
$2\text{Hg} \rightarrow \text{Hg}_2^{2+} + 2\text{e}^-$	-0.788
$\text{Fe}^{2+} \rightarrow \text{Fe}^{3+} + \text{e}^-$	-0.771
$4(\text{OH})^- \rightarrow \text{O}_2 + 2\text{H}_2\text{O} + 4\text{e}^-$	-0.401
$\text{Cu} \rightarrow \text{Cu}^{2+} + 2\text{e}^-$	-0.337
$\text{Sn}^{2+} \rightarrow \text{Sn}^{4+} + 2\text{e}^-$	-0.150
$\text{H}_2 \rightarrow 2\text{H}^+ + 2\text{e}^-$	0.000
$\text{Pb} \rightarrow \text{Pb}^{2+} + 2\text{e}^-$	+0.126
$\text{Sn} \rightarrow \text{Sn}^{2+} + 2\text{e}^-$	+0.136
$\text{Ni} \rightarrow \text{Ni}^{2+} + 2\text{e}^-$	+0.250
$\text{Co} \rightarrow \text{Co}^{2+} + 2\text{e}^-$	+0.277
$\text{Cd} \rightarrow \text{Cd}^{2+} + 2\text{e}^-$	+0.403
$\text{Fe} \rightarrow \text{Fe}^{2+} + 2\text{e}^-$	+0.440
$\text{Cr} \rightarrow \text{Cr}^{3+} + 3\text{e}^-$	+0.744
$\text{Zn} \rightarrow \text{Zn}^{2+} + 2\text{e}^-$	+0.763
$\text{Al} \rightarrow \text{Al}^{3+} + 3\text{e}^-$	+1.662
$\text{Mg} \rightarrow \text{Mg}^{2+} + 2\text{e}^-$	+2.363
$\text{Na} \rightarrow \text{Na}^+ + \text{e}^-$	+2.714
$\text{K} \rightarrow \text{K}^+ + \text{e}^-$	+2.925
* Measured at 25°C. Reactions are written as anode half-cells. Arrows are reversed for cathode half-cells.	

Flinn, Richard A., and Paul K. Trojan, *Engineering Materials and Their Applications*, 4th ed., Houghton Mifflin Company, 1990.

NOTE: In some chemistry texts, the reactions and the signs of the values (in this table) are reversed; for example, the half-cell potential of zinc is given as -0.763 volt for the reaction  $\text{Zn}^{2+} + 2\text{e}^- \rightarrow \text{Zn}$ . When the potential  $E_o$  is positive, the reaction proceeds spontaneously as written.

# MATERIALS SCIENCE/STRUCTURE OF MATTER

## ATOMIC BONDING

### Primary Bonds

- Ionic (e.g., salts, metal oxides)
- Covalent (e.g., within polymer molecules)
- Metallic (e.g., metals)

## CORROSION

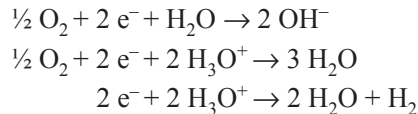
A table listing the standard electromotive potentials of metals is shown on the previous page.

For corrosion to occur, there must be an anode and a cathode in electrical contact in the presence of an electrolyte.

### Anode Reaction (Oxidation) of a Typical Metal, M



### Possible Cathode Reactions (Reduction)



When dissimilar metals are in contact, the more electropositive one becomes the anode in a corrosion cell. Different regions of carbon steel can also result in a corrosion reaction: e.g., cold-worked regions are anodic to noncold-worked; different oxygen concentrations can cause oxygen-deficient regions to become cathodic to oxygen-rich regions; grain boundary regions are anodic to bulk grain; in multiphase alloys, various phases may not have the same galvanic potential.

## DIFFUSION

### Diffusion Coefficient

$$D = D_0 e^{-Q/(RT)}, \text{ where}$$

$D$  = diffusion coefficient

$D_0$  = proportionality constant

$Q$  = activation energy

$R$  = gas constant [8.314 J/(mol•K)]

$T$  = absolute temperature

## THERMAL AND MECHANICAL PROCESSING

*Cold working* (plastically deforming) a metal increases strength and lowers ductility.

Raising temperature causes (1) recovery (stress relief), (2) recrystallization, and (3) grain growth. *Hot working* allows these processes to occur simultaneously with deformation.

*Quenching* is rapid cooling from elevated temperature, preventing the formation of equilibrium phases.

In steels, quenching austenite [FCC ( $\gamma$ ) iron] can result in martensite instead of equilibrium phases—ferrite [BCC ( $\alpha$ ) iron] and cementite (iron carbide).

## PROPERTIES OF MATERIALS

### Electrical

Capacitance: The charge-carrying capacity of an insulating material

Charge held by a capacitor

$$q = CV$$

$q$  = charge

$C$  = capacitance

$V$  = voltage

Capacitance of a parallel plate capacitor

$$C = \frac{\epsilon A}{d}$$

$C$  = capacitance

$\epsilon$  = permittivity of material

$A$  = cross-sectional area of the plates

$d$  = distance between the plates

$\epsilon$  is also expressed as the product of the dielectric constant ( $\kappa$ ) and the permittivity of free space ( $\epsilon_0 = 8.85 \times 10^{-12}$  F/m)

Resistivity: The material property that determines the resistance of a resistor

Resistivity of a material within a resistor

$$\rho = \frac{RA}{L}$$

$\rho$  = resistivity of the material

$R$  = resistance of the resistor

$A$  = cross-sectional area of the resistor

$L$  = length of the resistor

Conductivity is the reciprocal of the resistivity

*Photoelectric effect*—electrons are emitted from matter (metals and nonmetallic solids, liquids or gases) as a consequence of their absorption of energy from electromagnetic radiation of very short wavelength and high frequency.

*Piezoelectric effect*—the electromechanical and the electrical state in crystalline materials.

### Mechanical

Strain is defined as change in length per unit length; for pure tension the following apply:

Engineering strain

$$\epsilon = \frac{\Delta L}{L_0}$$

$\epsilon$  = engineering strain

$\Delta L$  = change in length

$L_0$  = initial length

True strain

$$\epsilon_T = \frac{dL}{L}$$

$\epsilon_T$  = engineering strain

$dL$  = differential change in length

$L$  = initial length

$\epsilon_T = \ln(1 + \epsilon)$

### Properties of Metals

Metal	Symbol	Atomic Weight	Density $\rho$ (kg/m <sup>3</sup> ) Water = 1000	Melting Point (°C)	Melting Point (°F)	Specific Heat [J/(kg·K)]	Electrical Resistivity (10 <sup>-8</sup> $\Omega \cdot m$ ) at 0°C (273.2 K)	Heat Conductivity $\lambda$ [W/(m·K)] at 0°C (273.2 K)
Aluminum	Al	26.98	2,698	660	1,220	895.9	2.5	236
Antimony	Sb	121.75	6,692	630	1,166	209.3	39	25.5
Arsenic	As	74.92	5,776	subl. 613	subl. 1,135	347.5	26	–
Barium	Ba	137.33	3,594	710	1,310	284.7	36	–
Beryllium	Be	9.012	1,846	1,285	2,345	2,051.5	2.8	218
Bismuth	Bi	208.98	9,803	271	519	125.6	107	8.2
Cadmium	Cd	112.41	8,647	321	609	234.5	6.8	97
Caesium	Cs	132.91	1,900	29	84	217.7	18.8	36
Calcium	Ca	40.08	1,530	840	1,544	636.4	3.2	–
Cerium	Ce	140.12	6,711	800	1,472	188.4	7.3	11
Chromium	Cr	52	7,194	1,860	3,380	406.5	12.7	96.5
Cobalt	Co	58.93	8,800	1,494	2,721	431.2	5.6	105
Copper	Cu	63.54	8,933	1,084	1,983	389.4	1.55	403
Gallium	Ga	69.72	5,905	30	86	330.7	13.6	41
Gold	Au	196.97	19,281	1,064	1,947	129.8	2.05	319
Indium	In	114.82	7,290	156	312	238.6	8	84
Iridium	Ir	192.22	22,550	2,447	4,436	138.2	4.7	147
Iron	Fe	55.85	7,873	1,540	2,804	456.4	8.9	83.5
Lead	Pb	207.2	11,343	327	620	129.8	19.2	36
Lithium	Li	6.94	533	180	356	4,576.2	8.55	86
Magnesium	Mg	24.31	1,738	650	1,202	1,046.7	3.94	157
Manganese	Mn	54.94	7,473	1,250	2,282	502.4	138	8
Mercury	Hg	200.59	13,547	–39	–38	142.3	94.1	7.8
Molybdenum	Mo	95.94	10,222	2,620	4,748	272.1	5	139
Nickel	Ni	58.69	8,907	1,455	2,651	439.6	6.2	94
Niobium	Nb	92.91	8,578	2,425	4,397	267.9	15.2	53
Osmium	Os	190.2	22,580	3,030	5,486	129.8	8.1	88
Palladium	Pd	106.4	11,995	1,554	2,829	230.3	10	72
Platinum	Pt	195.08	21,450	1,772	3,221	134	9.81	72
Potassium	K	39.09	862	63	145	753.6	6.1	104
Rhodium	Rh	102.91	12,420	1,963	3,565	242.8	4.3	151
Rubidium	Rb	85.47	1,533	38.8	102	330.7	11	58
Ruthenium	Ru	101.07	12,360	2,310	4,190	255.4	7.1	117
Silver	Ag	107.87	10,500	961	1,760	234.5	1.47	428
Sodium	Na	22.989	966	97.8	208	1,235.1	4.2	142
Strontium	Sr	87.62	2,583	770	1,418	–	20	–
Tantalum	Ta	180.95	16,670	3,000	5,432	150.7	12.3	57
Thallium	Tl	204.38	11,871	304	579	138.2	10	10
Thorium	Th	232.04	11,725	1,700	3,092	117.2	14.7	54
Tin	Sn	118.69	7,285	232	449	230.3	11.5	68
Titanium	Ti	47.88	4,508	1,670	3,038	527.5	39	22
Tungsten	W	183.85	19,254	3,387	6,128	142.8	4.9	177
Uranium	U	238.03	19,050	1,135	2,075	117.2	28	27
Vanadium	V	50.94	6,090	1,920	3,488	481.5	18.2	31
Zinc	Zn	65.38	7,135	419	786	393.5	5.5	117
Zirconium	Zr	91.22	6,507	1,850	3,362	284.7	40	23

### Some Extrinsic, Elemental Semiconductors

Element	Dopant	Periodic table group of dopant	Maximum solid solubility of dopant (atoms/m <sup>3</sup> )
Si	B	III A	$600 \times 10^{24}$
	Al	III A	$20 \times 10^{24}$
	Ga	III A	$40 \times 10^{24}$
	P	V A	$1,000 \times 10^{24}$
	As	V A	$2,000 \times 10^{24}$
Ge	Sb	V A	$70 \times 10^{24}$
	Al	III A	$400 \times 10^{24}$
	Ga	III A	$500 \times 10^{24}$
	In	III A	$4 \times 10^{24}$
	As	V A	$80 \times 10^{24}$
	Sb	V A	$10 \times 10^{24}$

### Impurity Energy Levels for Extrinsic Semiconductors

Semiconductor	Dopant	$E_g - E_d$ (eV)	$E_a$ (eV)
Si	P	0.044	—
	As	0.049	—
	Sb	0.039	—
	Bi	0.069	—
	B	—	0.045
	Al	—	0.057
	Ga	—	0.065
	In	—	0.160
	Tl	—	0.260
Ge	P	0.012	—
	As	0.013	—
	Sb	0.096	—
	B	—	0.010
	Al	—	0.010
	Ga	—	0.010
	In	—	0.011
	Tl	—	0.01
GaAs	Se	0.005	—
	Te	0.003	—
	Zn	—	0.024
	Cd	—	0.021

Stress is defined as force per unit area; for pure tension the following apply:

Engineering stress

$$\sigma = \frac{F}{A_0}$$

$\sigma$  = engineering stress

$F$  = applied force

$A_0$  = initial cross-sectional area

True stress

$$\sigma_T = \frac{F}{A}$$

$\sigma_T$  = true stress

$F$  = applied force

$A$  = actual cross-sectional area

The Elastic Modulus (also called Modulus, Modulus of Elasticity, Young's Modulus) describes the relationship between engineering and engineering strain during elastic loading. Hooke's Law applies in such a case.

$\sigma = E\epsilon$  where  $E$  is the elastic modulus.

Key mechanical properties obtained from a tensile test curve:

$\sigma(\epsilon)$

Elastic modulus

Ductility (also called percent elongation) permanent engineering strain after failure

Ultimate tensile strength (also called tensile strength) maximum engineering stress

Yield strength: engineering stress at which permanent deformation is first observed, calculated by 0.2% offset method.

Other mechanical properties:

Creep: time dependent deformation under load. Usually measured by strain rate. For steady-state creep this is:

$$\frac{d\epsilon}{dt} = A\sigma^n e^{-\frac{Q}{RT}}$$

$A$  = pre-exponential constant

$n$  = stress sensitivity

$Q$  = activation energy for creep

$R$  = ideal gas law constant

$T$  = absolute temperature

Fatigue: time dependent failure under cyclic load. Fatigue life is the number of cycles to failure. The endurance limit is the stress below which fatigue failure is unlikely.

Fracture toughness: the combination of applied stress and the crack length in a brittle material. It is the stress intensity when the material will fail.

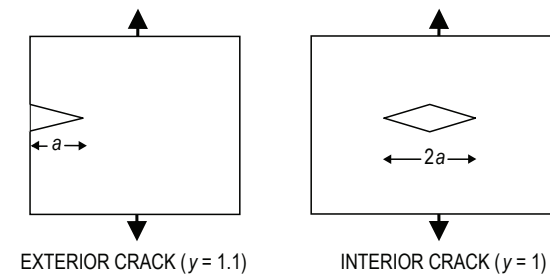
$$K_{IC} = Y\sigma\sqrt{\pi a}$$

$K_{IC}$  = fracture toughness

$\sigma$  = applied engineering stress

$a$  = crack length

$Y$  = geometrical factor



The critical value of stress intensity at which catastrophic crack propagation occurs,  $K_{IC}$ , is a material property.

♦ W.R. Runyan and S.B. Watelski, in *Handbook of Materials and Processes for Electronics*, C.A. Harper, ed., McGraw-Hill, 1970.



## Representative Values of Fracture Toughness

Material	$K_{Ic}$ (MPa·m <sup>1/2</sup> )	$K_{Ic}$ (ksi·in <sup>1/2</sup> )
A1 2014-T651	24.2	22
A1 2024-T3	44	40
52100 Steel	14.3	13
4340 Steel	46	42
Alumina	4.5	4.1
Silicon Carbide	3.5	3.2

## RELATIONSHIP BETWEEN HARDNESS AND TENSILE STRENGTH

For plain carbon steels, there is a general relationship between Brinell hardness and tensile strength as follows:

$$TS(\text{psi}) \simeq 500 \text{ BHN}$$

$$TS(\text{MPa}) \simeq 3.5 \text{ BHN}$$

## ASTM GRAIN SIZE

$$S_V = 2P_L$$

$$N_{(0.0645 \text{ mm}^2)} = 2^{(n-1)}$$

$$\frac{N_{\text{actual}}}{\text{Actual Area}} = \frac{N}{(0.0645 \text{ mm}^2)}, \text{ where}$$

$S_V$  = grain-boundary surface per unit volume

$P_L$  = number of points of intersection per unit length between the line and the boundaries

$N$  = number of grains observed in a area of 0.0645 mm<sup>2</sup>

$n$  = grain size (nearest integer > 1)

## COMPOSITE MATERIALS

$$\rho_c = \sum f_i \rho_i$$

$$C_c = \sum f_i c_i$$

$$\left[ \sum \frac{f_i}{E_i} \right]^{-1} \leq E_c \leq \sum f_i E_i$$

$$\sigma_c = \sum f_i \sigma_i$$

$\rho_c$  = density of composite

$C_c$  = heat capacity of composite per unit volume

$E_c$  = Young's modulus of composite

$f_i$  = volume fraction of individual material

$c_i$  = heat capacity of individual material per unit volume

$E_i$  = Young's modulus of individual material

$\sigma_c$  = strength parallel to fiber direction

Also, for axially oriented, long, fiber-reinforced composites, the strains of the two components are equal.

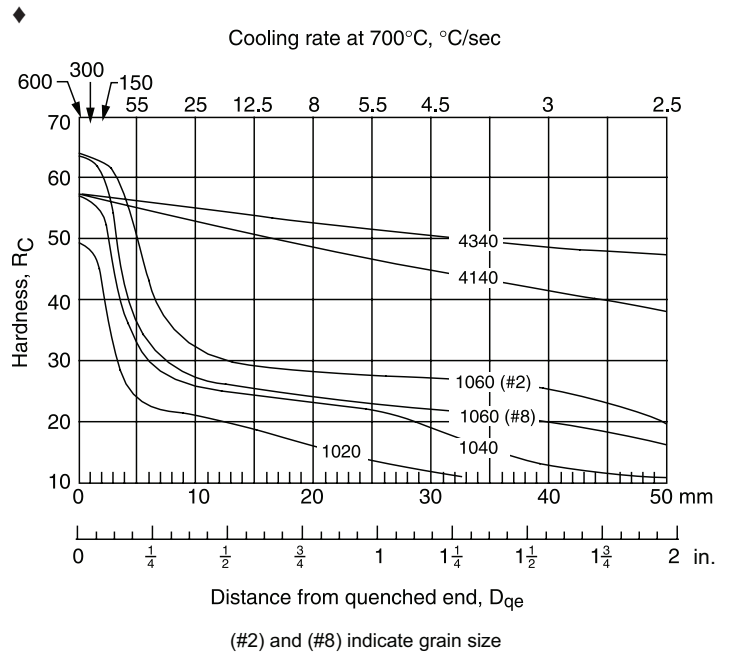
$$(\Delta L/L)_1 = (\Delta L/L)_2$$

$\Delta L$  = change in length of the composite

$L$  = original length of the composite

Hardness: Resistance to penetration: Measured by denting a material under known load and measuring the size of the dent.

Hardenability: The “ease” with which hardness can be obtained.



**JOMINY HARDENABILITY CURVES FOR SIX STEELS**

♦ Van Vlack, L., *Elements of Materials Science & Engineering*, Addison-Wesley, 1989.

The following two graphs show cooling curves for four different positions in the bar.

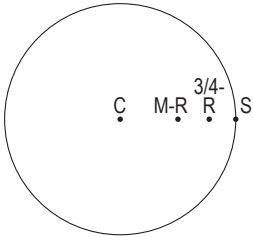
C = Center

M-R = Halfway between center and surface

$\frac{3}{4}$ -R = 75% of the distance between the center and the surface

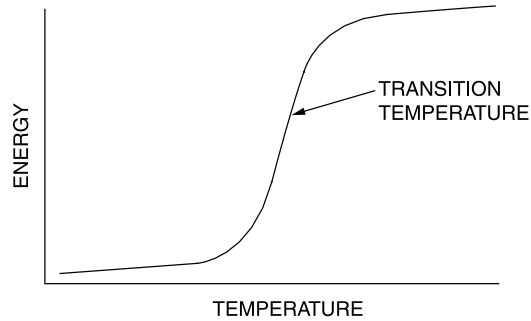
S = Surface

These positions are shown in the following figure.



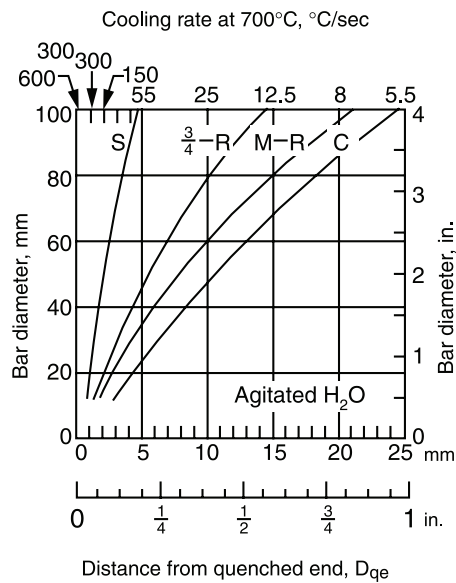
## Impact Test

The *Charpy Impact Test* is used to find energy required to fracture and to identify ductile to brittle transition.

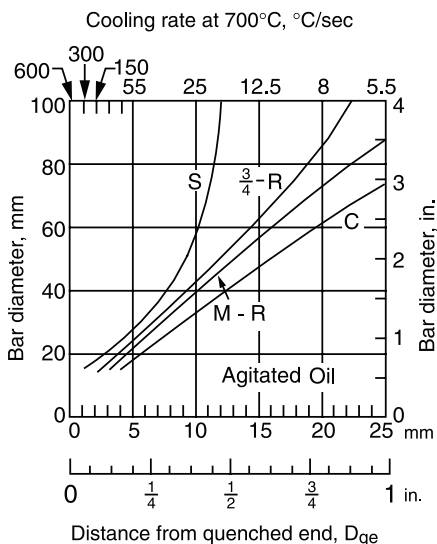


Impact tests determine the amount of energy required to cause failure in standardized test samples. The tests are repeated over a range of temperatures to determine the *ductile to brittle transition temperature*.

♦ Van Vlack, L., *Elements of Materials Science & Engineering*, Addison-Wesley, 1989.

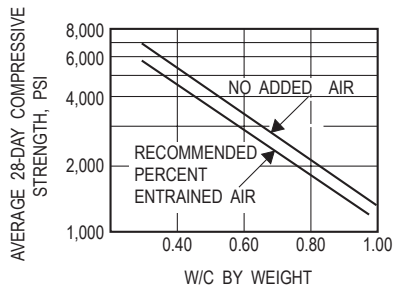


## COOLING RATES FOR BARS QUENCHED IN AGITATED WATER



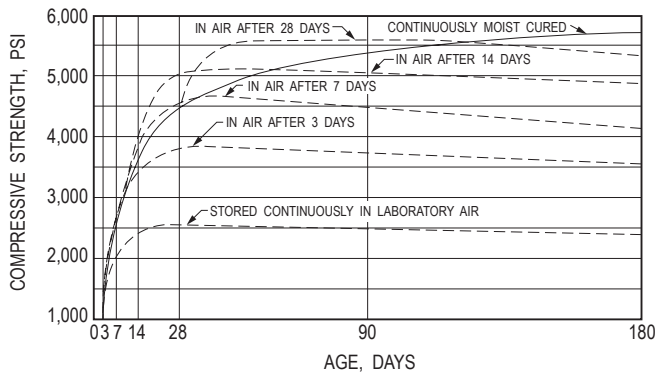
## COOLING RATES FOR BARS QUENCHED IN AGITATED OIL

## Concrete



Concrete strength decreases with increases in water-cement ratio for concrete with and without entrained air.

Water-cement (W/C) ratio is the primary factor affecting the strength of concrete. The figure above shows how W/C expressed as a ratio of weight of water and cement by weight of concrete mix affects the compressive strength of both air-entrained and non air-entrained concrete.



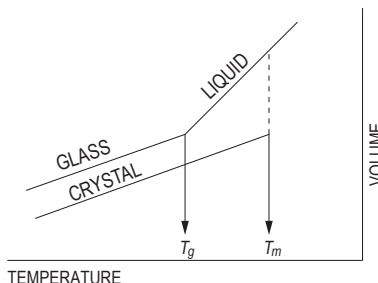
Concrete compressive strength varies with moist-curing conditions. Mixes tested had a water-cement ratio of 0.50, a slump of 3.5 in., cement content of 556 lb/yd<sup>3</sup>, sand content of 36%, and air content of 4%.

Water content affects workability. However, an increase in water without a corresponding increase in cement reduces the concrete strength. Superplasticizers are the most typical way to increase workability. Air entrainment is used to improve durability.

## Amorphous Materials

Amorphous materials such as glass are non-crystalline solids. Thermoplastic polymers are either semicrystalline or amorphous.

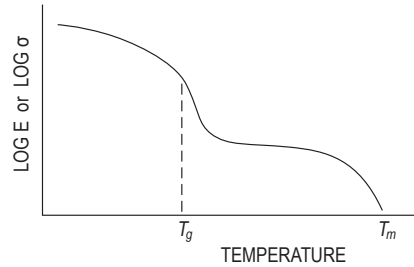
Below the glass transition temperature ( $T_g$ ) the amorphous material will be a brittle solid.



The volume temperature curve as shown above is often used to show the difference between amorphous and crystalline solids.

## Polymers

Polymers are classified as thermoplastics that can be melted and reformed. Thermosets cannot be melted and reformed.



The above curve shows the temperature dependent strength ( $\sigma$ ) or modulus ( $E$ ) for a thermoplastic polymer.

## Polymer Additives

Chemicals and compounds are added to polymers to improve properties for commercial use. These substances, such as plasticizers, improve formability during processing, while others increase strength or durability.

Examples of common additives are:

Plasticizers: vegetable oils, low molecular weight polymers or monomers

Fillers: talc, chopped glass fibers

Flame retardants: halogenated paraffins, zinc borate, chlorinated phosphates

Ultraviolet or visible light resistance: carbon black

Oxidation resistance: phenols, aldehydes

## Thermal Properties

The thermal expansion coefficient is the ratio of engineering strain to the change in temperature.

$$\alpha = \frac{\epsilon}{\Delta T}$$

$\alpha$  = thermal expansion coefficient

$\epsilon$  = engineering strain

$\Delta T$  = change in temperature

Specific heat (also called heat capacity) is the amount of heat required to raise the temperature of something or an amount of something by 1 degree.

At constant pressure the amount of heat ( $Q$ ) required to increase the temperature of something by  $\Delta T$  is  $C_p \Delta T$ , where  $C_p$  is the constant pressure heat capacity.

At constant volume the amount of heat ( $Q$ ) required to increase the temperature of something by  $\Delta T$  is  $C_v \Delta T$ , where  $C_v$  is the constant volume heat capacity.

An object can have a heat capacity that would be expressed as energy/degree.

The heat capacity of a material can be reported as energy/degree per unit mass or per unit volume.

♦ *Concrete Manual*, 8th ed., U.S. Bureau of Reclamation, 1975.

♦ Merritt, Frederick S., *Standard Handbook for Civil Engineers*, 3rd ed., McGraw-Hill, 1983.

## BINARY PHASE DIAGRAMS

Allows determination of (1) what phases are present at equilibrium at any temperature and average composition, (2) the compositions of those phases, and (3) the fractions of those phases.

Eutectic reaction (liquid  $\rightarrow$  two solid phases)

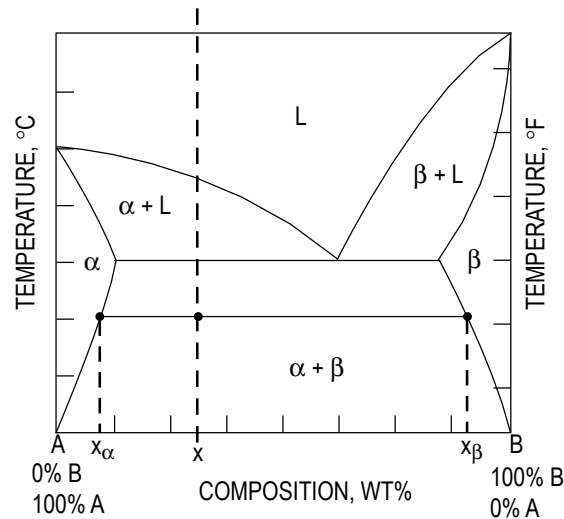
Eutectoid reaction (solid  $\rightarrow$  two solid phases)

Peritectic reaction (liquid + solid  $\rightarrow$  solid)

Peritectoid reaction (two solid phases  $\rightarrow$  solid)

### Lever Rule

The following phase diagram and equations illustrate how the weight of each phase in a two-phase system can be determined:

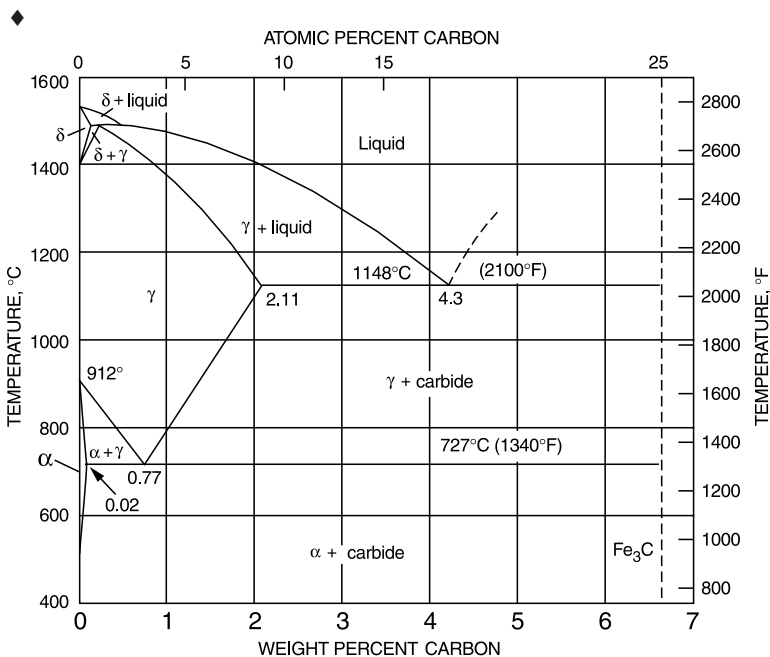


(In diagram, L = liquid.) If  $x$  = the average composition at temperature  $T$ , then

$$\text{wt}\% \alpha = \frac{x_{\beta} - x}{x_{\beta} - x_{\alpha}} \times 100$$

$$\text{wt}\% \beta = \frac{x - x_{\alpha}}{x_{\beta} - x_{\alpha}} \times 100$$

### Iron-Iron Carbide Phase Diagram



♦ Van Vlack, L., *Elements of Materials Science & Engineering*, Addison-Wesley, Boston, 1989.

# STATICS

## FORCE

A *force* is a *vector* quantity. It is defined when its (1) magnitude, (2) point of application, and (3) direction are known.

The vector form of a force is

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j}$$

## RESULTANT (TWO DIMENSIONS)

The *resultant*,  $F$ , of  $n$  forces with components  $F_{x,i}$  and  $F_{y,i}$  has the magnitude of

$$F = \left[ \left( \sum_{i=1}^n F_{x,i} \right)^2 + \left( \sum_{i=1}^n F_{y,i} \right)^2 \right]^{1/2}$$

The resultant direction with respect to the  $x$ -axis is

$$\theta = \arctan \left( \frac{\sum_{i=1}^n F_{y,i}}{\sum_{i=1}^n F_{x,i}} \right)$$

## RESOLUTION OF A FORCE

$$F_x = F \cos \theta_x; F_y = F \cos \theta_y; F_z = F \cos \theta_z$$

$$\cos \theta_x = F_x/F; \cos \theta_y = F_y/F; \cos \theta_z = F_z/F$$

Separating a force into components when the geometry of force is known and  $R = \sqrt{x^2 + y^2 + z^2}$

$$F_x = (x/R)F; \quad F_y = (y/R)F; \quad F_z = (z/R)F$$

## MOMENTS (COUPLES)

A system of two forces that are equal in magnitude, opposite in direction, and parallel to each other is called a *couple*. A *moment*  $\mathbf{M}$  is defined as the cross product of the *radius vector*  $\mathbf{r}$  and the *force*  $\mathbf{F}$  from a point to the line of action of the force.

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}; \quad \begin{aligned} M_x &= yF_z - zF_y, \\ M_y &= zF_x - xF_z, \text{ and} \\ M_z &= xF_y - yF_x. \end{aligned}$$

## SYSTEMS OF FORCES

$$\mathbf{F} = \sum \mathbf{F}_n$$

$$\mathbf{M} = \sum (\mathbf{r}_n \times \mathbf{F}_n)$$

## Equilibrium Requirements

$$\sum \mathbf{F}_n = 0$$

$$\sum \mathbf{M}_n = 0$$

## CENTROIDS OF MASSES, AREAS, LENGTHS, AND VOLUMES

The following formulas are for discrete masses, areas, lengths, and volumes:

$$\mathbf{r}_c = \sum m_n \mathbf{r}_n / \sum m_n, \text{ where}$$

$m_n$  = the *mass of each particle* making up the system,

$\mathbf{r}_n$  = the *radius vector* to each particle from a selected reference point, and

$\mathbf{r}_c$  = the *radius vector* to the *centroid of the total mass* from the selected reference point.

The *moment of area* ( $M_a$ ) is defined as

$$M_{ay} = \sum x_n a_n$$

$$M_{ax} = \sum y_n a_n$$

The *centroid of area* is defined as

$$x_{ac} = M_{ay}/A = \sum x_n a_n/A$$

$$y_{ac} = M_{ax}/A = \sum y_n a_n/A$$

where  $A = \sum a_n$

## MOMENT OF INERTIA

The *moment of inertia*, or the second moment of area, is defined as

$$I_y = \int x^2 dA$$

$$I_x = \int y^2 dA$$

The *polar moment of inertia*  $J$  of an area about a point is equal to the sum of the moments of inertia of the area about any two perpendicular axes in the area and passing through the same point.

$$\begin{aligned} I_z &= J = I_y + I_x = \int (x^2 + y^2) dA \\ &= r_p^2 A, \text{ where} \end{aligned}$$

### Moment of Inertia Parallel Axis Theorem

The moment of inertia of an area about any axis is defined as the moment of inertia of the area about a parallel centroidal axis plus a term equal to the area multiplied by the square of the perpendicular distance  $d$  from the centroidal axis to the axis in question.

$$I'_x = I_{x_c} + d_x^2 A$$

$$I'_y = I_{y_c} + d_y^2 A, \text{ where}$$

$d_x, d_y$  = distance between the two axes in question,

$I_{x_c}, I_{y_c}$  = the moment of inertia about the centroidal axis, and

$I'_x, I'_y$  = the moment of inertia about the new axis.

### Radius of Gyration

The *radius of gyration*  $r_p, r_x, r_y$  is the distance from a reference axis at which all of the area can be considered to be concentrated to produce the moment of inertia.

$$r_x = \sqrt{I_x/A}; \quad r_y = \sqrt{I_y/A}; \quad r_p = \sqrt{J/A}$$

### Product of Inertia

The *product of inertia* ( $I_{xy}$ , etc.) is defined as:

$$I_{xy} = \int xy dA, \text{ with respect to the } xy\text{-coordinate system,}$$

The *parallel-axis theorem* also applies:

$$I'_{xy} = I_{x_c y_c} + d_x d_y A \text{ for the } xy\text{-coordinate system, etc.}$$

where

$d_x$  =  $x$ -axis distance between the two axes in question, and

$d_y$  =  $y$ -axis distance between the two axes in question.

### FRICTION

The largest frictional force is called the *limiting friction*.

Any further increase in applied forces will cause motion.

$$F \leq \mu_s N, \text{ where}$$

$F$  = friction force,

$\mu_s$  = *coefficient of static friction*, and

$N$  = normal force between surfaces in contact.

### SCREW THREAD

For a *screw-jack, square thread*,

$$M = Pr \tan (\alpha \pm \phi), \text{ where}$$

+ is for screw tightening,

– is for screw loosening,

$M$  = external moment applied to axis of screw,

$P$  = load on jack applied along and on the line of the axis,

$r$  = the mean thread radius,

$\alpha$  = the *pitch angle* of the thread, and

$\mu = \tan \phi$  = the appropriate coefficient of friction.

### BELT FRICTION

$$F_1 = F_2 e^{\mu \theta}, \text{ where}$$

$F_1$  = force being applied in the direction of impending motion,

$F_2$  = force applied to resist impending motion,

$\mu$  = coefficient of static friction, and

$\theta$  = the total *angle of contact* between the surfaces expressed in radians.

### STATICALLY DETERMINATE TRUSS

#### Plane Truss: Method of Joints

The method consists of solving for the forces in the members by writing the two equilibrium equations for each joint of the truss.

$$\Sigma F = 0 \text{ and } \Sigma M = 0, \text{ where}$$

$F$  = horizontal forces and member components and

$F$  = vertical forces and member components.

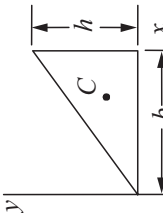
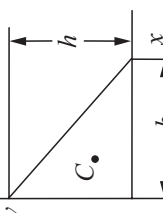
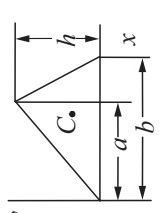
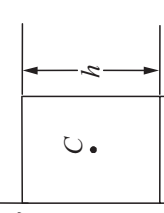
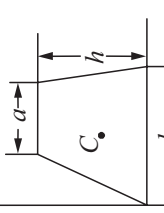
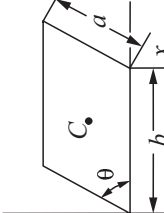
#### Plane Truss: Method of Sections

The method consists of drawing a free-body diagram of a portion of the truss in such a way that the unknown truss member force is exposed as an external force.

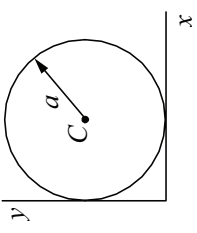
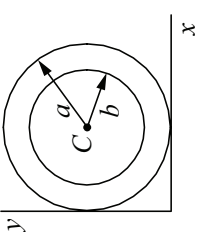
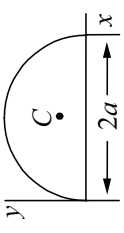
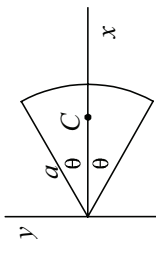
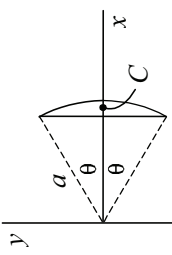
### CONCURRENT FORCES

A concurrent-force system is one in which the lines of action of the applied forces all meet at one point.

A *two-force* body in static equilibrium has two applied forces that are equal in magnitude, opposite in direction, and collinear.

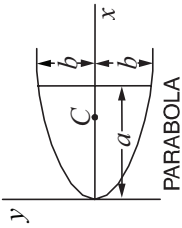
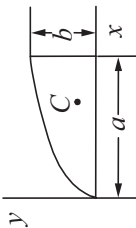
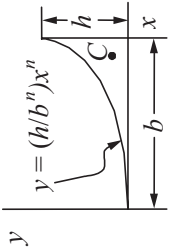
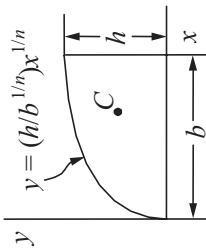
Figure	Area & Centroid	Area Moment of Inertia	(Radius of Gyration) <sup>2</sup>	Product of Inertia
	$A = bh/2$ $x_c = 2b/3$ $y_c = h/3$	$I_{x_c} = bh^3/36$ $I_{y_c} = b^3h/36$ $I_x = bh^3/12$ $I_y = b^3h/4$	$r_{x_c}^2 = h^2/18$ $r_{y_c}^2 = b^2/18$ $r_x^2 = h^2/6$ $r_y^2 = b^2/2$	$I_{x_c y_c} = Abh/36 = b^2h^2/72$ $I_{xy} = Abh/4 = b^2h^2/8$
	$A = bh/2$ $x_c = b/3$ $y_c = h/3$	$I_{x_c} = bh^3/36$ $I_{y_c} = b^3h/36$ $I_x = bh^3/12$ $I_y = b^3h/12$	$r_{x_c}^2 = h^2/18$ $r_{y_c}^2 = b^2/18$ $r_x^2 = h^2/6$ $r_y^2 = b^2/6$	$I_{x_c y_c} = -Abh/36 = -b^2h^2/72$ $I_{xy} = Abh/12 = b^2h^2/24$
	$A = bh/2$ $x_c = (a+b)/3$ $y_c = h/3$	$I_{x_c} = bh^3/36$ $I_{y_c} = [bh(b^2 - ab + a^2)]/36$ $I_x = bh^3/12$ $I_y = [bh(b^2 + ab + a^2)]/12$	$r_{x_c}^2 = h^2/18$ $r_{y_c}^2 = (b^2 - ab + a^2)/18$ $r_x^2 = h^2/6$ $r_y^2 = (b^2 + ab + a^2)/6$	$I_{x_c y_c} = [Ah(2a - b)]/36$ $I_{xy} = [bh^2(2a - b)]/72$ $I_{xy} = [Ah(2a + b)]/12$ $I_{xy} = [bh^2(2a + b)]/24$
	$A = bh$ $x_c = b/2$ $y_c = h/2$	$I_{x_c} = bh^3/12$ $I_{y_c} = b^3h/12$ $I_x = bh^3/3$ $I_y = b^3h/3$ $J = [bh(b^2 + h^2)]/12$	$r_{x_c}^2 = h^2/12$ $r_{y_c}^2 = b^2/12$ $r_x^2 = h^2/3$ $r_y^2 = b^2/3$ $r_p^2 = (b^2 + h^2)/12$	$I_{x_c y_c} = 0$ $I_{xy} = Abh/4 = b^2h^2/4$
	$A = h(a+b)/2$ $y_c = \frac{h(2a+b)}{3(a+b)}$	$I_{x_c} = \frac{h^3(a^2 + 4ab + b^2)}{36(a+b)}$ $I_x = \frac{h^3(3a+b)}{12}$	$r_{x_c}^2 = \frac{h^2(a^2 + 4ab + b^2)}{18(a+b)}$ $r_x^2 = \frac{h^2(3a+b)}{6(a+b)}$	
	$A = ab \sin \theta$ $x_c = (b + a \cos \theta)/2$ $y_c = (a \sin \theta)/2$	$I_{x_c} = (a^3b \sin^3 \theta)/12$ $I_{y_c} = [ab \sin \theta (b^2 + a^2 \cos^2 \theta)]/12$ $I_x = (a^3b \sin^3 \theta)/3$ $I_y = [ab \sin \theta (b + a \cos \theta)^2]/3 - (a^2b^2 \sin \theta \cos \theta)/6$	$r_{x_c}^2 = (a \sin \theta)^2/12$ $r_{y_c}^2 = (b^2 + a^2 \cos^2 \theta)/12$ $r_x^2 = (a \sin \theta)^2/3$ $r_y^2 = (b + a \cos \theta)^2/3 - (ab \cos \theta)/6$	$I_{x_c y_c} = (a^3b \sin^2 \theta \cos \theta)/12$

Housner, George W., and Donald E. Hudson, *Applied Mechanics Dynamics*, D. Van Nostrand Company, Inc., Princeton, NJ, 1959. Table reprinted by permission of G.W. Housner & D.E. Hudson.

Figure	Area & Centroid	Area Moment of Inertia	(Radius of Gyration) <sup>2</sup>	Product of Inertia
	$A = \pi a^2$ $x_c = a$ $y_c = a$	$I_{x_c} = I_{y_c} = \pi a^4 / 4$ $I_x = I_y = 5\pi a^4 / 4$ $J = \pi a^4 / 2$	$r_{x_c}^2 = r_{y_c}^2 = a^2 / 4$ $r_x^2 = r_y^2 = 5a^2 / 4$ $r_p^2 = a^2 / 2$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2$
	$A = \pi(a^2 - b^2)$ $x_c = a$ $y_c = a$	$I_{x_c} = I_{y_c} = \pi(a^4 - b^4) / 4$ $I_x = I_y = \frac{5\pi a^4}{4} - \pi a^2 b^2 - \frac{\pi b^4}{4}$ $J = \pi(a^4 - b^4) / 2$	$r_{x_c}^2 = r_{y_c}^2 = (a^2 + b^2) / 4$ $r_x^2 = r_y^2 = (5a^2 + b^2) / 4$ $r_p^2 = (a^2 + b^2) / 2$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2 = \pi a^2(a^2 - b^2)$
	$A = \pi a^2 / 2$ $x_c = a$ $y_c = 4a / (3\pi)$	$I_{x_c} = \frac{a^4(9\pi^2 - 64)}{72\pi}$ $I_{y_c} = \pi a^4 / 8$ $I_x = \pi a^4 / 8$ $I_y = 5\pi a^4 / 8$	$r_{x_c}^2 = \frac{a^2(9\pi^2 - 64)}{36\pi^2}$ $r_{y_c}^2 = a^2 / 4$ $r_x^2 = a^2 / 4$ $r_y^2 = 5a^2 / 4$	$I_{x_c y_c} = 0$ $I_{xy} = 2a^4 / 3$
 CIRCULAR SECTOR	$A = a^2 \theta$ $x_c = \frac{2a \sin \theta}{3 \theta}$ $y_c = 0$	$I_x = a^4 (\theta - \sin \theta \cos \theta) / 4$ $I_y = a^4 (\theta + \sin \theta \cos \theta) / 4$	$r_x^2 = \frac{a^2 (\theta - \sin \theta \cos \theta)}{4 \theta}$ $r_y^2 = \frac{a^2 (\theta + \sin \theta \cos \theta)}{4 \theta}$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
 CIRCULAR SEGMENT	$A = a^2 \left[ \theta - \frac{\sin 2\theta}{2} \right]$ $x_c = \frac{2a}{3} \frac{\sin^3 \theta}{\theta - \sin \theta \cos \theta}$ $y_c = 0$	$I_x = \frac{Aa^2}{4} \left[ 1 - \frac{2\sin^3 \theta \cos \theta}{3\theta - 3\sin \theta \cos \theta} \right]$ $I_y = \frac{Aa^2}{4} \left[ 1 + \frac{2\sin^3 \theta \cos \theta}{\theta - \sin \theta \cos \theta} \right]$	$r_x^2 = \frac{a^2}{4} \left[ 1 - \frac{2\sin^3 \theta \cos \theta}{3\theta - 3\sin \theta \cos \theta} \right]$ $r_y^2 = \frac{a^2}{4} \left[ 1 + \frac{2\sin^3 \theta \cos \theta}{\theta - \sin \theta \cos \theta} \right]$	$I_{x_c y_c} = 0$ $I_{xy} = 0$

Housner, George W., and Donald E. Hudson, *Applied Mechanics Dynamics*, D. Van Nostrand Company, Inc., Princeton, NJ, 1959. Table reprinted by permission of G.W. Housner & D.E. Hudson.



Figure	Area & Centroid	Area Moment of Inertia	(Radius of Gyration) <sup>2</sup>	Product of Inertia
 <p>PARABOLA</p>	$A = 4ab/3$ $x_c = 3a/5$ $y_c = 0$	$I_{x_c} = I_x = 4ab^3/15$ $I_{y_c} = 16a^3b/175$ $I_y = 4a^3b/7$	$r_{x_c}^2 = r_x^2 = b^2/5$ $r_{y_c}^2 = 12a^2/175$ $r_y^2 = 3a^2/7$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
 <p>HALF A PARABOLA</p>	$A = 2ab/3$ $x_c = 3a/5$ $y_c = 3b/8$	$I_x = 2ab^3/15$ $I_y = 2ba^3/7$	$r_x^2 = b^2/5$ $r_y^2 = 3a^2/7$	$I_{xy} = Aab/4 = a^2b^2$
 <p>n<sup>th</sup> DEGREE PARABOLA</p>	$A = bh/(n+1)$ $x_c = \frac{n+1}{n+2}b$ $y_c = \frac{h}{2} \frac{n+1}{2n+1}$	$I_x = \frac{bh^3}{3(n+1)}$ $I_y = \frac{hb^3}{n+3}$	$r_x^2 = \frac{h^2(n+1)}{3(3n+1)}$ $r_y^2 = \frac{n+1}{n+3}b^2$	
 <p>n<sup>th</sup> DEGREE PARABOLA</p>	$A = \frac{n}{n+1}bh$ $x_c = \frac{n+1}{2n+1}b$ $y_c = \frac{n+1}{2(n+2)}h$	$I_x = \frac{n}{3(n+3)}bh^3$ $I_y = \frac{n}{3n+1}b^3h$	$r_x^2 = \frac{n+1}{3(n+1)}h^2$ $r_y^2 = \frac{n+1}{3n+1}b^2$	

Housner, George W., and Donald E. Hudson, *Applied Mechanics Dynamics*, D. Van Nostrand Company, Inc., Princeton, NJ, 1959. Table reprinted by permission of G.W. Housner & D.E. Hudson.

# DYNAMICS

## COMMON NOMENCLATURE

$t$  = time  
 $s$  = position  
 $v$  = velocity  
 $a$  = acceleration  
 $a_n$  = normal acceleration  
 $a_t$  = tangential acceleration  
 $\theta$  = angle  
 $\omega$  = angular velocity  
 $\alpha$  = angular acceleration  
 $\Omega$  = angular velocity of  $x, y, z$  reference axis  
 $\dot{\Omega}$  = angular acceleration of reference axis  
 $r_{A/B}$  = relative position of "A" with respect to "B"  
 $v_{A/B}$  = relative velocity of "A" with respect to "B"  
 $a_{A/B}$  = relative acceleration of "A" with respect to "B"

## PARTICLE KINEMATICS

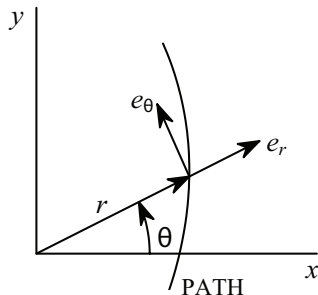
Kinematics is the study of motion without consideration of the mass of, or the forces acting on, the system. For particle motion, let  $\mathbf{r}(t)$  be the position vector of the particle in an inertial reference frame. The velocity and acceleration of the particle are defined, respectively, as

$$\begin{aligned}
 \mathbf{v} &= d\mathbf{r}/dt \\
 \mathbf{a} &= d\mathbf{v}/dt, \text{ where} \\
 \mathbf{v} &= \text{the instantaneous velocity} \\
 \mathbf{a} &= \text{the instantaneous acceleration} \\
 t &= \text{time}
 \end{aligned}$$

## Cartesian Coordinates

$$\begin{aligned}
 \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\
 \mathbf{v} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \\
 \mathbf{a} &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}, \text{ where} \\
 \dot{x} &= dx/dt = v_x, \text{ etc.} \\
 \ddot{x} &= d^2x/dt^2 = a_x, \text{ etc.}
 \end{aligned}$$

## Radial and Transverse Components for Planar Motion



Unit vectors  $\mathbf{e}_\theta$  and  $\mathbf{e}_r$  are, respectively, normal to and collinear with the position vector  $\mathbf{r}$ . Thus:

$$\begin{aligned}
 \mathbf{r} &= r\mathbf{e}_r \\
 \mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \\
 \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta, \text{ where} \\
 r &= \text{the radial distance} \\
 \theta &= \text{the angle between the } x \text{ axis and } \mathbf{e}_r \\
 \dot{r} &= dr/dt, \text{ etc.}, \ddot{r} = d^2r/dt^2, \text{ etc.}
 \end{aligned}$$

## ◆ Particle Rectilinear Motion

Variable $a$	Constant $a = a_c$
$a = \frac{dv}{dt}$	$v = v_0 + a_c t$
$v = \frac{ds}{dt}$	$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$
$a ds = v dv$	$v^2 = v_0^2 + 2a_c(s - s_0)$

## ◆ Particle Curvilinear Motion

$x, y, z$ Coordinates	$r, \theta, z$ Coordinates
$v_x = \dot{x}$ $a_x = \ddot{x}$	$v_r = \dot{r}$ $a_r = \ddot{r} - r\dot{\theta}^2$
$v_y = \dot{y}$ $a_y = \ddot{y}$	$v_\theta = r\dot{\theta}$ $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$v_z = \dot{z}$ $a_z = \ddot{z}$	$v_z = \dot{z}$ $a_z = \ddot{z}$

## $n, t, b$ Coordinates

$v = \dot{s}$	$a_t = \dot{v} = \frac{dv}{ds}$
$a_n = \frac{v^2}{\rho}$	$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{ d^2y/dx^2 }$

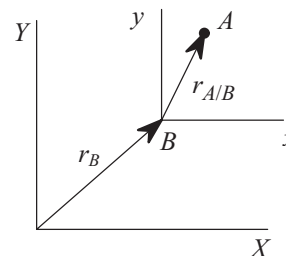
## ◆ Relative Motion

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \quad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

## Relative Motion

The equations for the relative position, velocity, and acceleration may be written as

## Translating Axis

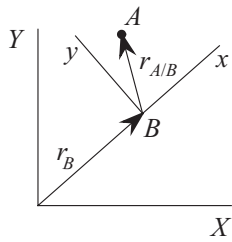


$$\begin{aligned}
 \mathbf{r}_A &= \mathbf{r}_B + \mathbf{r}_{A/B} \\
 \mathbf{v}_A &= \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B} = \mathbf{v}_B + \mathbf{v}_{A/B} \\
 \mathbf{a}_A &= \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}) = \mathbf{a}_B + \mathbf{a}_{A/B}
 \end{aligned}$$

where  $\boldsymbol{\omega}$  and  $\boldsymbol{\alpha}$  are, respectively, the angular velocity and angular acceleration of the relative position vector  $\mathbf{r}_{A/B}$ .

◆ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

### Rotating Axis



$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

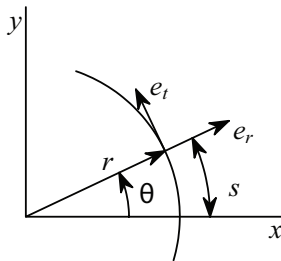
$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{A/B}$$

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\omega} \times \mathbf{v}_{A/B} + \mathbf{a}_{A/B}$$

where  $\boldsymbol{\omega}$  and  $\boldsymbol{\alpha}$  are, respectively, the total angular velocity and acceleration of the relative position vector  $\mathbf{r}_{A/B}$ .

### Plane Circular Motion

A special case of transverse and radial components is for constant radius rotation about the origin, or plane circular motion.



Here the vector quantities are defined as

$$\mathbf{r} = r\mathbf{e}_r$$

$$\mathbf{v} = r\omega\mathbf{e}_t$$

$$\mathbf{a} = (-r\omega^2)\mathbf{e}_r + r\alpha\mathbf{e}_t, \text{ where}$$

$r$  = the radius of the circle

$\theta$  = the angle between the  $x$  and  $\mathbf{e}_r$  axes

The magnitudes of the angular velocity and acceleration, respectively, are defined as

$$\omega = \dot{\theta}$$

$$\alpha = \dot{\omega} = \ddot{\theta}$$

Arc length, tangential velocity, and tangential acceleration, respectively, are

$$s = r\theta$$

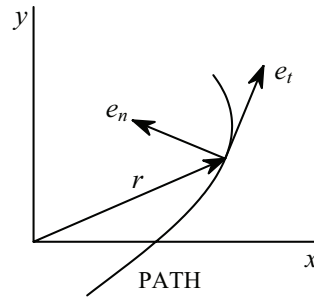
$$v_t = r\omega$$

$$a_t = r\alpha$$

The normal acceleration is given by

$$a_n = -r\omega^2 \text{ (towards the center of the circle)}$$

### Normal and Tangential Components



Unit vectors  $\mathbf{e}_t$  and  $\mathbf{e}_n$  are, respectively, tangent and normal to the path with  $\mathbf{e}_n$  pointing to the center of curvature. Thus

$$\mathbf{v} = v(t)\mathbf{e}_t$$

$$\mathbf{a} = a(t)\mathbf{e}_t + (v_t^2/\rho)\mathbf{e}_n, \text{ where}$$

$\rho$  = instantaneous radius of curvature

### Constant Acceleration

The equations for the velocity and displacement when acceleration is a constant are given as

$$a(t) = a_0$$

$$v(t) = a_0(t - t_0) + v_0$$

$$s(t) = a_0(t - t_0)^2/2 + v_0(t - t_0) + s_0, \text{ where}$$

$s$  = distance along the line of travel

$s_0$  = displacement at time  $t_0$

$v$  = velocity along the direction of travel

$v_0$  = velocity at time  $t_0$

$a_0$  = constant acceleration

$t$  = time

$t_0$  = some initial time

For a free-falling body,  $a_0 = -g$  (downward).

An additional equation for velocity as a function of position may be written as

$$v^2 = v_0^2 + 2a_0(s - s_0)$$

For constant angular acceleration, the equations for angular velocity and displacement are

$$\alpha(t) = \alpha_0$$

$$\omega(t) = \alpha_0(t - t_0) + \omega_0$$

$$\theta(t) = \alpha_0(t - t_0)^2/2 + \omega_0(t - t_0) + \theta_0, \text{ where}$$

$\theta$  = angular displacement

$\theta_0$  = angular displacement at time  $t_0$

$\omega$  = angular velocity

$\omega_0$  = angular velocity at time  $t_0$

$\alpha_0$  = constant angular acceleration

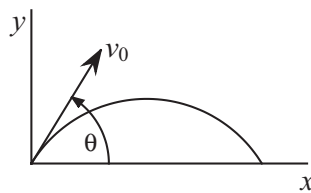
$t$  = time

$t_0$  = some initial time

An additional equation for angular velocity as a function of angular position may be written as

$$\omega^2 = \omega_0^2 + 2\alpha_0(\theta - \theta_0)$$

## Projectile Motion



The equations for common projectile motion may be obtained from the constant acceleration equations as

$$\begin{aligned} a_x &= 0 \\ v_x &= v_0 \cos(\theta) \\ x &= v_0 \cos(\theta)t + x_0 \\ a_y &= -g \\ v_y &= -gt + v_0 \sin(\theta) \\ y &= -gt^2/2 + v_0 \sin(\theta)t + y_0 \end{aligned}$$

## Non-constant Acceleration

When non-constant acceleration,  $a(t)$ , is considered, the equations for the velocity and displacement may be obtained from

$$\begin{aligned} v(t) &= \int_{t_0}^t a(\tau) d\tau + v_{t_0} \\ s(t) &= \int_{t_0}^t v(\tau) d\tau + s_{t_0} \end{aligned}$$

For variable angular acceleration

$$\begin{aligned} \omega(t) &= \int_{t_0}^t \alpha(\tau) d\tau + \omega_{t_0} \\ \theta(t) &= \int_{t_0}^t \omega(\tau) d\tau + \theta_{t_0} \end{aligned}$$

## CONCEPT OF WEIGHT

$$\begin{aligned} W &= mg, \text{ where} \\ W &= \text{weight, N (lbf)} \\ m &= \text{mass, kg (lbf-sec}^2/\text{ft)} \\ g &= \text{local acceleration of gravity, m/s}^2 \text{ (ft/sec}^2\text{)} \end{aligned}$$

## PARTICLE KINETICS

Newton's second law for a particle is

$$\Sigma \mathbf{F} = d(m\mathbf{v})/dt, \text{ where}$$

$$\begin{aligned} \Sigma \mathbf{F} &= \text{the sum of the applied forces acting on the particle} \\ m &= \text{the mass of the particle} \\ \mathbf{v} &= \text{the velocity of the particle} \end{aligned}$$

For constant mass,

$$\Sigma \mathbf{F} = m d\mathbf{v}/dt = m\mathbf{a}$$

## One-Dimensional Motion of a Particle (Constant Mass)

When motion exists only in a single dimension then, without loss of generality, it may be assumed to be in the  $x$  direction, and

$$a_x = F_x/m, \text{ where}$$

$F_x$  = the resultant of the applied forces, which in general can depend on  $t$ ,  $x$ , and  $v_x$ .

If  $F_x$  only depends on  $t$ , then

$$\begin{aligned} a_x(t) &= F_x(t)/m \\ v_x(t) &= \int_{t_0}^t a_x(\tau) d\tau + v_{xt0} \\ x(t) &= \int_{t_0}^t v_x(\tau) d\tau + x_{t0} \end{aligned}$$

If the force is constant (i.e. independent of time, displacement, and velocity) then

$$\begin{aligned} a_x &= F_x/m \\ v_x &= a_x(t - t_0) + v_{xt0} \\ x &= a_x(t - t_0)^2/2 + v_{xt0}(t - t_0) + x_{t0} \end{aligned}$$

## Normal and Tangential Kinetics for Planar Problems

When working with normal and tangential directions, the scalar equations may be written as

$$\begin{aligned} \Sigma F_t &= ma_t = m dv_t/dt \text{ and} \\ \Sigma F_n &= ma_n = m(v_t^2/\rho) \end{aligned}$$

## ◆ Principle of Work and Energy

$$T_1 + U_{1-2} = T_2$$

## ◆ Kinetic Energy

Particle	$T = \frac{1}{2}mv^2$
Rigid Body (Plane Motion)	$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

## ◆ Work

$$\begin{aligned} \text{Variable force} \quad U_F &= \int F \cos \theta ds \\ \text{Constant force} \quad U_F &= (F_c \cos \theta) \Delta s \\ \text{Weight} \quad U_W &= -W \Delta y \\ \text{Spring} \quad U_s &= -\left(\frac{1}{2}ks_2^2 - ks_1^2\right) \\ \text{Couple moment} \quad U_M &= M \Delta \theta \end{aligned}$$

## ◆ Power and Efficiency

$$P = \frac{dU}{dt} = \mathbf{F} \cdot \mathbf{v} \quad \epsilon = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{U_{\text{out}}}{U_{\text{in}}}$$

## ◆ Conservation of Energy Theorem

$$T_1 + V_1 = T_2 + V_2$$

◆ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

### ♦ Potential Energy

$V = V_g + V_e$ , where  $V_g = \pm W_y$ ,  $V_e = + 1/2 ks^2$

### Work and Energy

Work  $W$  is defined as

$$W = \int \mathbf{F} \cdot d\mathbf{r}$$

### Kinetic Energy

The kinetic energy of a particle is the work done by an external agent in accelerating the particle from rest to a velocity  $\mathbf{v}$ . Thus

$$T = mv^2/2$$

In changing the velocity from  $\mathbf{v}_1$  to  $\mathbf{v}_2$ , the change in kinetic energy is

$$T_2 - T_1 = m(v_2^2 - v_1^2)/2$$

### Potential Energy

The work done by an external agent in the presence of a conservative field is termed the change in potential energy.

### Potential Energy in Gravity Field

$$U = mgh, \text{ where}$$

$h$  = the elevation above some specified datum.

### Elastic Potential Energy

For a linear elastic spring with modulus, stiffness, or spring constant, the force in the spring is

$$F_s = kx, \text{ where}$$

$x$  = the change in length of the spring from the undeformed length of the spring.

The potential energy stored in the spring when compressed or extended by an amount  $x$  is

$$U = kx^2/2$$

In changing the deformation in the spring from position  $x_1$  to  $x_2$ , the change in the potential energy stored in the spring is

$$U_2 - U_1 = k(x_2^2 - x_1^2)/2$$

### Principle of Work and Energy

If  $T_i$  and  $U_i$  are, respectively, the kinetic and potential energy of a particle at state  $i$ , then for conservative systems (no energy dissipation or gain), the law of conservation of energy is

$$T_2 + U_2 = T_1 + U_1$$

If nonconservative forces are present, then the work done by these forces must be accounted for. Hence

$$T_2 + U_2 = T_1 + U_1 + W_{1 \rightarrow 2}, \text{ where}$$

$W_{1 \rightarrow 2}$  = the work done by the nonconservative forces in moving between state 1 and state 2. Care must be exercised during computations to correctly compute the algebraic sign of the work term. If the forces serve to increase the energy of the system,  $W_{1 \rightarrow 2}$  is positive. If the forces, such as friction, serve to dissipate energy,  $W_{1 \rightarrow 2}$  is negative.

### Impulse and Momentum

#### Linear

Assuming constant mass, the equation of motion of a particle may be written as

$$m d\mathbf{v}/dt = \mathbf{F}$$

$$m d\mathbf{v} = \mathbf{F} dt$$

For a system of particles, by integrating and summing over the number of particles, this may be expanded to

$$\Sigma m_i (\mathbf{v}_i)_{t_2} = \Sigma m_i (\mathbf{v}_i)_{t_1} + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i dt$$

The term on the left side of the equation is the linear momentum of a system of particles at time  $t_2$ . The first term on the right side of the equation is the linear momentum of a system of particles at time  $t_1$ . The second term on the right side of the equation is the impulse of the force  $\mathbf{F}$  from time  $t_1$  to  $t_2$ . It should be noted that the above equation is a vector equation. Component scalar equations may be obtained by considering the momentum and force in a set of orthogonal directions.

#### Angular Momentum or Moment of Momentum

The angular momentum or the moment of momentum about point  $O$  for a particle is defined as

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}, \text{ or}$$

$$\mathbf{H}_O = I_O \omega$$

Taking the time derivative of the above, the equation of motion may be written as

$$\dot{\mathbf{H}}_O = d(I_O \omega)/dt = \mathbf{M}, \text{ where}$$

$\mathbf{M}$  is the moment applied to the particle. Now by integrating and summing over a system of any number of particles, this may be expanded to

$$\Sigma (\mathbf{H}_{Oi})_{t_2} = \Sigma (\mathbf{H}_{Oi})_{t_1} + \Sigma \int_{t_1}^{t_2} \mathbf{M}_{Oi} dt$$

The term on the left side of the equation is the angular momentum of a system of particles at time  $t_2$ . The first term on the right side of the equation is the angular momentum of a system of particles at time  $t_1$ . The second term on the right side of the equation is the angular impulse of the moment  $\mathbf{M}$  from time  $t_1$  to  $t_2$ .

♦ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

## Impact

During an impact, momentum is conserved while energy may or may not be conserved. For direct central impact with no external forces

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2, \text{ where}$$

$m_1, m_2$  = the masses of the two bodies

$v_1, v_2$  = the velocities of the bodies just before impact

$v'_1, v'_2$  = the velocities of the bodies just after impact

For impacts, the relative velocity expression is

$$e = \frac{(v'_2)_n - (v'_1)_n}{(v_1)_n - (v_2)_n}, \text{ where}$$

$e$  = coefficient of restitution

$(v_i)_n$  = the velocity normal to the plane of impact just **before** impact

$(v'_i)_n$  = the velocity normal to the plane of impact just **after** impact

The value of  $e$  is such that

$0 \leq e \leq 1$ , with limiting values

$e = 1$ , perfectly elastic (energy conserved)

$e = 0$ , perfectly plastic (no rebound)

Knowing the value of  $e$ , the velocities after the impact are given as

$$(v'_1)_n = \frac{m_2 (v_2)_n (1 + e) + (m_1 - e m_2) (v_1)_n}{m_1 + m_2}$$

$$(v'_2)_n = \frac{m_1 (v_1)_n (1 + e) - (e m_1 - m_2) (v_2)_n}{m_1 + m_2}$$

## Friction

The Laws of Friction are

1. The total friction force  $F$  that can be developed is independent of the magnitude of the area of contact.
2. The total friction force  $F$  that can be developed is proportional to the normal force  $N$ .
3. For low velocities of sliding, the total frictional force that can be developed is practically independent of the velocity, although experiments show that the force  $F$  necessary to initiate slip is greater than that necessary to maintain the motion.

The formula expressing the Laws of Friction is

$$F \leq \mu N, \text{ where}$$

$\mu$  = the coefficient of friction.

In general

$$F < \mu_s N, \text{ no slip occurring}$$

$$F = \mu_s N, \text{ at the point of impending slip}$$

$$F = \mu_k N, \text{ when slip is occurring}$$

Here,

$\mu_s$  = the coefficient of static friction

$\mu_k$  = the coefficient of kinetic friction

The coefficient of kinetic friction is often approximated as 75% of the coefficient of static friction.

## PLANE MOTION OF A RIGID BODY

### Kinematics of a Rigid Body

#### Rigid Body Rotation

For rigid body rotation  $\theta$

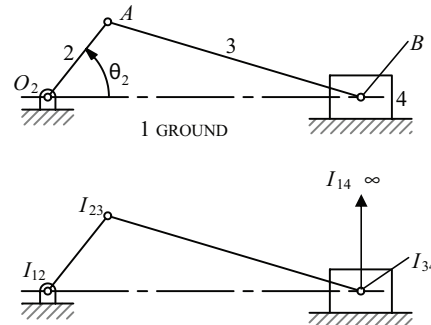
$$\omega = d\theta/dt$$

$$\alpha = d\omega/dt$$

$$\alpha d\theta = \omega d\omega$$

#### Instantaneous Center of Rotation (Instant Centers)

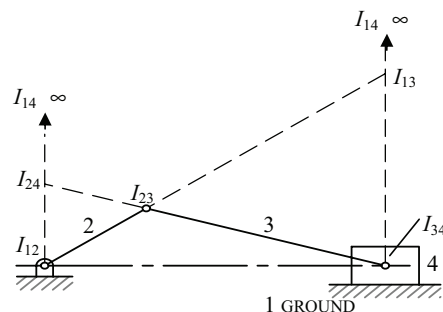
An instantaneous center of rotation (instant center) is a point, common to two bodies, at which each has the same velocity (magnitude and direction) at a given instant. It is also a point in space about which a body rotates, instantaneously.



The figure shows a fourbar slider-crank. Link 2 (the crank) rotates about the fixed center,  $O_2$ . Link 3 couples the crank to the slider (link 4), which slides against ground (link 1). Using the definition of an instant center (IC), we see that the pins at  $O_2, A$ , and  $B$  are ICs that are designated  $I_{12}, I_{23}$ , and  $I_{34}$ . The easily observable IC is  $I_{14}$ , which is located at infinity with its direction perpendicular to the interface between links 1 and 4 (the direction of sliding). To locate the remaining two ICs (for a fourbar) we must make use of Kennedy's rule.

*Kennedy's Rule: When three bodies move relative to one another they have three instantaneous centers, all of which lie on the same straight line.*

To apply this rule to the slider-crank mechanism, consider links 1, 2, and 3 whose ICs are  $I_{12}, I_{23}$ , and  $I_{13}$ , all of which lie on a straight line. Consider also links 1, 3, and 4 whose ICs are  $I_{13}, I_{34}$ , and  $I_{14}$ , all of which lie on a straight line. Extending the line through  $I_{12}$  and  $I_{23}$  and the line through  $I_{34}$  and  $I_{14}$  to their intersection locates  $I_{13}$ , which is common to the two groups of links that were considered.



Similarly, if body groups 1, 2, 4 and 2, 3, 4 are considered, a line drawn through known ICs  $I_{12}$  and  $I_{14}$  to the intersection of a line drawn through known ICs  $I_{23}$  and  $I_{34}$  locates  $I_{24}$ .

The number of ICs,  $c$ , for a given mechanism is related to the number of links,  $n$ , by

$$c = \frac{n(n-1)}{2}$$

### Kinetics of a Rigid Body

In general, Newton's second law for a rigid body, with constant mass and mass moment of inertia, in plane motion may be written in vector form as

$$\Sigma \mathbf{F} = m \mathbf{a}_c$$

$$\Sigma \mathbf{M}_c = I_c \alpha$$

$$\Sigma \mathbf{M}_p = I_c \alpha + \mathbf{r}_{pc} \times m \mathbf{a}_c, \text{ where}$$

$\mathbf{F}$  are forces and  $\mathbf{a}_c$  is the acceleration of the body's mass center both in the plane of motion,  $\mathbf{M}_c$  are moments and  $\alpha$  is the angular acceleration both about an axis normal to the plane of motion,  $I_c$  is the mass moment of inertia about the normal axis through the mass center, and  $\mathbf{r}_{pc}$  is a vector from point  $p$  to point  $c$ .

♦ **Mass Moment of Inertia**  $I = \int r^2 dm$

*Parallel-Axis Theorem*  $I = I_G + md^2$

*Radius of Gyration*  $k = \sqrt{\frac{I}{m}}$

### ♦ Equations of Motion

<p><i>Rigid Body</i></p> <p><i>(Plane Motion)</i></p>	$\Sigma F_x = m(a_G)_x$ $\Sigma F_y = m(a_G)_y$ $\Sigma M_G = I_G \alpha \text{ or } \Sigma M_p = \Sigma (M_k)_p$
---	---

### Mass Moment of Inertia

The definitions for the mass moments of inertia are

$$I_x = \int (y^2 + z^2) dm$$

$$I_y = \int (x^2 + z^2) dm$$

$$I_z = \int (x^2 + y^2) dm$$

A table listing moment of inertia formulas for some standard shapes is at the end of this section.

### Parallel-Axis Theorem

The mass moments of inertia may be calculated about any axis through the application of the above definitions. However, once the moments of inertia have been determined about an axis passing through a body's mass center, it may be transformed to another parallel axis. The transformation equation is

$$I_{new} = I_c + md^2, \text{ where}$$

$I_{new}$  = the mass moment of inertia about any specified axis

$I_c$  = the mass moment of inertia about an axis that is parallel to the above specified axis but passes through the body's mass center

$m$  = the mass of the body

$d$  = the normal distance from the body's mass center to the above-specified axis

### Mass Radius of Gyration

The mass radius of gyration is defined as

$$r_m = \sqrt{I/m}$$

Without loss of generality, the body may be assumed to be in the  $x$ - $y$  plane. The scalar equations of motion may then be written as

$$\Sigma F_x = ma_{xc}$$

$$\Sigma F_y = ma_{yc}$$

$$\Sigma M_{zc} = I_{zc} \alpha, \text{ where}$$

$zc$  indicates the  $z$  axis passing through the body's mass center,  $a_{xc}$  and  $a_{yc}$  are the acceleration of the body's mass center in the  $x$  and  $y$  directions, respectively, and  $\alpha$  is the angular acceleration of the body about the  $z$  axis.

### Rotation about an Arbitrary Fixed Axis

#### ♦ Rigid Body Motion About a Fixed Axis

<i>Variable</i> $\alpha$	<i>Constant</i> $\alpha = \alpha_c$
$\alpha = \frac{d\omega}{dt}$	$\omega = \omega_0 + \alpha_c t$
$\omega = \frac{d\theta}{dt}$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha_c t^2$
$\omega d\omega = \alpha d\theta$	$\omega^2 = \omega_0^2 + 2\alpha_c (\theta - \theta_0)$

For rotation about some arbitrary fixed axis  $q$

$$\Sigma M_q = I_q \alpha$$

If the applied moment acting about the fixed axis is constant then integrating with respect to time, from  $t = 0$  yields

$$\begin{aligned} \alpha &= M_q / I_q \\ \omega &= \omega_0 + \alpha t \\ \theta &= \theta_0 + \omega_0 t + \alpha t^2 / 2 \end{aligned}$$

where  $\omega_0$  and  $\theta_0$  are the values of angular velocity and angular displacement at time  $t = 0$ , respectively.

♦ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.



The change in kinetic energy is the work done in accelerating the rigid body from  $\omega_0$  to  $\omega$

$$I_q \omega^2/2 = I_q \omega_0^2/2 + \int_{\theta_0}^{\theta} M_q d\theta$$

### Principles of Work Energy

In general the kinetic energy for a rigid body may be written as

$$T = mv^2/2 + I_c \omega^2/2$$

For motion in the  $xy$  plane this reduces to

$$T = m(v_{cx}^2 + v_{cy}^2)/2 + I_c \omega^2/2$$

For motion about an instant center,

$$T = I_{IC} \omega^2/2$$

### ◆ Principle of Angular Impulse and Momentum

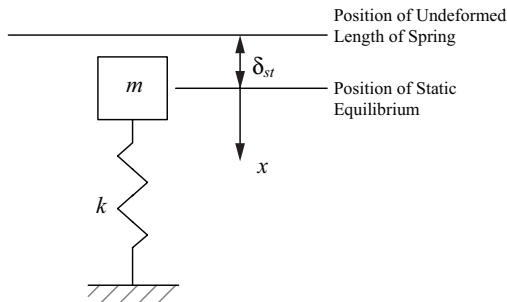
<i>Rigid Body</i>	$(\mathbf{H}_G)_1 + \sum \int \mathbf{M}_G dt = (\mathbf{H}_G)_2$
<i>(Plane Motion)</i>	<p>where <math>\mathbf{H}_G = I_G \omega</math></p> <p><math>(\mathbf{H}_O)_1 + \sum \int \mathbf{M}_O dt = (\mathbf{H}_O)_2</math></p> <p>where <math>\mathbf{H}_O = I_O \omega</math></p>

### ◆ Conservation of Angular Momentum

$$\Sigma(\text{syst. } \mathbf{H})_1 = \Sigma(\text{syst. } \mathbf{H})_2$$

### Free Vibration

The figure illustrates a single degree-of-freedom system.



The equation of motion may be expressed as

$$m\ddot{x} = mg - k(x + \delta_{st})$$

where  $m$  is mass of the system,  $k$  is the spring constant of the system,  $\delta_{st}$  is the static deflection of the system, and  $x$  is the displacement of the system from static equilibrium.

From statics it may be shown that

$$mg = k\delta_{st}$$

thus the equation of motion may be written as

$$m\ddot{x} + kx = 0, \text{ or}$$

$$\ddot{x} + (k/m)x = 0$$

The solution of this differential equation is

$$x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

where  $\omega_n = \sqrt{k/m}$  is the undamped natural circular frequency and  $C_1$  and  $C_2$  are constants of integration whose values are determined from the initial conditions.

If the initial conditions are denoted as  $x(0) = x_0$  and

$\dot{x}(0) = v_0$ , then

$$x(t) = x_0 \cos(\omega_n t) + (v_0/\omega_n) \sin(\omega_n t)$$

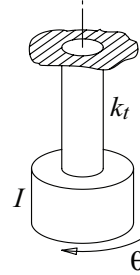
It may also be shown that the undamped natural frequency may be expressed in terms of the static deflection of the system as

$$\omega_n = \sqrt{g/\delta_{st}}$$

The undamped natural period of vibration may now be written as

$$\tau_n = 2\pi/\omega_n = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\sqrt{\frac{g}{\delta_{st}}}}$$

### Torsional Vibration



For torsional free vibrations it may be shown that the differential equation of motion is

$$\ddot{\theta} + (k_t/I)\theta = 0, \text{ where}$$

$\theta$  = the angular displacement of the system

$k_t$  = the torsional stiffness of the massless rod

$I$  = the mass moment of inertia of the end mass

The solution may now be written in terms of the initial conditions  $\theta(0) = \theta_0$  and  $\dot{\theta}(0) = \dot{\theta}_0$  as

$$\theta(t) = \theta_0 \cos(\omega_n t) + (\dot{\theta}_0/\omega_n) \sin(\omega_n t)$$

where the undamped natural circular frequency is given by

$$\omega_n = \sqrt{k_t/I}$$

The torsional stiffness of a solid round rod with associated polar moment-of-inertia  $J$ , length  $L$ , and shear modulus of elasticity  $G$  is given by

$$k_t = GJ/L$$

Thus the undamped circular natural frequency for a system with a solid round supporting rod may be written as

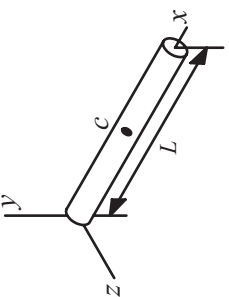
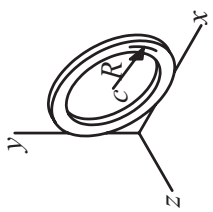
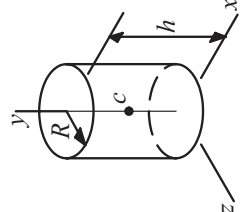
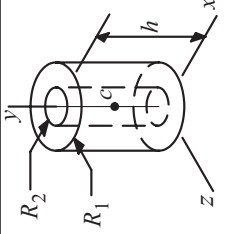
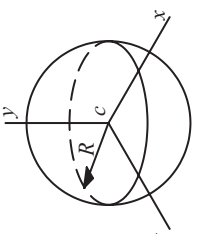
$$\omega_n = \sqrt{GJ/IL}$$

Similar to the linear vibration problem, the undamped natural period may be written as

$$\tau_n = 2\pi/\omega_n = \frac{2\pi}{\sqrt{\frac{k_t}{I}}} = \frac{2\pi}{\sqrt{\frac{GJ}{IL}}}$$

◆ Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.



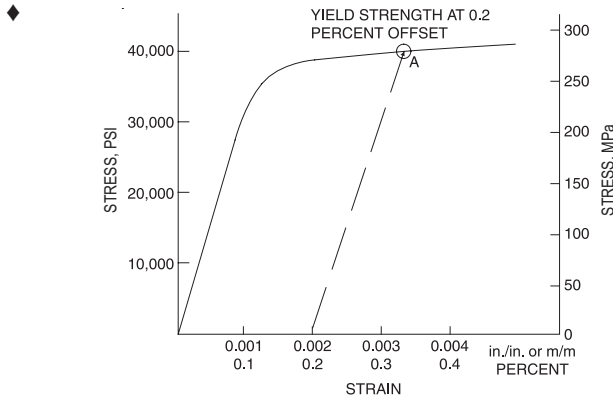
Figure	Mass & Centroid	Mass Moment of Inertia	(Radius of Gyration) <sup>2</sup>	Product of Inertia
	$M = \rho LA$ $x_c = L/2$ $y_c = 0$ $z_c = 0$ $A$ = cross-sectional area of rod $\rho$ = mass/vol.	$I_x = I_{x_c} = 0$ $I_{y_c} = I_{z_c} = ML^2/12$ $I_y = I_z = ML^2/3$	$r_x^2 = r_{x_c}^2 = 0$ $r_{y_c}^2 = r_{z_c}^2 = L^2/12$ $r_y^2 = r_z^2 = L^2/3$	$I_{x_c y_c}, \text{ etc.} = 0$ $I_{xy}, \text{ etc.} = 0$
	$M = 2\pi R \rho A$ $x_c = R$ = mean radius $y_c = R$ = mean radius $z_c = 0$ $A$ = cross-sectional area of ring $\rho$ = mass/vol.	$I_{x_c} = I_{y_c} = MR^2/2$ $I_{z_c} = MR^2$ $I_x = I_y = 3MR^2/2$ $I_z = 3MR^2$	$r_{x_c}^2 = r_{y_c}^2 = R^2/2$ $r_{z_c}^2 = R^2$ $r_x^2 = r_y^2 = 3R^2/2$ $r_z^2 = 3R^2$	$I_{x_c y_c}, \text{ etc.} = 0$ $I_{z_c z_c} = MR^2$ $I_{xz} = I_{yz} = 0$
	$M = \pi R^2 \rho h$ $x_c = 0$ $y_c = h/2$ $z_c = 0$ $\rho$ = mass/vol.	$I_{x_c} = I_{z_c} = M(3R^2 + h^2)/12$ $I_{y_c} = I_y = MR^2/2$ $I_x = I_z = M(3R^2 + 4h^2)/12$	$r_{x_c}^2 = r_{z_c}^2 = (3R^2 + h^2)/12$ $r_{y_c}^2 = r_y^2 = R^2/2$ $r_x^2 = r_z^2 = (3R^2 + 4h^2)/12$	$I_{x_c y_c}, \text{ etc.} = 0$ $I_{xy}, \text{ etc.} = 0$
	$M = \pi(R_1^2 - R_2^2)\rho h$ $x_c = 0$ $y_c = h/2$ $z_c = 0$ $\rho$ = mass/vol.	$I_{x_c} = I_{z_c} = M(3R_1^2 + 3R_2^2 + h^2)/12$ $I_{y_c} = I_y = M(R_1^2 + R_2^2)/2$ $I_x = I_z = M(3R_1^2 + 3R_2^2 + 4h^2)/12$	$r_{x_c}^2 = r_{z_c}^2 = (3R_1^2 + 3R_2^2 + h^2)/12$ $r_{y_c}^2 = r_y^2 = (R_1^2 + R_2^2)/2$ $r_x^2 = r_z^2 = (3R_1^2 + 3R_2^2 + 4h^2)/12$	$I_{x_c y_c}, \text{ etc.} = 0$ $I_{xy}, \text{ etc.} = 0$
	$M = \frac{4}{3}\pi R^3 \rho$ $x_c = 0$ $y_c = 0$ $z_c = 0$ $\rho$ = mass/vol.	$I_{x_c} = I_x = 2MR^2/5$ $I_{y_c} = I_y = 2MR^2/5$ $I_{z_c} = I_z = 2MR^2/5$	$r_{x_c}^2 = r_x^2 = 2R^2/5$ $r_{y_c}^2 = r_y^2 = 2R^2/5$ $r_{z_c}^2 = r_z^2 = 2R^2/5$	$I_{x_c y_c}, \text{ etc.} = 0$

Housner, George W., and Donald E. Hudson. *Applied Mechanics Dynamics*. D. Van Nostrand Company, Inc., Princeton, NJ, 1959. Table reprinted by permission of G.W. Housner & D.E. Hudson.

# MECHANICS OF MATERIALS

## UNIAXIAL STRESS-STRAIN

### Stress-Strain Curve for Mild Steel



The slope of the linear portion of the curve equals the modulus of elasticity.

## DEFINITIONS

### Engineering Strain

$$\epsilon = \Delta L / L_o, \text{ where}$$

$\epsilon$  = engineering strain (units per unit)

$\Delta L$  = change in length (units) of member

$L_o$  = original length (units) of member

### Percent Elongation

$$\% \text{ Elongation} = \left( \frac{\Delta L}{L_o} \right) \times 100$$

### Percent Reduction in Area (RA)

The % reduction in area from initial area,  $A_i$ , to final area,  $A_f$ , is:

$$\% RA = \left( \frac{A_i - A_f}{A_i} \right) \times 100$$

### Shear Stress-Strain

$$\gamma = \tau / G, \text{ where}$$

$\gamma$  = shear strain

$\tau$  = shear stress

$G$  = shear modulus (constant in linear torsion-rotation relationship)

$$G = \frac{E}{2(1 + \nu)}, \text{ where}$$

$E$  = modulus of elasticity (Young's modulus)

$\nu$  = Poisson's ratio

= - (lateral strain)/(longitudinal strain)

## Uniaxial Loading and Deformation

$$\sigma = P/A, \text{ where}$$

$\sigma$  = stress on the cross section

$P$  = loading

$A$  = cross-sectional area

$$\epsilon = \delta / L, \text{ where}$$

$\delta$  = elastic longitudinal deformation

$L$  = length of member

$$E = \sigma / \epsilon = \frac{P/A}{\delta/L}$$

$$\delta = \frac{PL}{AE}$$

True stress is load divided by actual cross-sectional area whereas engineering stress is load divided by the initial area.

## THERMAL DEFORMATIONS

$$\delta_t = \alpha L (T - T_o), \text{ where}$$

$\delta_t$  = deformation caused by a change in temperature

$\alpha$  = temperature coefficient of expansion

$L$  = length of member

$T$  = final temperature

$T_o$  = initial temperature

## CYLINDRICAL PRESSURE VESSEL

### Cylindrical Pressure Vessel

For internal pressure only, the stresses at the inside wall are:

$$\sigma_t = P_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad \sigma_r = -P_i$$

For external pressure only, the stresses at the outside wall are:

$$\sigma_t = -P_o \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad \sigma_r = -P_o, \text{ where}$$

$\sigma_t$  = tangential (hoop) stress

$\sigma_r$  = radial stress

$P_i$  = internal pressure

$P_o$  = external pressure

$r_i$  = inside radius

$r_o$  = outside radius

For vessels with end caps, the axial stress is:

$$\sigma_a = P_i \frac{r_i^2}{r_o^2 - r_i^2}$$

$\sigma_t$ ,  $\sigma_r$ , and  $\sigma_a$  are principal stresses.

♦ Flinn, Richard A., and Paul K. Trojan, *Engineering Materials & Their Applications*, 4th ed., Houghton Mifflin Co., Boston, 1990.

When the thickness of the cylinder wall is about one-tenth or less of inside radius, the cylinder can be considered as thin-walled. In which case, the internal pressure is resisted by the hoop stress and the axial stress.

$$\sigma_t = \frac{P_i r}{t} \quad \text{and} \quad \sigma_a = \frac{P_i r}{2t}$$

where  $t$  = wall thickness and  $r = \frac{r_i + r_o}{2}$ .

## STRESS AND STRAIN

### Principal Stresses

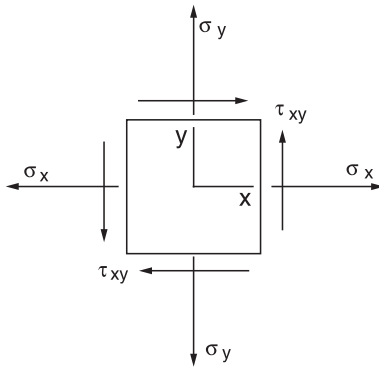
For the special case of a *two-dimensional* stress state, the equations for principal stress reduce to

$$\sigma_a, \sigma_b = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_c = 0$$

The two nonzero values calculated from this equation are temporarily labeled  $\sigma_a$  and  $\sigma_b$  and the third value  $\sigma_c$  is always zero in this case. Depending on their values, the three roots are then labeled according to the convention:

*algebraically largest* =  $\sigma_1$ , *algebraically smallest* =  $\sigma_3$ , *other* =  $\sigma_2$ . A typical 2D stress element is shown below with all indicated components shown in their positive sense.



### Mohr's Circle – Stress, 2D

To construct a Mohr's circle, the following sign conventions are used.

1. Tensile normal stress components are plotted on the horizontal axis and are considered positive. Compressive normal stress components are negative.
2. For constructing Mohr's circle only, shearing stresses are plotted above the normal stress axis when the pair of shearing stresses, acting on opposite and parallel faces of an element, forms a clockwise couple. Shearing stresses are plotted below the normal axis when the shear stresses form a counterclockwise couple.

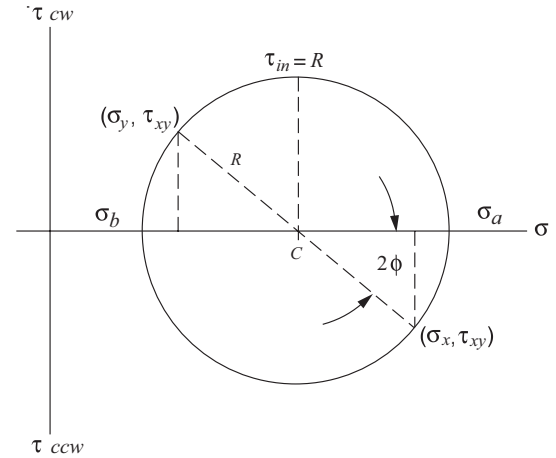
The circle drawn with the center on the normal stress (horizontal) axis with center,  $C$ , and radius,  $R$ , where

$$C = \frac{\sigma_x + \sigma_y}{2}, \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The two nonzero principal stresses are then:

$$\sigma_a = C + R$$

$$\sigma_b = C - R$$



The maximum *inplane* shear stress is  $\tau_{in} = R$ . However, the maximum shear stress considering three dimensions is always

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}.$$

### Hooke's Law

Three-dimensional case:

$$\begin{aligned} \epsilon_x &= (1/E)[\sigma_x - \nu(\sigma_y + \sigma_z)] & \gamma_{xy} &= \tau_{xy}/G \\ \epsilon_y &= (1/E)[\sigma_y - \nu(\sigma_z + \sigma_x)] & \gamma_{yz} &= \tau_{yz}/G \\ \epsilon_z &= (1/E)[\sigma_z - \nu(\sigma_x + \sigma_y)] & \gamma_{zx} &= \tau_{zx}/G \end{aligned}$$

Plane stress case ( $\sigma_z = 0$ ):

$$\begin{aligned} \epsilon_x &= (1/E)(\sigma_x - \nu\sigma_y) \\ \epsilon_y &= (1/E)(\sigma_y - \nu\sigma_x) \\ \epsilon_z &= -(1/E)(\nu\sigma_x + \nu\sigma_y) \end{aligned} \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Uniaxial case ( $\sigma_y = \sigma_z = 0$ ):  $\sigma_x = E\epsilon_x$  or  $\sigma = E\epsilon$ , where

$\epsilon_x, \epsilon_y, \epsilon_z$  = normal strain

$\sigma_x, \sigma_y, \sigma_z$  = normal stress

$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$  = shear strain

$\tau_{xy}, \tau_{yz}, \tau_{zx}$  = shear stress

$E$  = modulus of elasticity

$G$  = shear modulus

$\nu$  = Poisson's ratio

♦ Crandall, S.H., and N.C. Dahl, *An Introduction to Mechanics of Solids*, McGraw-Hill, New York, 1959.

## TORSION

Torsion stress in circular solid or thick-walled ( $t > 0.1 r$ ) shafts:

$$\tau = \frac{Tr}{J}$$

where  $J$  = polar moment of inertia

## TORSIONAL STRAIN

$$\gamma_{\phi z} = \lim_{\Delta z \rightarrow 0} r(\Delta\phi/\Delta z) = r(d\phi/dz)$$

The shear strain varies in direct proportion to the radius, from zero strain at the center to the greatest strain at the outside of the shaft.  $d\phi/dz$  is the twist per unit length or the rate of twist.

$$\tau_{\phi z} = G\gamma_{\phi z} = Gr(d\phi/dz)$$

$$T = G(d\phi/dz) \int_A r^2 dA = GJ(d\phi/dz)$$

$$\phi = \int_0^L \frac{T}{GJ} dz = \frac{TL}{GJ}, \text{ where}$$

$\phi$  = total angle (radians) of twist

$T$  = torque

$L$  = length of shaft

$T/\phi$  gives the *twisting moment per radian of twist*. This is called the *torsional stiffness* and is often denoted by the symbol  $k$  or  $c$ .

## For Hollow, Thin-Walled Shafts

$$\tau = \frac{T}{2A_m t}, \text{ where}$$

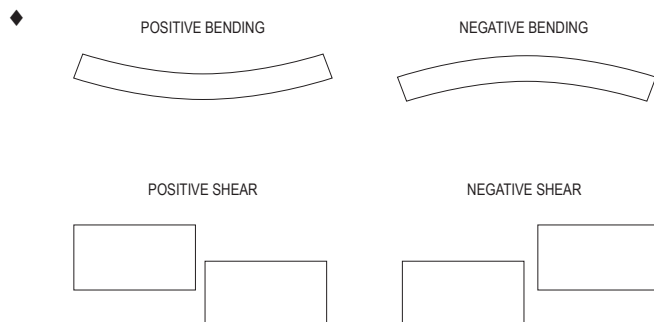
$t$  = thickness of shaft wall

$A_m$  = the total mean area enclosed by the shaft measured to the midpoint of the wall.

## BEAMS

### Shearing Force and Bending Moment Sign Conventions

1. The bending moment is *positive* if it produces bending of the beam *concave upward* (compression in top fibers and tension in bottom fibers).
2. The shearing force is *positive* if the *right portion of the beam tends to shear downward with respect to the left*.



The relationship between the load ( $w$ ), shear ( $V$ ), and moment ( $M$ ) equations are:

$$w(x) = -\frac{dV(x)}{dx}$$

$$V = \frac{dM(x)}{dx}$$

$$V_2 - V_1 = \int_{x_1}^{x_2} [-w(x)] dx$$

$$M_2 - M_1 = \int_{x_1}^{x_2} V(x) dx$$

## Stresses in Beams

The normal stress in a beam due to bending:

$$\sigma_x = -My/I, \text{ where}$$

$M$  = the moment at the section

$I$  = the *moment of inertia* of the cross section

$y$  = the distance from the neutral axis to the fiber location above or below the neutral axis

The maximum normal stresses in a beam due to bending:

$$\sigma_x = \pm Mc/I, \text{ where}$$

$c$  = distance from the neutral axis to the outermost fiber of a symmetrical beam section.

$$\sigma_x = -M/s, \text{ where}$$

$s$  =  $I/c$ : the elastic section modulus of the beam.

Transverse shear stress:

$$\tau_{xy} = VQ/(Ib), \text{ where}$$

$V$  = shear force

$Q = A'\bar{y}'$ , where

$A'$  = area above the layer (or plane) upon which the desired transverse shear stress acts

$\bar{y}'$  = distance from neutral axis to area centroid

$b$  = width or thickness of the cross-section

Transverse shear flow:

$$q = VQ/I$$

♦ Timoshenko, S., and Gleason H. MacCullough, *Elements of Strengths of Materials*, K. Van Nostrand Co./Wadsworth Publishing Co., 1949.

## Deflection of Beams

Using  $1/\rho = M/(EI)$ ,

$$EI \frac{d^2 y}{dx^2} = M, \text{ differential equation of deflection curve}$$

$$EI \frac{d^3 y}{dx^3} = dM(x)/dx = V$$

$$EI \frac{d^4 y}{dx^4} = dV(x)/dx = -w$$

Determine the deflection curve equation by double integration (apply boundary conditions applicable to the deflection and/or slope).

$$EI (dy/dx) = \int M(x) dx$$

$$EI y = \int \int M(x) dx dx$$

The constants of integration can be determined from the physical geometry of the beam.

## Composite Sections

The bending stresses in a beam composed of dissimilar materials (material 1 and material 2) where  $E_1 > E_2$  are:

$$\sigma_1 = -nMy/I_T$$

$$\sigma_2 = -My/I_T, \text{ where}$$

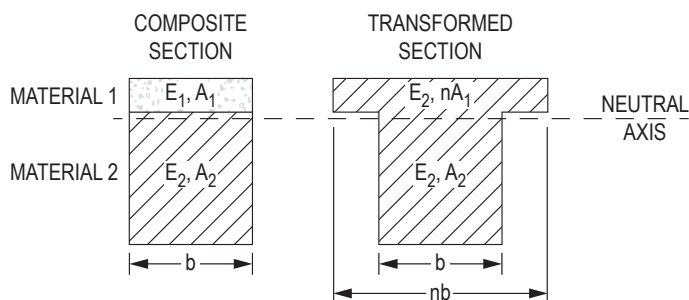
$I_T$  = the *moment of inertia* of the *transformed* section

$n$  = the modular ratio  $E_1/E_2$

$E_1$  = elastic modulus of material 1

$E_2$  = elastic modulus of material 2

The composite section is transformed into a section composed of a single material. The centroid and then the moment of inertia are found on the *transformed* section for use in the bending stress equations.



## COLUMNS

Critical axial load for long column subject to buckling:

Euler's Formula

$$P_{cr} = \frac{\pi^2 EI}{(K\ell)^2}, \text{ where}$$

$\ell$  = unbraced column length

$K$  = effective-length factor to account for end supports

Theoretical effective-length factors for columns include:

Pinned-pinned,  $K = 1.0$

Fixed-fixed,  $K = 0.5$

Fixed-pinned,  $K = 0.7$

Fixed-free,  $K = 2.0$

Critical buckling stress for long columns:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(K\ell/r)^2}, \text{ where}$$

$r$  = *radius of gyration*  $\sqrt{I/A}$

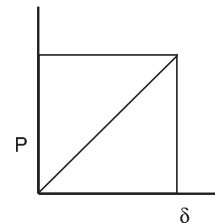
$K\ell/r$  = effective *slenderness ratio* for the column

## ELASTIC STRAIN ENERGY

If the strain remains within the elastic limit, the work done during deflection (extension) of a member will be transformed into potential energy and can be recovered.

If the final load is  $P$  and the corresponding elongation of a tension member is  $\delta$ , then the total energy  $U$  stored is equal to the work  $W$  done during loading.

$$U = W = P\delta/2$$



The strain energy per unit volume is

$$u = U/AL = \sigma^2/2E \quad (\text{for tension})$$

## MATERIAL PROPERTIES

**Table 1 - Typical Material Properties**  
(Use these values if the specific alloy and temper are not listed on Table 2 below)

Material	Modulus of Elasticity, E [Mpsi (GPa)]	Modulus of Rigidity, G [Mpsi (GPa)]	Poisson's Ratio, $\nu$	Coefficient of Thermal Expansion, $\alpha$ [ $10^{-6}/^{\circ}\text{F}$ ( $10^{-6}/^{\circ}\text{C}$ )]	Density, $\rho$ [lb/in <sup>3</sup> (Mg/m <sup>3</sup> )]
Steel	29.0 (200.0)	11.5 (80.0)	0.30	6.5 (11.7)	0.282 (7.8)
Aluminum	10.0 (69.0)	3.8 (26.0)	0.33	13.1 (23.6)	0.098 (2.7)
Cast Iron	14.5 (100.0)	6.0 (41.4)	0.21	6.7 (12.1)	0.246–0.282 (6.8–7.8)
Wood (Fir)	1.6 (11.0)	0.6 (4.1)	0.33	1.7 (3.0)	—
Brass	14.8–18.1 (102–125)	5.8 (40)	0.33	10.4 (18.7)	0.303–0.313 (8.4–8.7)
Copper	17 (117)	6.5 (45)	0.36	9.3 (16.6)	0.322 (8.9)
Bronze	13.9–17.4 (96–120)	6.5 (45)	0.34	10.0 (18.0)	0.278–0.314 (7.7–8.7)
Magnesium	6.5 (45)	2.4 (16.5)	0.35	14 (25)	0.061 (1.7)
Glass	10.2 (70)	—	0.22	5.0 (9.0)	0.090 (2.5)
Polystyrene	0.3 (2)	—	0.34	38.9 (70.0)	0.038 (1.05)
Polyvinyl Chloride (PVC)	<0.6 (<4)	—	—	28.0 (50.4)	0.047 (1.3)
Alumina Fiber	58 (400)	—	—	—	0.141 (3.9)
Aramide Fiber	18.1 (125)	—	—	—	0.047 (1.3)
Boron Fiber	58 (400)	—	—	—	0.083 (2.3)
Beryllium Fiber	43.5 (300)	—	—	—	0.069 (1.9)
BeO Fiber	58 (400)	—	—	—	0.108 (3.0)
Carbon Fiber	101.5 (700)	—	—	—	0.083 (2.3)
Silicon Carbide Fiber	58 (400)	—	—	—	0.116 (3.2)

**Table 2 - Average Mechanical Properties of Typical Engineering Materials**  
(U.S. Customary Units)  
(Use these values for the specific alloys and temperature listed. For all other materials refer to Table 1 above.)

Materials	Specific Weight $\gamma$ (lb/in <sup>3</sup> )	Modulus of Elasticity E (10 <sup>3</sup> ksi)	Modulus of Rigidity G (10 <sup>3</sup> ksi)	Yield Strength (ksi)			Ultimate Strength (ksi)			% Elongation in 2 in. specimen	Poisson's Ratio $\nu$	Coef. of Therm. Expansion $\alpha$ (10 <sup>-6</sup> )/°F
				Tens.	Comp.	Shear	Tens.	Comp.	Shear			
Metallic												
Aluminum Wrought Alloys	0.101	10.6	3.9	60	60	25	68	68	42	10	0.35	12.8
2014-T6 6061-T6	0.098	10.0	3.7	37	37	19	42	42	27	12	0.35	13.1
Cast Iron Alloys	0.260	10.0	3.9	—	—	—	26	97	—	0.6	0.28	6.70
Gray ASTM 20 Malleable ASTM A-197	0.263	25.0	9.8	—	—	—	40	83	—	5	0.28	6.60
Copper Alloys	0.316	14.6	5.4	11.4	11.4	—	35	35	—	35	0.35	9.80
Red Brass C83400 Bronze C86100	0.319	15.0	5.6	50	50	—	95	95	—	20	0.34	9.60
Magnesium Alloy [ Am 1004-T611]	0.066	6.48	2.5	22	22	—	40	40	22	1	0.30	14.3
Steel Alloys	0.284	29.0	11.0	36	36	—	58	58	—	30	0.32	6.60
Structural A36 Stainless 304	0.284	28.0	11.0	30	30	—	75	75	—	40	0.27	9.60
Tool L2	0.295	29.0	11.0	102	102	—	116	116	—	22	0.32	6.50
Titanium Alloy [ Ti-6Al-4V]	0.160	17.4	6.4	134	134	—	145	145	—	16	0.36	5.20
Nonmetallic												
Concrete	0.086	3.20	—	—	—	1.8	—	—	—	—	0.15	6.0
Low Strength High Strength	0.086	4.20	—	—	—	5.5	—	—	—	—	0.15	6.0
Plastic Reinforced	0.0524	19.0	—	—	—	—	104	70	10.2	2.8	0.34	—
Kevlar 49 30% Glass	0.0524	10.5	—	—	—	—	13	19	—	—	0.34	—
Wood Select Structural Grade	0.017	1.90	—	—	—	—	0.30 <sup>c</sup>	3.78 <sup>d</sup>	0.90 <sup>d</sup>	—	0.29 <sup>c</sup>	—
Douglas Fir White Spruce	0.130	1.40	—	—	—	—	0.36 <sup>c</sup>	5.18 <sup>d</sup>	0.97 <sup>d</sup>	—	0.31 <sup>c</sup>	—

<sup>a</sup> SPECIFIC VALUES MAY VARY FOR A PARTICULAR MATERIAL DUE TO ALLOY OR MINERAL COMPOSITION, MECHANICAL WORKING OF THE SPECIMEN, OR HEAT TREATMENT. FOR A MORE EXACT VALUE REFERENCE BOOKS FOR THE MATERIAL SHOULD BE CONSULTED.

<sup>b</sup> THE YIELD AND ULTIMATE STRENGTHS FOR DUCTILE MATERIALS CAN BE ASSUMED EQUAL FOR BOTH TENSION AND COMPRESSION.

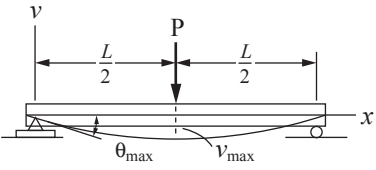
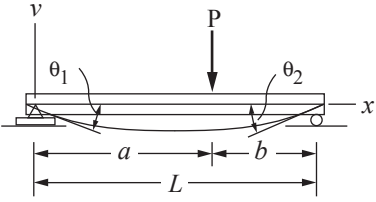
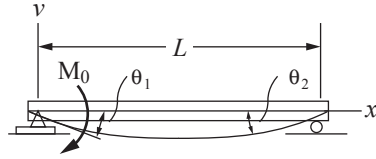
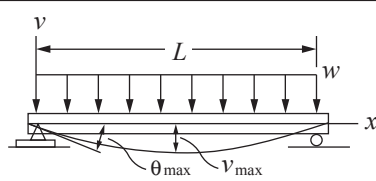
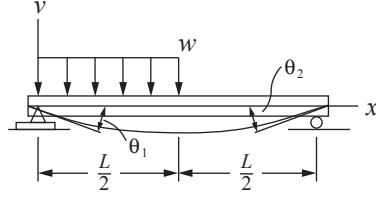
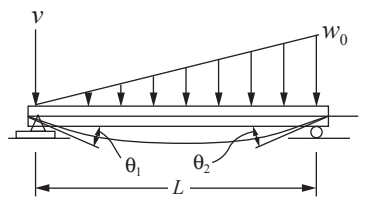
<sup>c</sup> MEASURED PERPENDICULAR TO THE GRAIN.

<sup>d</sup> MEASURED PARALLEL TO THE GRAIN.

<sup>e</sup> DEFORMATION MEASURED PERPENDICULAR TO THE GRAIN WHEN THE LOAD IS APPLIED ALONG THE GRAIN.

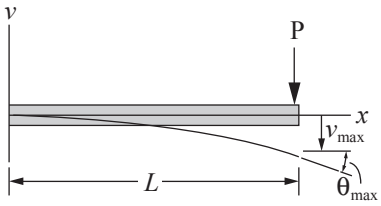
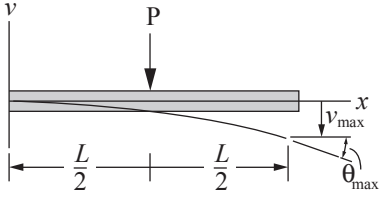
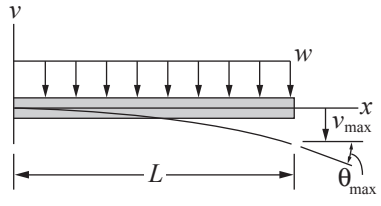
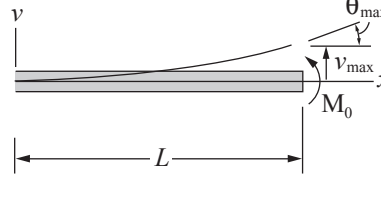
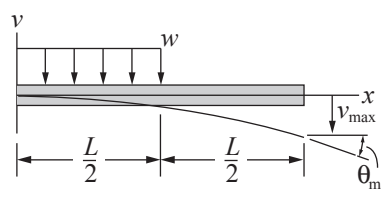
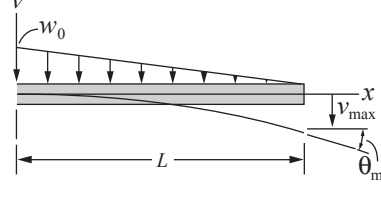
Hibbeler, R.C., *Mechanics of Materials*, 4th ed., Prentice Hall, 2000.

# Simply Supported Beam Slopes and Deflections

BEAM	SLOPE	DEFLECTION	ELASTIC CURVE
	$\theta_{\max} = \frac{-PL^2}{16EI}$	$v_{\max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI} (3L^2 - 4x^2)$ $0 \leq x \leq L/2$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v \Big _{x=a} = \frac{-Pba}{6EIL} (L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL} (L^2 - b^2 - x^2)$ $0 \leq x \leq a$
	$\theta_1 = \frac{-M_0L}{3EI}$ $\theta_2 = \frac{M_0L}{6EI}$	$v_{\max} = \frac{-M_0L^2}{\sqrt{243EI}}$	$v = \frac{-M_0x}{6EIL} (x^2 - 3Lx + 2L^2)$
	$\theta_{\max} = \frac{-wL^3}{24EI}$	$v_{\max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI} (x^3 - 3Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v \Big _{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{\max} = -0.006563 \frac{wL^4}{EI}$ $\text{at } x = 0.4598L$	$v = \frac{-wx}{384EI} (16x^3 - 24Lx^2 + 9L^3)$ $0 \leq x \leq L/2$ $v = \frac{-wL}{384EI} (8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \leq x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{\max} = -0.00652 \frac{w_0L^4}{EI}$ $\text{at } x = 0.5193$	$v = \frac{-w_0x}{360EIL} (3x^4 - 10L^2x^2 + 7L^4)$

Hibbeler, R.C., *Mechanics of Materials*, 4th ed., Prentice Hall, 2000.

# Cantilevered Beam Slopes and Deflections

BEAM	SLOPE	DEFLECTION	ELASTIC CURVE
	$\theta_{\max} = \frac{-PL^2}{2EI}$	$v_{\max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L-x)$
	$\theta_{\max} = \frac{-PL^2}{8EI}$	$v_{\max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{6EI}\left(\frac{3}{2}L-x\right) \quad 0 \leq x \leq L/2$ $v = \frac{-PL^2}{24EI}\left(3x-\frac{1}{2}L\right) \quad L/2 \leq x \leq L$
	$\theta_{\max} = \frac{-wL^3}{6EI}$	$v_{\max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2-4Lx+6L^2)$
	$\theta_{\max} = \frac{M_0L}{EI}$	$v_{\max} = \frac{M_0L^2}{2EI}$	$v = \frac{M_0x^2}{2EI}$
	$\theta_{\max} = \frac{-wL^3}{48EI}$	$v_{\max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2-2Lx+\frac{3}{2}L^2\right) \quad 0 \leq x \leq L/2$ $v = \frac{-wL^3}{192EI}(4x-L/2) \quad L/2 \leq x \leq L$
	$\theta_{\max} = \frac{-w_0L^3}{24EI}$	$v_{\max} = \frac{-w_0L^4}{30EI}$	$v = \frac{-w_0x^2}{120EI}(10L^3-10L^2x+5Lx^2-x^3)$

Hibbeler, R.C., *Mechanics of Materials*, 4th ed., Prentice Hall, 2000.



# THERMODYNAMICS

## PROPERTIES OF SINGLE-COMPONENT SYSTEMS

### Nomenclature

1. Intensive properties are independent of mass.
2. Extensive properties are proportional to mass.
3. Specific properties are lowercase (extensive/mass).

### State Functions (properties)

Absolute Pressure, $P$	(lbf/in <sup>2</sup> or Pa)
Absolute Temperature, $T$	(°R or K)
Volume, $V$	(ft <sup>3</sup> or m <sup>3</sup> )
Specific Volume, $v = V/m$	(ft <sup>3</sup> /lbm or m <sup>3</sup> /kg)
Internal Energy, $U$	(Btu or kJ)
Specific Internal Energy, $u = U/m$	(Btu/lbm or kJ/kg)
Enthalpy, $H$	(Btu or kJ)
Specific Enthalpy, $h = u + Pv = H/m$	(Btu/lbm or kJ/kg)
Entropy, $S$	(Btu/°R or kJ/K)
Specific Entropy, $s = S/m$	[Btu/(lbm·°R) or kJ/(kg·K)]
Gibbs Free Energy, $g = h - Ts$	(Btu/lbm or kJ/kg)
Helmholtz Free Energy, $a = u - Ts$	(Btu/lbm or kJ/kg)

For a single-phase pure component, specification of any two intensive, independent properties is sufficient to fix all the rest.

Heat Capacity at Constant Pressure,

$$c_p = \left( \frac{\partial h}{\partial T} \right)_P \quad [\text{Btu}/(\text{lbm} \cdot ^\circ\text{R}) \text{ or } \text{kJ}/(\text{kg} \cdot \text{K})]$$

Heat Capacity at Constant Volume,

$$c_v = \left( \frac{\partial u}{\partial T} \right)_v \quad [\text{Btu}/(\text{lbm} \cdot ^\circ\text{R}) \text{ or } \text{kJ}/(\text{kg} \cdot \text{K})]$$

The steam tables in this section provide  $T$ ,  $P$ ,  $v$ ,  $u$ ,  $h$ , and  $s$  data for saturated and superheated water.

A  $P$ - $h$  diagram for refrigerant HFC-134a providing  $T$ ,  $P$ ,  $v$ ,  $h$ , and  $s$  data in a graphical format is included in this section.

Thermal and physical property tables for selected gases, liquids, and solids are included in this section.

### Properties for Two-Phase (vapor-liquid) Systems

Quality  $x$  (for liquid-vapor systems at saturation) is defined as the mass fraction of the vapor phase:

$$x = m_g / (m_g + m_f), \text{ where}$$

$m_g$  = mass of vapor

$m_f$  = mass of liquid

Specific volume of a two-phase system can be written:

$$v = xv_g + (1 - x)v_f \text{ or } v = v_f + xv_{fg}, \text{ where}$$

$v_f$  = specific volume of saturated liquid

$v_g$  = specific volume of saturated vapor

$v_{fg}$  = specific volume change upon vaporization

$$= v_g - v_f$$

Similar expressions exist for  $u$ ,  $h$ , and  $s$ :

$$u = xu_g + (1 - x)u_f \text{ or } u = u_f + xv_{fg}$$

$$h = xh_g + (1 - x)h_f \text{ or } h = h_f + xv_{fg}$$

$$s = xs_g + (1 - x)s_f \text{ or } s = s_f + xs_{fg}$$

## PVT BEHAVIOR

### Ideal Gas

For an ideal gas,  $Pv = RT$  or  $PV = mRT$ , and

$$P_1v_1/T_1 = P_2v_2/T_2, \text{ where}$$

$P$  = pressure

$v$  = specific volume

$m$  = mass of gas

$R$  = gas constant

$T$  = absolute temperature

$V$  = volume

$R$  is specific to each gas but can be found from

$$R = \frac{\bar{R}}{(\text{mol. wt})}, \text{ where}$$

$\bar{R}$  = the universal gas constant

$$= 1,545 \text{ ft} \cdot \text{lbf}/(\text{lbmol} \cdot ^\circ\text{R}) = 8,314 \text{ J}/(\text{kmol} \cdot \text{K}).$$

$$= 8.314 \text{ kPa} \cdot \text{m}^3/(\text{kmol} \cdot \text{K}) = 0.08206 \text{ L} \cdot \text{atm}/(\text{mole} \cdot \text{K})$$

For ideal gases,  $c_p - c_v = R$

Ideal gas behavior is characterized by:

- no intermolecular interactions
- molecules occupy zero volume

The properties of an ideal gas reflect those of a single molecule and are attributable entirely to the structure of the molecule and the system  $T$ .

For ideal gases:

$$\left( \frac{\partial h}{\partial P} \right)_T = 0 \quad \left( \frac{\partial u}{\partial v} \right)_T = 0$$

For cold air standard, heat capacities are assumed to be constant at their room temperature values. In that case, the following are true:

$$\Delta u = c_v \Delta T; \quad \Delta h = c_p \Delta T$$

$$\Delta s = c_p \ln (T_2/T_1) - R \ln (P_2/P_1)$$

$$\Delta s = c_v \ln (T_2/T_1) + R \ln (v_2/v_1)$$

Also, for constant entropy processes:

$$\frac{P_2}{P_1} = \left( \frac{v_1}{v_2} \right)^k; \quad \frac{T_2}{T_1} = \left( \frac{P_2}{P_1} \right)^{\frac{k-1}{k}}$$

$$\frac{T_2}{T_1} = \left( \frac{v_1}{v_2} \right)^{k-1}, \text{ where } k = c_p/c_v$$

## Ideal Gas Mixtures

$i = 1, 2, \dots, n$  constituents. Each constituent is an ideal gas.

Mole Fraction:

$$x_i = N_i/N; N = \sum N_i; \sum x_i = 1$$

where  $N_i$  = number of moles of component  $i$

$N$  = total moles in the mixture

Mass Fraction:  $y_i = m_i/m; m = \sum m_i; \sum y_i = 1$

Molecular Weight:  $M = m/N = \sum x_i M_i$

To convert *mole fractions*  $x_i$  to *mass fractions*  $y_i$ :

$$y_i = \frac{x_i M_i}{\sum (x_i M_i)}$$

To convert *mass fractions* to *mole fractions*:

$$x_i = \frac{y_i/M_i}{\sum (y_i/M_i)}$$

Partial Pressures:  $P_i = \frac{m_i R_i T}{V}$  and  $P = \sum P_i$

Partial Volumes:  $V_i = \frac{m_i R_i T}{P}$  and  $V = \sum V_i$

where  $P, V, T$  = the pressure, volume, and temperature of the mixture and  $R_i = R/M_i$

Combining the above generates the following additional expressions for mole fraction.

$$x_i = P_i/P = V_i/V$$

Other Properties:

$$u = \sum (y_i u_i); h = \sum (y_i h_i); s = \sum (y_i s_i)$$

$u_i$  and  $h_i$  are evaluated at  $T$

$s_i$  is evaluated at  $T$  and  $P_i$

## Real Gas

Most gases exhibit ideal gas behavior when the system pressure is less than 3 atm since the distance between molecules is large enough to produce negligible molecular interactions. The behavior of a real gas deviates from that of an ideal gas at higher pressures due to molecular interactions.

For a real gas,  $Pv = ZRT$  where  $Z$  = compressibility factor.

$Z = 1$  for an ideal gas;  $Z \neq 1$  for a real gas

## Equations of State (EOS)

EOS are used to quantify  $PvT$  behavior

Ideal Gas EOS (applicable only to ideal gases)

$$P = \left( \frac{RT}{v} \right)$$

Generalized Compressibility EOS (applicable to all systems as gases, liquids, and/or solids)

$$P = \left( \frac{RT}{v} \right) Z$$

Virial EOS (applicable only to gases)

$$P = \left( \frac{RT}{v} \right) \left( 1 + \frac{B}{v} + \frac{C}{v^2} + \dots \right) \text{ where } B, C, \dots$$

are virial coefficients obtained from  $PvT$  measurements or statistical mechanics.

Cubic EOS (theoretically motivated with intent to predict gas and liquid thermodynamic properties)

$$P = \frac{RT}{v-b} - \frac{a(T)}{(v+c_1b)(v+c_2b)}$$

where  $a(T)$ ,  $b$ , and  $c_1$  and  $c_2$  are species specific.

An example of a cubic EOS is the Van der Waals equation with constants based on the critical point:

$$\left( P + \frac{a}{\bar{v}^2} \right) (\bar{v} - b) = \bar{R}T$$

where  $a = \left( \frac{27}{64} \right) \left( \frac{\bar{R}^2 T_c^2}{P_c} \right)$ ,  $b = \frac{\bar{R} T_c}{8 P_c}$

where  $P_c$  and  $T_c$  are the pressure and temperature at the critical point, respectively, and  $\bar{v}$  is the molar specific volume.

EOS are used to predict:

- $P, v$ , or  $T$  when two of the three are specified
- other thermodynamic properties based on analytic manipulation of the EOS
- mixture properties using appropriate mixing rules to create a pseudo-component that mimics the mixture properties

The Theorem of Corresponding States asserts that all normal fluids have the same value of  $Z$  at the same reduced temperature  $T_r$  and pressure  $P_r$ .

$$T_r = \frac{T}{T_c} \quad P_r = \frac{P}{P_c}$$

where  $T_c$  and  $P_c$  are the critical temperature and pressure, respectively, expressed in absolute units. This is captured in the Generalized Compressibility Factor chart.

## FIRST LAW OF THERMODYNAMICS

The *First Law of Thermodynamics* is a statement of conservation of energy in a thermodynamic system. The net energy crossing the system boundary is equal to the change in energy inside the system.

*Heat*  $Q$  ( $q = Q/m$ ) is *energy transferred* due to temperature difference and is considered positive if it is inward or added to the system.

*Work*  $W$  ( $w = W/m$ ) is considered *positive if it is outward or work done by the system*.

## Closed Thermodynamic System

No mass crosses system boundary

$$Q - W = \Delta U + \Delta KE + \Delta PE$$

where  $\Delta U$  = change in internal energy

$\Delta KE$  = change in kinetic energy

$\Delta PE$  = change in potential energy

Energy can cross the boundary only in the form of heat or work. Work can be boundary work,  $w_b$ , or other work forms (electrical work, etc.)

*Reversible boundary work* is given by  $w_b = \int P dv$ .

### Special Cases of Closed Systems (with no change in kinetic or potential energy)

Constant System Pressure process (**Charles' Law**):

$$w_b = P\Delta v$$

(ideal gas)  $T/v = \text{constant}$

Constant Volume process:

$$w_b = 0$$

(ideal gas)  $T/P = \text{constant}$

Isentropic process (ideal gas):

$$Pv^k = \text{constant}$$

$$w = (P_2v_2 - P_1v_1)/(1 - k)$$

$$= R(T_2 - T_1)/(1 - k)$$

Constant Temperature process (**Boyle's Law**):

(ideal gas)  $Pv = \text{constant}$

$$w_b = RT \ln(v_2/v_1) = RT \ln(P_1/P_2)$$

Polytropic process (ideal gas):

$$Pv^n = \text{constant}$$

$$w = (P_2v_2 - P_1v_1)/(1 - n), n \neq 1$$

## Open Thermodynamic System

Mass crosses the system boundary.

There is flow work ( $Pv$ ) done by mass entering the system.

The reversible flow work is given by:

$$w_{rev} = - \int v dP + \Delta KE + \Delta PE$$

First Law applies whether or not processes are reversible.

Open System First Law (energy balance)

$$\begin{aligned} \Sigma \dot{m}_i \left[ h_i + V_i^2/2 + gZ_i \right] - \Sigma \dot{m}_e \left[ h_e + V_e^2/2 + gZ_e \right] \\ + \dot{Q}_{in} - \dot{W}_{net} = d(m_s u_s)/dt, \text{ where} \end{aligned}$$

$\dot{W}_{net}$  = rate of net or shaft work

$\dot{m}$  = mass flow rate (subscripts  $i$  and  $e$  refer to inlet and exit states of system)

$g$  = acceleration of gravity

$Z$  = elevation

$V$  = velocity

$m_s$  = mass of fluid within the system

$u_s$  = specific internal energy of system

$\dot{Q}_{in}$  = rate of heat transfer (neglecting kinetic and potential energy of the system)

### Special Cases of Open Systems (with no change in kinetic or potential energy)

Constant Volume process:

$$w_{rev} = -v(P_2 - P_1)$$

Constant System Pressure process:

$$w_{rev} = 0$$

Constant Temperature process:

(ideal gas)  $Pv = \text{constant}$

$$w_{rev} = RT \ln(v_2/v_1) = RT \ln(P_1/P_2)$$

Isentropic process (ideal gas):

$$Pv^k = \text{constant}$$

$$w_{rev} = k(P_2v_2 - P_1v_1)/(1 - k)$$

$$= kR(T_2 - T_1)/(1 - k)$$

$$w_{rev} = \frac{k}{k-1} RT_1 \left[ 1 - \left( \frac{P_2}{P_1} \right)^{(k-1)/k} \right]$$

Polytropic process (ideal gas):

$$Pv^n = \text{constant}$$

$$w_{rev} = n(P_2v_2 - P_1v_1)/(1 - n)$$

## Steady-Flow Systems

The system does not change state with time. This assumption is valid for steady operation of turbines, pumps, compressors, throttling valves, nozzles, and heat exchangers, including boilers and condensers.

$$\Sigma \dot{m} \left( h_i + V_i^2/2 + gZ_i \right) - \Sigma \dot{m}_e \left( h_e + V_e^2/2 + gZ_e \right) + \dot{Q}_{in} - \dot{W}_{out} = 0$$

and

$$\Sigma \dot{m}_i = \Sigma \dot{m}_e$$

where

$\dot{m}$  = mass flow rate (subscripts  $i$  and  $e$  refer to inlet and exit states of system)

$g$  = acceleration of gravity

$Z$  = elevation

$V$  = velocity

$\dot{Q}$  = rate of heat transfer

$\dot{W}$  = rate of work

### Special Cases of Steady-Flow Energy Equation

Nozzles, Diffusers: Velocity terms are significant. No elevation change, no heat transfer, and no work. Single mass stream.

$$h_i + V_i^2/2 = h_e + V_e^2/2$$

$$\text{Isentropic Efficiency (nozzle)} = \frac{V_e^2 - V_i^2}{2(h_i - h_{es})}, \text{ where}$$

$h_{es}$  = enthalpy at isentropic exit state.

Turbines, Pumps, Compressors: Often considered adiabatic (no heat transfer). Velocity terms usually can be ignored. There are significant work terms and a single mass stream.

$$h_i = h_e + w$$

$$\text{Isentropic Efficiency (turbine)} = \frac{h_i - h_e}{h_i - h_{es}}$$

$$\text{Isentropic Efficiency (compressor, pump)} = \frac{h_{es} - h_i}{h_e - h_i}$$

For pump only,  $h_{es} - h_i = v_i(p_e - p_i)$

Throttling Valves and Throttling Processes: No work, no heat transfer, and single-mass stream. Velocity terms are often insignificant.

$$h_i = h_e$$

Boilers, Condensers, Evaporators, One Side in a Heat Exchanger: Heat transfer terms are significant. For a single-mass stream, the following applies:

$$h_i + q = h_e$$

Heat Exchangers: No heat loss to the surroundings or work. Two separate flow rates  $\dot{m}_1$  and  $\dot{m}_2$ :

$$\dot{m}_1(h_{1i} - h_{1e}) = \dot{m}_2(h_{2e} - h_{2i})$$

Mixers, Separators, Open or Closed Feedwater Heaters:

$$\Sigma \dot{m}_i h_i = \Sigma \dot{m}_e h_e \quad \text{and}$$

$$\Sigma \dot{m}_i = \Sigma \dot{m}_e$$

### BASIC CYCLES

Heat engines take in heat  $Q_H$  at a high temperature  $T_H$ , produce a net amount of work  $W$ , and reject heat  $Q_L$  at a low temperature  $T_L$ . The efficiency  $\eta$  of a heat engine is given by:

$$\eta = W/Q_H = (Q_H - Q_L)/Q_H$$

The most efficient engine possible is the *Carnot Cycle*. Its efficiency is given by:

$$\eta_c = (T_H - T_L)/T_H, \text{ where}$$

$T_H$  and  $T_L$  = absolute temperatures (Kelvin or Rankine).

The following heat-engine cycles are plotted on  $P$ - $v$  and  $T$ - $s$  diagrams in this section:

Carnot, Otto, Rankine

Refrigeration cycles are the reverse of heat-engine cycles. Heat is moved from low to high temperature requiring work,  $W$ . Cycles can be used either for refrigeration or as heat pumps.

*Coefficient of Performance (COP)* is defined as:

$$\text{COP} = Q_H/W \text{ for heat pumps, and as}$$

$$\text{COP} = Q_L/W \text{ for refrigerators and air conditioners.}$$

Upper limit of COP is based on reversed Carnot Cycle:

$$\text{COP}_c = T_H/(T_H - T_L) \text{ for heat pumps and}$$

$$\text{COP}_c = T_L/(T_H - T_L) \text{ for refrigeration.}$$

$$1 \text{ ton refrigeration} = 12,000 \text{ Btu/hr} = 3,516 \text{ W}$$

The following refrigeration cycles are plotted on  $T$ - $s$  diagrams in this section:

reversed rankine, two-stage refrigeration, air refrigeration

### PSYCHROMETRICS

Properties of an air-water vapor mixture at a fixed pressure are given in graphical form on a psychrometric chart as provided in this section. When the system pressure is 1 atm, an ideal-gas mixture is assumed.

The definitions that follow use subscript  $a$  for dry air and  $v$  for water vapor.

$P$  = pressure of the air-water mixture, normally 1 atm

$T$  = dry-bulb temp (air/water mixture temperature)

$P_a$  = partial pressure of dry air

$P_v$  = partial pressure of water vapor

$$P = P_a + P_v$$

*Specific Humidity* (absolute humidity, humidity ratio)  $\omega$ :

$$\omega = m_v/m_a, \text{ where}$$

$m_v$  = mass of water vapor

$m_a$  = mass of dry air

$$\omega = 0.622P_v/P_a = 0.622P_v/(P - P_v)$$

*Relative Humidity* (rh)  $\phi$ :

$$\phi = P_v/P_g, \text{ where}$$

$P_g$  = saturation pressure of water at  $T$ .

Enthalpy  $h$ :  $h = h_a + \omega h_v$

*Dew-Point Temperature*  $T_{dp}$ :

$$T_{dp} = T_{sat} \text{ at } P_g = P_v$$

*Wet-bulb temperature*  $T_{wb}$  is the temperature indicated by a thermometer covered by a wick saturated with liquid water and in contact with moving air.

*Humid Volume*: Volume of moist air/mass of dry air.

## SECOND LAW OF THERMODYNAMICS

Thermal Energy Reservoirs

$$\Delta S_{\text{reservoir}} = Q/T_{\text{reservoir}}, \text{ where}$$

$Q$  is measured with respect to the reservoir.

### Kelvin-Planck Statement of Second Law

No heat engine can operate in a cycle while transferring heat with a single heat reservoir.

*COROLLARY* to Kelvin-Planck: No heat engine can have a higher efficiency than a Carnot Cycle operating between the same reservoirs.

### Clausius' Statement of Second Law

No refrigeration or heat pump cycle can operate without a net work input.

*COROLLARY*: No refrigerator or heat pump can have a higher COP than a Carnot Cycle refrigerator or heat pump.

### Entropy

$$ds = (1/T) \delta q_{\text{rev}}$$

$$s_2 - s_1 = \int_1^2 (1/T) \delta q_{\text{rev}}$$

Inequality of Clausius

$$\oint (1/T) \delta q_{\text{rev}} \leq 0$$

$$\int_1^2 (1/T) \delta q \leq s_2 - s_1$$

Isothermal, Reversible Process

$$\Delta s = s_2 - s_1 = q/T$$

Isentropic Process

$$\Delta s = 0; ds = 0$$

A reversible adiabatic process is isentropic.

Adiabatic Process

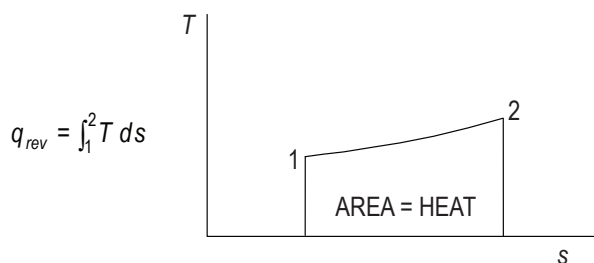
$$\delta q = 0; \Delta s \geq 0$$

Increase of Entropy Principle

$$\Delta s_{\text{total}} = \Delta s_{\text{system}} + \Delta s_{\text{surroundings}} \geq 0$$

$$\Delta \dot{s}_{\text{total}} = \sum \dot{m}_{\text{out}} s_{\text{out}} - \sum \dot{m}_{\text{in}} s_{\text{in}} - \sum (\dot{q}_{\text{external}}/T_{\text{external}}) \geq 0$$

Temperature-Entropy ( $T$ - $s$ ) Diagram



Entropy Change for Solids and Liquids

$$ds = c (dT/T)$$

$$s_2 - s_1 = \int c (dT/T) = c_{\text{mean}} \ln (T_2/T_1),$$

where  $c$  equals the heat capacity of the solid or liquid.

## Exergy (Availability)

Exergy (also known as availability) is the maximum possible work that can be obtained from a cycle of a heat engine. The maximum possible work is obtained in a reversible process.

### Closed-System Exergy (Availability)

(no chemical reactions)

$$\phi = (u - u_L) - T_L(s - s_L) + p_L(v - v_L)$$

where the subscript  $L$  designates environmental conditions and  $\phi$  is availability function.

$$w_{\text{max}} = w_{\text{rev}} = \phi_1 - \phi_2$$

### Open-System Exergy (Availability)

$$\Psi = (h - h_L) - T_L(s - s_L) + V^2/2 + gZ$$

where  $V$  is velocity,  $g$  is acceleration of gravity,  $Z$  is elevation and  $\Psi$  is availability function.

$$w_{\text{max}} = w_{\text{rev}} = \Psi_1 - \Psi_2$$

## Gibbs Free Energy, $\Delta G$

Energy released or absorbed in a reaction occurring reversibly at constant pressure and temperature.

## Helmholtz Free Energy, $\Delta A$

Energy released or absorbed in a reaction occurring reversibly at constant volume and temperature.

## Irreversibility, $I$

$$I = w_{\text{rev}} - w_{\text{actual}} = T_L \Delta s_{\text{total}}$$

## HEATS OF REACTION

For a chemical reaction the associated energy can be defined in terms of heats of formation of the individual species  $\Delta H_f^\circ$  at the standard state

$$(\Delta H_r^\circ) = \sum_{\text{products}} v_i (\Delta H_f^\circ)_i - \sum_{\text{reactants}} v_i (\Delta H_f^\circ)_i$$

$v_i$  = stoichiometric coefficient for species " $i$ "

The standard state is 25°C and 1 bar.

The heat of formation is defined as the enthalpy change associated with the formation of a compound from its atomic species as they normally occur in nature [i.e.,  $\text{O}_2(\text{g})$ ,  $\text{H}_2(\text{g})$ ,  $\text{C}(\text{solid})$ , etc.]

The heat of reaction varies with the temperature as follows:

$$\Delta H_r^\circ(T) = \Delta H_r^\circ(T_{\text{ref}}) + \int_{T_{\text{ref}}}^T \Delta c_p dT$$

where  $T_{\text{ref}}$  is some reference temperature (typically 25°C or 298 K), and:

$$\Delta c_p = \sum_{\text{products}} v_i c_{p,i} - \sum_{\text{reactants}} v_i c_{p,i}$$

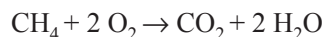
and  $c_{p,i}$  is the molar heat capacity of component  $i$ .

The heat of reaction for a combustion process using oxygen is also known as the heat of combustion. The principal products are  $\text{CO}_2(\text{g})$  and  $\text{H}_2\text{O}(\text{l})$ .



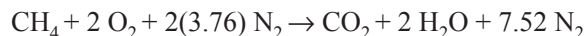
## Combustion Processes

First, the combustion equation should be written and balanced. For example, for the stoichiometric combustion of methane in oxygen:



### Combustion in Air

For each mole of oxygen, there will be 3.76 moles of nitrogen. For stoichiometric combustion of methane in air:



### Combustion in Excess Air

The excess oxygen appears as oxygen on the right side of the combustion equation.

### Incomplete Combustion

Some carbon is burned to create carbon monoxide (CO).

*Molar Air-Fuel Ratio*,  $\overline{A/F} = \frac{\text{No. of moles of air}}{\text{No. of moles of fuel}}$

*Air-Fuel Ratio*,  $A/F = \frac{\text{Mass of air}}{\text{Mass of fuel}} = (\overline{A/F}) \left( \frac{M_{\text{air}}}{M_{\text{fuel}}} \right)$

*Stoichiometric* (theoretical) air-fuel ratio is the air-fuel ratio calculated from the stoichiometric combustion equation.

Percent Theoretical Air =  $\frac{(A/F)_{\text{actual}}}{(A/F)_{\text{stoichiometric}}} \times 100$

Percent Excess Air =  $\frac{(A/F)_{\text{actual}} - (A/F)_{\text{stoichiometric}}}{(A/F)_{\text{stoichiometric}}} \times 100$

## VAPOR-LIQUID EQUILIBRIUM (VLE)

### Henry's Law at Constant Temperature

At equilibrium, the partial pressure of a gas is proportional to its concentration in a liquid. Henry's Law is valid for low concentrations; i.e.,  $x \approx 0$ .

$$P_i = Py_i = hx_i, \text{ where}$$

$h$  = Henry's Law constant

$P_i$  = partial pressure of a gas in contact with a liquid

$x_i$  = mol fraction of the gas in the liquid

$y_i$  = mol fraction of the gas in the vapor

$P$  = total pressure

### Raoult's Law for Vapor-Liquid Equilibrium

Valid for concentrations near 1; i.e.,  $x_i \approx 1$  at low pressure (ideal gas behavior)

$$P_i = x_i P_i^*, \text{ where}$$

$P_i$  = partial pressure of component  $i$

$x_i$  = mol fraction of component  $i$  in the liquid

$P_i^*$  = vapor pressure of pure component  $i$  at the temperature of the mixture

## Rigorous Vapor-Liquid Equilibrium

For a multicomponent mixture at equilibrium

$$\hat{f}_i^V = \hat{f}_i^L, \text{ where}$$

$\hat{f}_i^V$  = fugacity of component  $i$  in the vapor phase

$\hat{f}_i^L$  = fugacity of component  $i$  in the liquid phase

Fugacities of component  $i$  in a mixture are commonly calculated in the following ways:

For a liquid  $\hat{f}_i^L = x_i \gamma_i f_i^L$ , where

$x_i$  = mole fraction of component  $i$

$\gamma_i$  = activity coefficient of component  $i$

$f_i^L$  = fugacity of pure liquid component  $i$

For a vapor  $\hat{f}_i^V = y_i \hat{\Phi}_i P$ , where

$y_i$  = mole fraction of component  $i$  in the vapor

$\hat{\Phi}_i$  = fugacity coefficient of component  $i$  in the vapor

$P$  = system pressure

The activity coefficient  $\gamma_i$  is a correction for liquid phase non-ideality. Many models have been proposed for  $\gamma_i$  such as the Van Laar model:

$$\ln \gamma_1 = A_{12} \left( 1 + \frac{A_{12} x_1}{A_{21} x_2} \right)^{-2}$$

$$\ln \gamma_2 = A_{21} \left( 1 + \frac{A_{21} x_2}{A_{12} x_1} \right)^{-2}, \text{ where}$$

$\gamma_1$  = activity coefficient of component 1 in a two-component system

$\gamma_2$  = activity coefficient of component 2 in a two-component system

$A_{12}, A_{21}$  = constants, typically fitted from experimental data

The pure component fugacity is calculated as:

$$f_i^L = \Phi_i^{\text{sat}} P_i^{\text{sat}} \exp \left\{ v_i^L (P - P_i^{\text{sat}}) / (RT) \right\}, \text{ where}$$

$\Phi_i^{\text{sat}}$  = fugacity coefficient of pure saturated  $i$

$P_i^{\text{sat}}$  = saturation pressure of pure  $i$

$v_i^L$  = specific volume of pure liquid  $i$

$R$  = Ideal Gas Law Constant

$T$  = absolute temperature

Often at system pressures close to atmospheric:

$$f_i^L \cong P_i^{\text{sat}}$$

The fugacity coefficient  $\hat{\Phi}_i$  for component  $i$  in the vapor is calculated from an equation of state (e.g., Virial).

Sometimes it is approximated by a pure component value from a correlation. Often at pressures close to atmospheric,  $\hat{\Phi}_i = 1$ . The fugacity coefficient is a correction for vapor phase non-ideality.

For sparingly soluble gases the liquid phase is sometimes represented as:

$$\hat{f}_i^L = x_i k_i$$

where  $k_i$  is a constant set by experiment (Henry's constant). Sometimes other concentration units are used besides mole fraction with a corresponding change in  $k_i$ .

## PHASE RELATIONS

*Clapeyron Equation* for phase transitions:

$$\left(\frac{dP}{dT}\right)_{sat} = \frac{h_{fg}}{T v_{fg}} = \frac{s_{fg}}{v_{fg}}, \text{ where}$$

$h_{fg}$  = enthalpy change for phase transitions

$v_{fg}$  = volume change

$s_{fg}$  = entropy change

$T$  = absolute temperature

$(dP/dT)_{sat}$  = slope of phase transition (e.g., vapor-liquid) saturation line

*Clausius-Clapeyron Equation*

This equation results if it is assumed that (1) the volume change ( $v_{fg}$ ) can be replaced with the vapor volume ( $v_g$ ), (2) the latter can be replaced with  $P/\bar{R}T$  from the ideal gas law, and (3)  $h_{fg}$  is independent of the temperature ( $T$ ).

$$\ln_e \left( \frac{P_2}{P_1} \right) = \frac{h_{fg}}{\bar{R}} \cdot \frac{T_2 - T_1}{T_1 T_2}$$

*Gibbs Phase Rule (non-reacting systems)*

$P + F = C + 2$ , where

$P$  = number of phases making up a system

$F$  = degrees of freedom

$C$  = number of components in a system

## CHEMICAL REACTION EQUILIBRIA

### Definitions

*Conversion*: moles reacted/moles fed

*Extent* – For each species in a reaction, the mole balance may be written:

$$\text{moles}_{i,out} = \text{moles}_{i,in} + v_i \xi \text{ where}$$

$\xi$  is the extent in moles and  $v_i$  is the stoichiometric coefficient of the  $i$ th species, the sign of which is negative for reactants and positive for products.

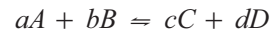
*Limiting reactant* – Reactant that would be consumed first if the reaction proceeded to completion. Other reactants are excess reactants.

*Selectivity* – Moles of desired product formed/moles of undesired product formed.

*Yield* – Moles of desired product formed/moles that would have been formed if there were no side reactions and the limiting reactant had reacted completely.

## Chemical Reaction Equilibrium

For the reaction



$$\Delta G^\circ = -RT \ln K_a$$

$$K_a = \frac{(\hat{a}_C^c)(\hat{a}_D^d)}{(\hat{a}_A^a)(\hat{a}_B^b)} = \prod_i (\hat{a}_i)^{v_i}, \text{ where}$$

$$\hat{a}_i = \text{activity of component } i = \frac{\hat{f}_i}{f_i^\circ}$$

$f_i^\circ$  = fugacity of pure  $i$  in its standard state at the equilibrium reaction temperature,  $T$

$v_i$  = stoichiometric coefficient of component  $i$

$\Delta G^\circ$  = standard Gibbs energy change of reaction

$K_a$  = chemical equilibrium constant

For mixtures of ideal gases:

$f_i^\circ$  = unit pressure, often 1 bar

$$\hat{f}_i = y_i P = p_i$$

where  $p_i$  = partial pressure of component  $i$ .

$$\text{Then } K_a = K_p = \frac{(p_C^c)(p_D^d)}{(p_A^a)(p_B^b)} = P^{c+d-a-b} \frac{(y_C^c)(y_D^d)}{(y_A^a)(y_B^b)}$$

For solids  $\hat{a}_i = 1$

For liquids  $\hat{a}_i = x_i \gamma_i$

The effect of temperature on the equilibrium constant is

$$\frac{d \ln K}{dT} = \frac{\Delta H^\circ}{RT^2}$$

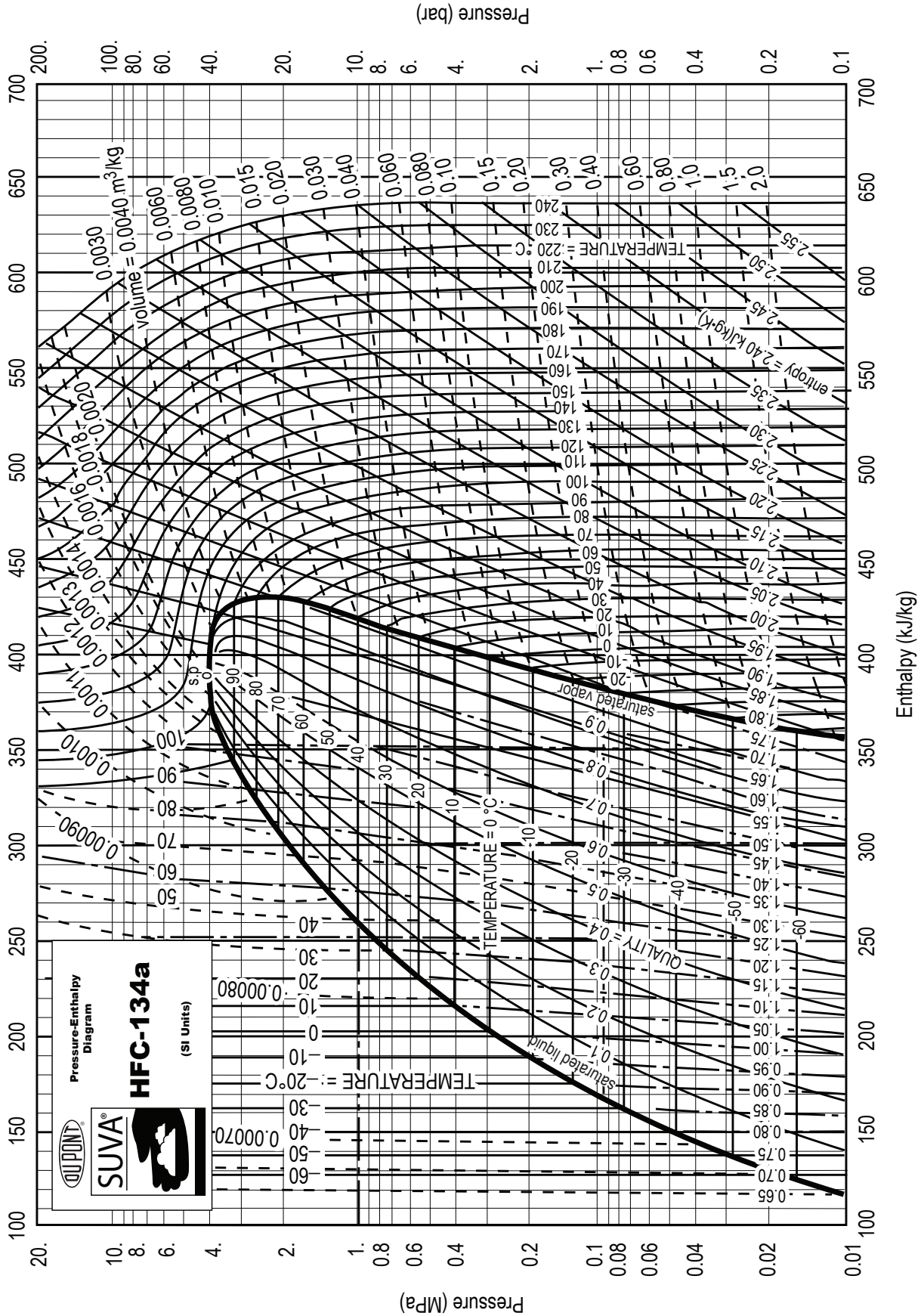
where  $\Delta H^\circ$  = standard enthalpy change of reaction.

<div>STEAM TABLES</div> <div>Saturated Water - Temperature Table</div>												
Temp. °C <i>T</i>	Sat. Press. kPa <i>P<sub>sat</sub></i>	Specific Volume m <sup>3</sup> /kg		Internal Energy kJ/kg			Enthalpy kJ/kg			Entropy kJ/(kg·K)		
		Sat. liquid <i>v<sub>f</sub></i>	Sat. vapor <i>v<sub>g</sub></i>	Sat. liquid <i>u<sub>f</sub></i>	Evap. <i>u<sub>fg</sub></i>	Sat. vapor <i>u<sub>g</sub></i>	Sat. liquid <i>h<sub>f</sub></i>	Evap. <i>h<sub>fg</sub></i>	Sat. vapor <i>h<sub>g</sub></i>	Sat. liquid <i>s<sub>f</sub></i>	Evap. <i>s<sub>fg</sub></i>	Sat. vapor <i>s<sub>g</sub></i>
0.01	0.6113	0.001 000	206.14	0.00	2375.3	2375.3	0.01	2501.3	2501.4	0.0000	9.1562	9.1562
5	0.8721	0.001 000	147.12	20.97	2361.3	2382.3	20.98	2489.6	2510.6	0.0761	8.9496	9.0257
10	1.2276	0.001 000	106.38	42.00	2347.2	2389.2	42.01	2477.7	2519.8	0.1510	8.7498	8.9008
15	1.7051	0.001 001	77.93	62.99	2333.1	2396.1	62.99	2465.9	2528.9	0.2245	8.5569	8.7814
20	2.339	0.001 002	57.79	83.95	2319.0	2402.9	83.96	2454.1	2538.1	0.2966	8.3706	8.6672
25	3.169	0.001 003	43.36	104.88	2304.9	2409.8	104.89	2442.3	2547.2	0.3674	8.1905	8.5580
30	4.246	0.001 004	32.89	125.78	2290.8	2416.6	125.79	2430.5	2556.3	0.4369	8.0164	8.4533
35	5.628	0.001 006	25.22	146.67	2276.7	2423.4	146.68	2418.6	2565.3	0.5053	7.8478	8.3531
40	7.384	0.001 008	19.52	167.56	2262.6	2430.1	167.57	2406.7	2574.3	0.5725	7.6845	8.2570
45	9.593	0.001 010	15.26	188.44	2248.4	2436.8	188.45	2394.8	2583.2	0.6387	7.5261	8.1648
50	12.349	0.001 012	12.03	209.32	2234.2	2443.5	209.33	2382.7	2592.1	0.7038	7.3725	8.0763
55	15.758	0.001 015	9.568	230.21	2219.9	2450.1	230.23	2370.7	2600.9	0.7679	7.2234	7.9913
60	19.940	0.001 017	7.671	251.11	2205.5	2456.6	251.13	2358.5	2609.6	0.8312	7.0784	7.9096
65	25.03	0.001 020	6.197	272.02	2191.1	2463.1	272.06	2346.2	2618.3	0.8935	6.9375	7.8310
70	31.19	0.001 023	5.042	292.95	2176.6	2469.6	292.98	2333.8	2626.8	0.9549	6.8004	7.7553
75	38.58	0.001 026	4.131	313.90	2162.0	2475.9	313.93	2321.4	2635.3	1.0155	6.6669	7.6824
80	47.39	0.001 029	3.407	334.86	2147.4	2482.2	334.91	2308.8	2643.7	1.0753	6.5369	7.6122
85	57.83	0.001 033	2.828	355.84	2132.6	2488.4	355.90	2296.0	2651.9	1.1343	6.4102	7.5445
90	70.14	0.001 036	2.361	376.85	2117.7	2494.5	376.92	2283.2	2660.1	1.1925	6.2866	7.4791
95	84.55	0.001 040	1.982	397.88	2102.7	2500.6	397.96	2270.2	2668.1	1.2500	6.1659	7.4159
MPa												
100	0.101 35	0.001 044	1.6729	418.94	2087.6	2506.5	419.04	2257.0	2676.1	1.3069	6.0480	7.3549
105	0.120 82	0.001 048	1.4194	440.02	2072.3	2512.4	440.15	2243.7	2683.8	1.3630	5.9328	7.2958
110	0.143 27	0.001 052	1.2102	461.14	2057.0	2518.1	461.30	2230.2	2691.5	1.4185	5.8202	7.2387
115	0.169 06	0.001 056	1.0366	482.30	2041.4	2523.7	482.48	2216.5	2699.0	1.4734	5.7100	7.1833
120	0.198 53	0.001 060	0.8919	503.50	2025.8	2529.3	503.71	2202.6	2706.3	1.5276	5.6020	7.1296
125	0.2321	0.001 065	0.7706	524.74	2009.9	2534.6	524.99	2188.5	2713.5	1.5813	5.4962	7.0775
130	0.2701	0.001 070	0.6685	546.02	1993.9	2539.9	546.31	2174.2	2720.5	1.6344	5.3925	7.0269
135	0.3130	0.001 075	0.5822	567.35	1977.7	2545.0	567.69	2159.6	2727.3	1.6870	5.2907	6.9777
140	0.3613	0.001 080	0.5089	588.74	1961.3	2550.0	589.13	2144.7	2733.9	1.7391	5.1908	6.9299
145	0.4154	0.001 085	0.4463	610.18	1944.7	2554.9	610.63	2129.6	2740.3	1.7907	5.0926	6.8833
150	0.4758	0.001 091	0.3928	631.68	1927.9	2559.5	632.20	2114.3	2746.5	1.8418	4.9960	6.8379
155	0.5431	0.001 096	0.3468	653.24	1910.8	2564.1	653.84	2098.6	2752.4	1.8925	4.9010	6.7935
160	0.6178	0.001 102	0.3071	674.87	1893.5	2568.4	675.55	2082.6	2758.1	1.9427	4.8075	6.7502
165	0.7005	0.001 108	0.2727	696.56	1876.0	2572.5	697.34	2066.2	2763.5	1.9925	4.7153	6.7078
170	0.7917	0.001 114	0.2428	718.33	1858.1	2576.5	719.21	2049.5	2768.7	2.0419	4.6244	6.6663
175	0.8920	0.001 121	0.2168	740.17	1840.0	2580.2	741.17	2032.4	2773.6	2.0909	4.5347	6.6256
180	1.0021	0.001 127	0.194 05	762.09	1821.6	2583.7	763.22	2015.0	2778.2	2.1396	4.4461	6.5857
185	1.1227	0.001 134	0.174 09	784.10	1802.9	2587.0	785.37	1997.1	2782.4	2.1879	4.3586	6.5465
190	1.2544	0.001 141	0.156 54	806.19	1783.8	2590.0	807.62	1978.8	2786.4	2.2359	4.2720	6.5079
195	1.3978	0.001 149	0.141 05	828.37	1764.4	2592.8	829.98	1960.0	2790.0	2.2835	4.1863	6.4698
200	1.5538	0.001 157	0.127 36	850.65	1744.7	2595.3	852.45	1940.7	2793.2	2.3309	4.1014	6.4323
205	1.7230	0.001 164	0.115 21	873.04	1724.5	2597.5	875.04	1921.0	2796.0	2.3780	4.0172	6.3952
210	1.9062	0.001 173	0.104 41	895.53	1703.9	2599.5	897.76	1900.7	2798.5	2.4248	3.9337	6.3585
215	2.104	0.001 181	0.094 79	918.14	1682.9	2601.1	920.62	1879.9	2800.5	2.4714	3.8507	6.3221
220	2.318	0.001 190	0.086 19	940.87	1661.5	2602.4	943.62	1858.5	2802.1	2.5178	3.7683	6.2861
225	2.548	0.001 199	0.078 49	963.73	1639.6	2603.9	966.78	1836.5	2803.3	2.5639	3.6863	6.2503
230	2.795	0.001 209	0.071 58	986.74	1617.2	2603.9	990.12	1813.8	2804.0	2.6099	3.6047	6.2146
235	3.060	0.001 219	0.065 37	1009.89	1594.2	2604.1	1013.62	1790.5	2804.2	2.6558	3.5233	6.1791
240	3.344	0.001 229	0.059 76	1033.21	1570.8	2604.0	1037.32	1766.5	2803.8	2.7015	3.4422	6.1437
245	3.648	0.001 240	0.054 71	1056.71	1546.7	2603.4	1061.23	1741.7	2803.0	2.7472	3.3612	6.1083
250	3.973	0.001 251	0.050 13	1080.39	1522.0	2602.4	1085.36	1716.2	2801.5	2.7927	3.2802	6.0730
255	4.319	0.001 263	0.045 98	1104.28	1596.7	2600.9	1109.73	1689.8	2799.5	2.8383	3.1992	6.0375
260	4.688	0.001 276	0.042 21	1128.39	1470.6	2599.0	1134.37	1662.5	2796.9	2.8838	3.1181	6.0019
265	5.081	0.001 289	0.038 77	1152.74	1443.9	2596.6	1159.28	1634.4	2793.6	2.9294	3.0368	5.9662
270	5.499	0.001 302	0.035 64	1177.36	1416.3	2593.7	1184.51	1605.2	2789.7	2.9751	2.9551	5.9301
275	5.942	0.001 317	0.032 79	1202.25	1387.9	2590.2	1210.07	1574.9	2785.0	3.0208	2.8730	5.8938
280	6.412	0.001 332	0.030 17	1227.46	1358.7	2586.1	1235.99	1543.6	2779.6	3.0668	2.7903	5.8571
285	6.909	0.001 348	0.027 77	1253.00	1328.4	2581.4	1262.31	1511.0	2773.3	3.1130	2.7070	5.8199
290	7.436	0.001 366	0.025 57	1278.92	1297.1	2576.0	1289.07	1477.1	2766.2	3.1594	2.6227	5.7821
295	7.993	0.001 384	0.023 54	1305.2	1264.7	2569.9	1316.3	1441.8	2758.1	3.2062	2.5375	5.7437
300	8.581	0.001 404	0.021 67	1332.0	1231.0	2563.0	1344.0	1404.9	2749.0	3.2534	2.4511	5.7045
305	9.202	0.001 425	0.019 948	1359.3	1195.9	2555.2	1372.4	1366.4	2738.7	3.3010	2.3633	5.6643
310	9.856	0.001 447	0.018 350	1387.1	1159.4	2546.4	1401.3	1326.0	2727.3	3.3493	2.2737	5.6230
315	10.547	0.001 472	0.016 867	1415.5	1121.1	2536.6	1431.0	1283.5	2714.5	3.3982	2.1821	5.5804
320	11.274	0.001 499	0.015 488	1444.6	1080.9	2525.5	1461.5	1238.6	2700.1	3.4480	2.0882	5.5362
330	12.845	0.001 561	0.012 996	1505.3	993.7	2498.9	1525.3	1140.6	2665.9	3.5507	1.8909	5.4417
340	14.586	0.001 638	0.010 797	1570.3	894.3	2464.6	1594.2	1027.9	2622.0	3.6594	1.6763	5.3357
350	16.513	0.001 740	0.008 813	1641.9	776.6	2418.4	1670.6	893.4	2563.9	3.7777	1.4335	5.2112
360	18.651	0.001 893	0.006 945	1725.2	626.3	2351.5	1760.5	720.3	2481.0	3.9147	1.1379	5.0526
370	21.03	0.002 213	0.004 925	1844.0	384.5	2228.5	1890.5	441.6	2332.1	4.1106	0.6865	4.7971
374.14	22.09	0.003 155	0.003 155	2029.6	0	2029.6	2099.3	0	2099.3	4.4298	0	4.4298



Superheated Water Tables								
$T$ Temp. °C	$v$ m <sup>3</sup> /kg	$u$ kJ/kg	$h$ kJ/kg	$s$ kJ/(kg·K)	$v$ m <sup>3</sup> /kg	$u$ kJ/kg	$h$ kJ/kg	$s$ kJ/(kg·K)
$p = 0.01 \text{ MPa (45.81}^\circ\text{C)}$					$p = 0.05 \text{ MPa (81.33}^\circ\text{C)}$			
Sat.	14.674	2437.9	2584.7	8.1502	3.240	2483.9	2645.9	7.5939
50	14.869	2443.9	2592.6	8.1749				
100	17.196	2515.5	2687.5	8.4479	3.418	2511.6	2682.5	7.6947
150	19.512	2587.9	2783.0	8.6882	3.889	2585.6	2780.1	7.9401
<b>200</b>	<b>21.825</b>	<b>2661.3</b>	<b>2879.5</b>	<b>8.9038</b>	<b>4.356</b>	<b>2659.9</b>	<b>2877.7</b>	<b>8.1580</b>
250	24.136	2736.0	2977.3	9.1002	4.820	2735.0	2976.0	8.3556
300	26.445	2812.1	3076.5	9.2813	5.284	2811.3	3075.5	8.5373
400	31.063	2968.9	3279.6	9.6077	6.209	2968.5	3278.9	8.8642
500	35.679	3132.3	3489.1	9.8978	7.134	3132.0	3488.7	9.1546
<b>600</b>	<b>40.295</b>	<b>3302.5</b>	<b>3705.4</b>	<b>10.1608</b>	<b>8.057</b>	<b>3302.2</b>	<b>3705.1</b>	<b>9.4178</b>
700	44.911	3479.6	3928.7	10.4028	8.981	3479.4	3928.5	9.6599
800	49.526	3663.8	4159.0	10.6281	9.904	3663.6	4158.9	9.8852
900	54.141	3855.0	4396.4	10.8396	10.828	3854.9	4396.3	10.0967
1000	58.757	4053.0	4640.6	11.0393	11.751	4052.9	4640.5	10.2964
<b>1100</b>	<b>63.372</b>	<b>4257.5</b>	<b>4891.2</b>	<b>11.2287</b>	<b>12.674</b>	<b>4257.4</b>	<b>4891.1</b>	<b>10.4859</b>
1200	67.987	4467.9	5147.8	11.4091	13.597	4467.8	5147.7	10.6662
1300	72.602	4683.7	5409.7	11.5811	14.521	4683.6	5409.6	10.8382
$p = 0.10 \text{ MPa (99.63}^\circ\text{C)}$					$p = 0.20 \text{ MPa (120.23}^\circ\text{C)}$			
Sat.	1.6940	2506.1	2675.5	7.3594	0.8857	2529.5	2706.7	7.1272
100	1.6958	2506.7	2676.2	7.3614				
150	1.9364	2582.8	2776.4	7.6134	0.9596	2576.9	2768.8	7.2795
200	2.172	2658.1	2875.3	7.8343	1.0803	2654.4	2870.5	7.5066
<b>250</b>	<b>2.406</b>	<b>2733.7</b>	<b>2974.3</b>	<b>8.0333</b>	<b>1.1988</b>	<b>2731.2</b>	<b>2971.0</b>	<b>7.7086</b>
300	2.639	2810.4	3074.3	8.2158	1.3162	2808.6	3071.8	7.8926
400	3.103	2967.9	3278.2	8.5435	1.5493	2966.7	3276.6	8.2218
500	3.565	3131.6	3488.1	8.8342	1.7814	3130.8	3487.1	8.5133
600	4.028	3301.9	3704.4	9.0976	2.013	3301.4	3704.0	8.7770
<b>700</b>	<b>4.490</b>	<b>3479.2</b>	<b>3928.2</b>	<b>9.3398</b>	<b>2.244</b>	<b>3478.8</b>	<b>3927.6</b>	<b>9.0194</b>
800	4.952	3663.5	4158.6	9.5652	2.475	3663.1	4158.2	9.2449
900	5.414	3854.8	4396.1	9.7767	2.705	3854.5	4395.8	9.4566
1000	5.875	4052.8	4640.3	9.9764	2.937	4052.5	4640.0	9.6563
1100	6.337	4257.3	4891.0	10.1659	3.168	4257.0	4890.7	9.8458
<b>1200</b>	<b>6.799</b>	<b>4467.7</b>	<b>5147.6</b>	<b>10.3463</b>	<b>3.399</b>	<b>4467.5</b>	<b>5147.5</b>	<b>10.0262</b>
1300	7.260	4683.5	5409.5	10.5183	3.630	4683.2	5409.3	10.1982
$p = 0.40 \text{ MPa (143.63}^\circ\text{C)}$					$p = 0.60 \text{ MPa (158.85}^\circ\text{C)}$			
Sat.	0.4625	2553.6	2738.6	6.8959	0.3157	2567.4	2756.8	6.7600
150	0.4708	2564.5	2752.8	6.9299				
200	0.5342	2646.8	2860.5	7.1706	0.3520	2638.9	2850.1	6.9665
250	0.5951	2726.1	2964.2	7.3789	0.3938	2720.9	2957.2	7.1816
<b>300</b>	<b>0.6548</b>	<b>2804.8</b>	<b>3066.8</b>	<b>7.5662</b>	<b>0.4344</b>	<b>2801.0</b>	<b>3061.6</b>	<b>7.3724</b>
350	0.7137	2884.6	3170.1	7.7324	0.4742	2881.2	3165.7	7.5464
400	0.7726	2964.4	3273.4	7.8985	0.5137	2962.1	3270.3	7.7079
500	0.8893	3129.2	3484.9	8.1913	0.5920	3127.6	3482.8	8.0021
600	1.0055	3300.2	3702.4	8.4558	0.6697	3299.1	3700.9	8.2674
<b>700</b>	<b>1.1215</b>	<b>3477.9</b>	<b>3926.5</b>	<b>8.6987</b>	<b>0.7472</b>	<b>3477.0</b>	<b>3925.3</b>	<b>8.5107</b>
800	1.2372	3662.4	4157.3	8.9244	0.8245	3661.8	4156.5	8.7367
900	1.3529	3853.9	4395.1	9.1362	0.9017	3853.4	4394.4	8.9486
1000	1.4685	4052.0	4639.4	9.3360	0.9788	4051.5	4638.8	9.1485
1100	1.5840	4256.5	4890.2	9.5256	1.0559	4256.1	4889.6	9.3381
<b>1200</b>	<b>1.6996</b>	<b>4467.0</b>	<b>5146.8</b>	<b>9.7060</b>	<b>1.1330</b>	<b>4466.5</b>	<b>5146.3</b>	<b>9.5185</b>
1300	1.8151	4682.8	5408.8	9.8780	1.2101	4682.3	5408.3	9.6906
$p = 0.80 \text{ MPa (170.43}^\circ\text{C)}$					$p = 1.00 \text{ MPa (179.91}^\circ\text{C)}$			
Sat.	0.2404	2576.8	2769.1	6.6628	0.194 44	2583.6	2778.1	6.5865
200	0.2608	2630.6	2839.3	6.8158	0.2060	2621.9	2827.9	6.6940
250	0.2931	2715.5	2950.0	7.0384	0.2327	2709.9	2942.6	6.9247
300	0.3241	2797.2	3056.5	7.2328	0.2579	2793.2	3051.2	7.1229
<b>350</b>	<b>0.3544</b>	<b>2878.2</b>	<b>3161.7</b>	<b>7.4089</b>	<b>0.2825</b>	<b>2875.2</b>	<b>3157.7</b>	<b>7.3011</b>
400	0.3843	2959.7	3267.1	7.5716	0.3066	2957.3	3263.9	7.4651
500	0.4433	3126.0	3480.6	7.8673	0.3541	3124.4	3478.5	7.7622
600	0.5018	3297.9	3699.4	8.1333	0.4011	3296.8	3697.9	8.0290
700	0.5601	3476.2	3924.2	8.3770	0.4478	3475.3	3923.1	8.2731
<b>800</b>	<b>0.6181</b>	<b>3661.1</b>	<b>4155.6</b>	<b>8.6033</b>	<b>0.4943</b>	<b>3660.4</b>	<b>4154.7</b>	<b>8.4996</b>
900	0.6761	3852.8	4393.7	8.8153	0.5407	3852.2	4392.9	8.7118
1000	0.7340	4051.0	4638.2	9.0153	0.5871	4050.5	4637.6	8.9119
1100	0.7919	4255.6	4889.1	9.2050	0.6335	4255.1	4888.6	9.1017
1200	0.8497	4466.1	5145.9	9.3855	0.6798	4465.6	5145.4	9.2822
<b>1300</b>	<b>0.9076</b>	<b>4681.8</b>	<b>5407.9</b>	<b>9.5575</b>	<b>0.7261</b>	<b>4681.3</b>	<b>5407.4</b>	<b>9.4543</b>

# **P-h Diagram for Refrigerant HFC-134a** (metric units)

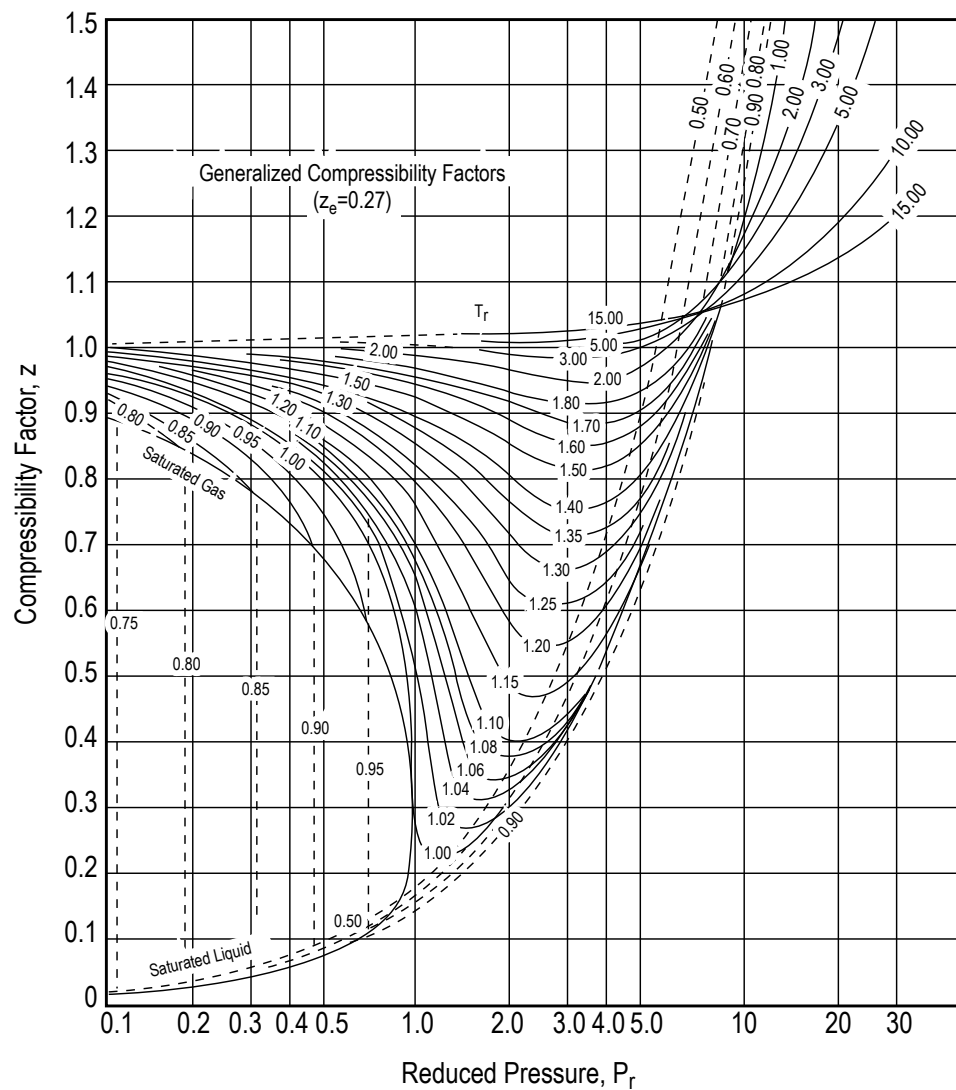


Data provided by DuPont Refrigerants, a division of E.I. duPont de Nemours and Co., Inc.

**Thermal and Physical Property Tables**  
(at room temperature)

GASES							
Substance	Mol wt	$c_p$		$c_v$		$k$	R
		kJ/(kg·K)	Btu/(lbm-°R)	kJ/(kg·K)	Btu/(lbm-°R)		kJ/(kg·K)
Gases							
Air	29	1.00	0.240	0.718	0.171	1.40	0.2870
Argon	40	0.520	0.125	0.312	0.0756	1.67	0.2081
Butane	58	1.72	0.415	1.57	0.381	1.09	0.1430
Carbon dioxide	44	0.846	0.203	0.657	0.158	1.29	0.1889
Carbon monoxide	28	1.04	0.249	0.744	0.178	1.40	0.2968
Ethane	30	1.77	0.427	1.49	0.361	1.18	0.2765
Helium	4	5.19	1.25	3.12	0.753	1.67	2.0769
Hydrogen	2	14.3	3.43	10.2	2.44	1.40	4.1240
Methane	16	2.25	0.532	1.74	0.403	1.30	0.5182
Neon	20	1.03	0.246	0.618	0.148	1.67	0.4119
Nitrogen	28	1.04	0.248	0.743	0.177	1.40	0.2968
Octane vapor	114	1.71	0.409	1.64	0.392	1.04	0.0729
Oxygen	32	0.918	0.219	0.658	0.157	1.40	0.2598
Propane	44	1.68	0.407	1.49	0.362	1.12	0.1885
Steam	18	1.87	0.445	1.41	0.335	1.33	0.4615

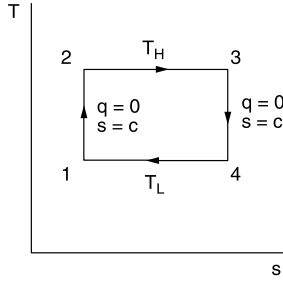
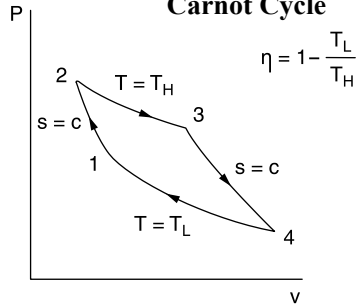
SELECTED LIQUIDS AND SOLIDS				
Substance	$c_p$		Density	
	kJ/(kg·K)	Btu/(lbm-°R)	kg/m <sup>3</sup>	lbm/ft <sup>3</sup>
Liquids				
Ammonia	4.80	1.146	602	38
Mercury	0.139	0.033	13,560	847
Water	4.18	1.000	997	62.4
Solids				
Aluminum	0.900	0.215	2,700	170
Copper	0.386	0.092	8,900	555
Ice (0°C; 32°F)	2.11	0.502	917	57.2
Iron	0.450	0.107	7,840	490
Lead	0.128	0.030	11,310	705



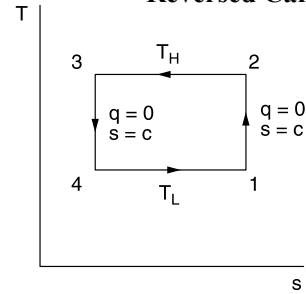
de Nevers, Noel, *Physical and Chemical Equilibrium for Chemical Engineers*, 2nd ed., Wiley, p. 308, © 2012.

## COMMON THERMODYNAMIC CYCLES

### Carnot Cycle



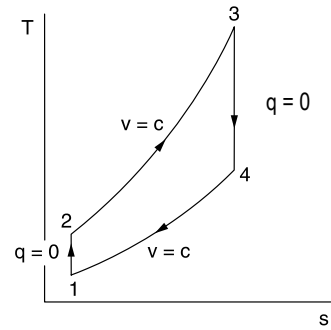
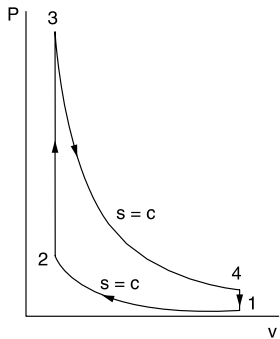
### Reversed Carnot



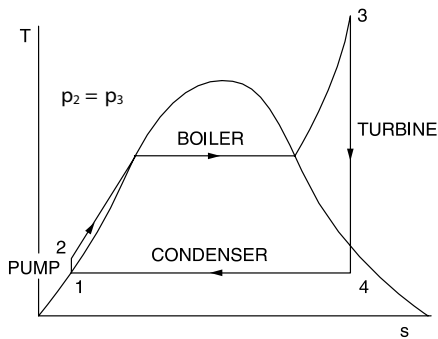
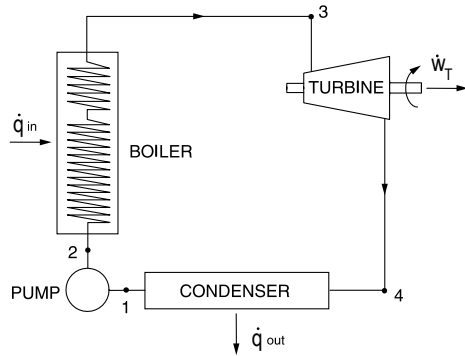
### Otto Cycle (Gasoline Engine)

$$\eta = 1 - r^{1-k}$$

$$r = v_1/v_2$$

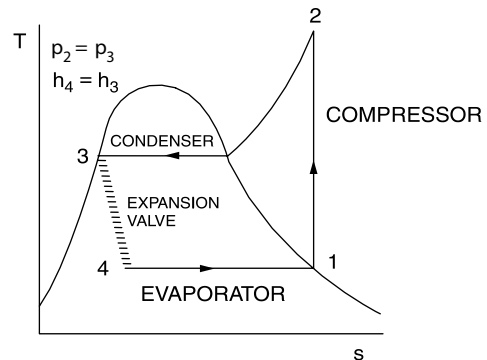
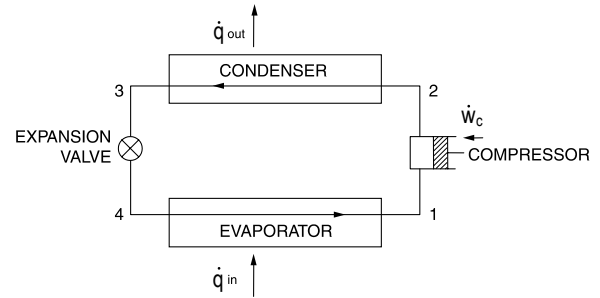


### Rankine Cycle



$$\eta = \frac{(h_3 - h_4) - (h_2 - h_1)}{h_3 - h_2}$$

### Refrigeration



$$\text{COP}_{\text{ref}} = \frac{h_1 - h_4}{h_2 - h_1}$$

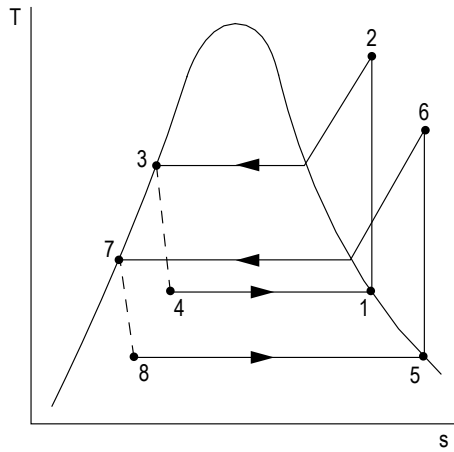
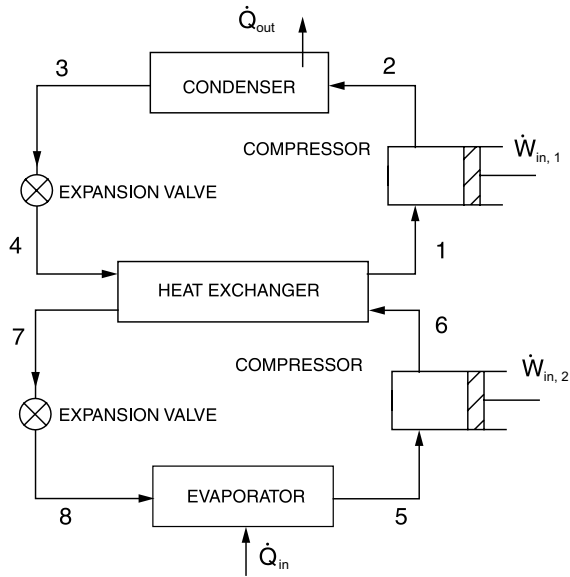
$$\text{COP}_{\text{HP}} = \frac{h_2 - h_3}{h_2 - h_1}$$

## REFRIGERATION AND HVAC

### Cycles

#### Refrigeration and HVAC

##### Two-Stage Cycle

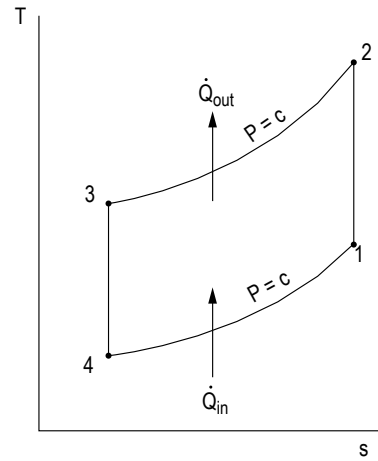
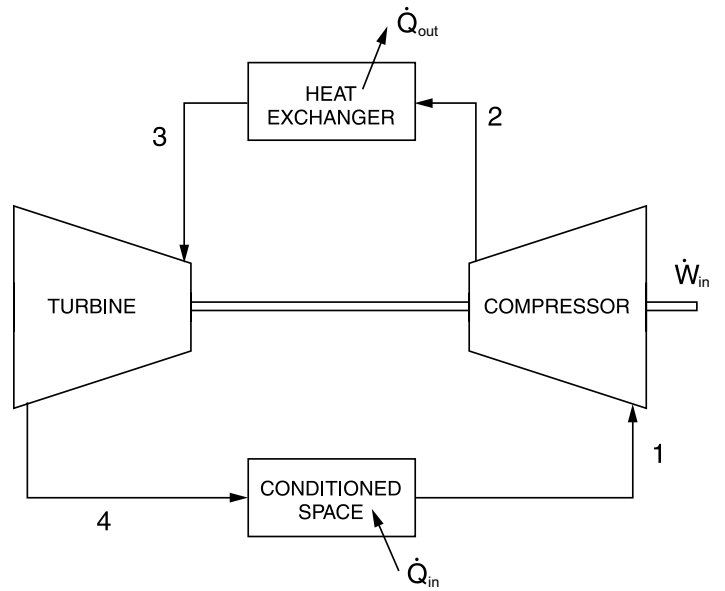


The following equations are valid if the mass flows are the same in each stage.

$$COP_{ref} = \frac{\dot{Q}_{in}}{\dot{W}_{in,1} + \dot{W}_{in,2}} = \frac{h_5 - h_8}{h_2 - h_1 + h_6 - h_5}$$

$$COP_{HP} = \frac{\dot{Q}_{out}}{\dot{W}_{in,1} + \dot{W}_{in,2}} = \frac{h_2 - h_3}{h_2 - h_1 + h_6 - h_5}$$

## Air Refrigeration Cycle



$$COP_{ref} = \frac{h_1 - h_4}{(h_2 - h_1) - (h_3 - h_4)}$$

$$COP_{HP} = \frac{h_2 - h_3}{(h_2 - h_1) - (h_3 - h_4)}$$

# ASHRAE Psychrometric Chart No. 1

(metric units)

© 2013 ASHRAE Psychrometric Chart No. 1

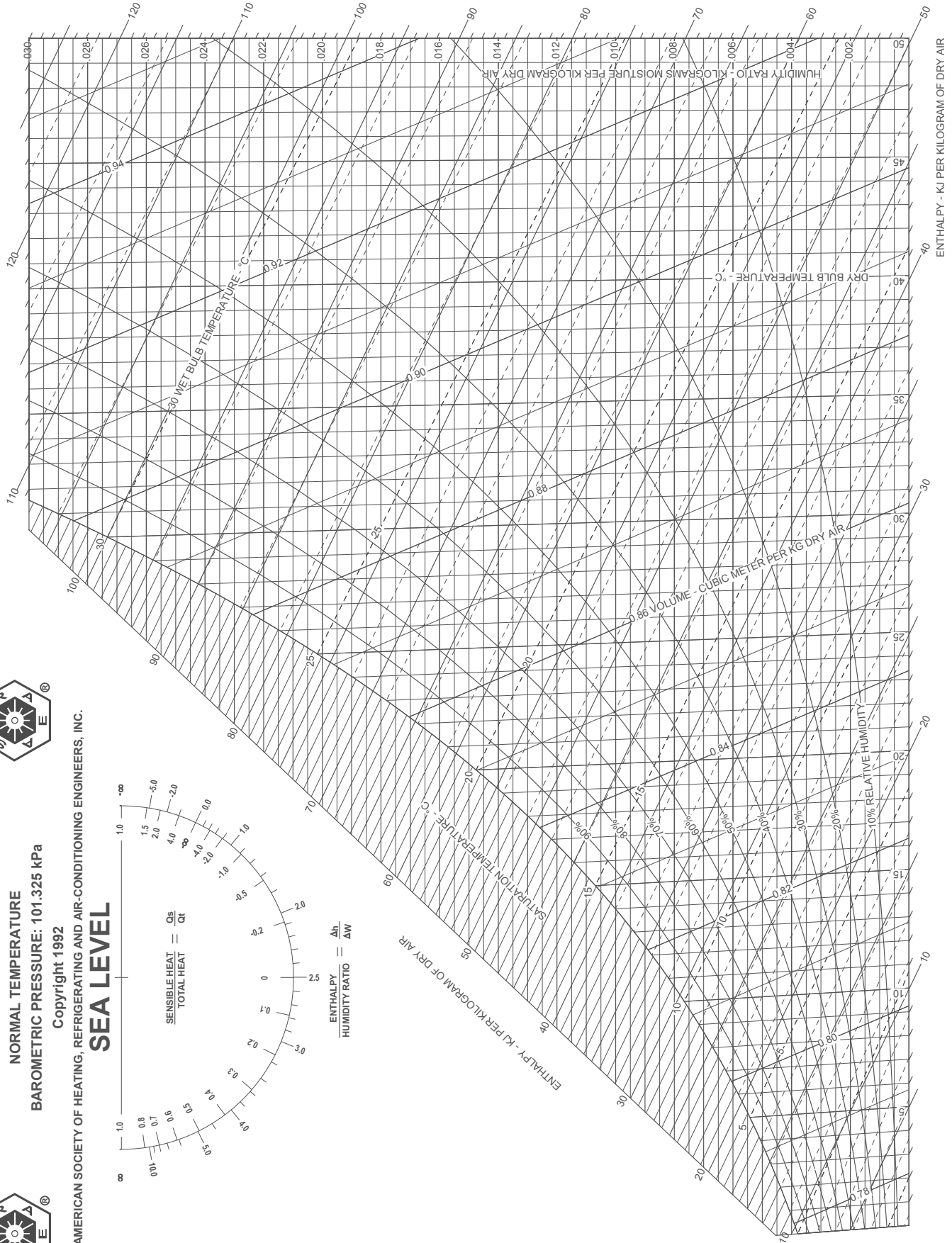
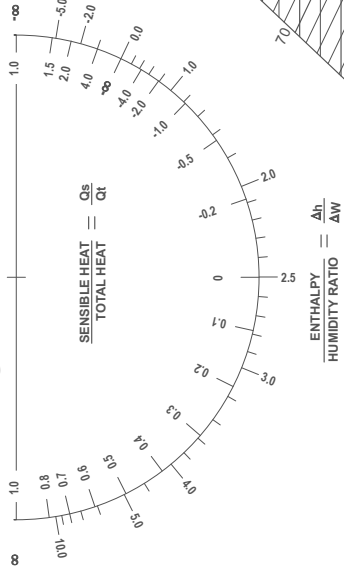


**ASHRAE PSYCHROMETRIC CHART NO. 1**  
 NORMAL TEMPERATURE  
 BAROMETRIC PRESSURE: 101.325 kPa

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AMERICAN SOCIETY OF HEATING, REFRIGERATING AND AIR-CONDITIONING ENGINEERS, INC.

**SEA LEVEL**





## © 2013 ASHRAE Psychrometric Chart No. 1

ASHRAE PSYCHROMETRIC CHART NO. 1  
NORMAL TEMPERATURE  
BAROMETRIC PRESSURE: 29.921 INCHES OF MERCURY  
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# FLUID MECHANICS

## DEFINITIONS

### Density, Specific Volume, Specific Weight, and Specific Gravity

The definitions of density, specific weight, and specific gravity follow:

$$\rho = \lim_{\Delta V \rightarrow 0} \Delta m / \Delta V$$

$$\gamma = \lim_{\Delta V \rightarrow 0} \Delta W / \Delta V$$

$$\gamma = \lim_{\Delta V \rightarrow 0} g \cdot \Delta m / \Delta V = \rho g$$

also  $SG = \gamma/\gamma_w = \rho/\rho_w$ , where

$\rho$  = density (also called *mass density*)

$\Delta m$  = mass of infinitesimal volume

$\Delta V$  = volume of infinitesimal object considered

$\gamma$  = *specific weight*

=  $\rho g$

$\Delta W$  = weight of an infinitesimal volume

$SG$  = *specific gravity*

$\rho_w$  = density of water at standard conditions

= 1,000 kg/m<sup>3</sup> (62.4 lbm/ft<sup>3</sup>)

$\gamma_w$  = specific weight of water at standard conditions

= 9,810 N/m<sup>3</sup> (62.4 lbf/ft<sup>3</sup>)

= 9,810 kg/(m<sup>2</sup>·s<sup>2</sup>)

### Stress, Pressure, and Viscosity

Stress is defined as

$$\tau(1) = \lim_{\Delta A \rightarrow 0} \Delta F / \Delta A, \text{ where}$$

$\tau(1)$  = surface stress vector at point 1

$\Delta F$  = force acting on infinitesimal area  $\Delta A$

$\Delta A$  = infinitesimal area at point 1

$$\tau_n = -P$$

$$\tau_t = \mu (dv/dy) \text{ (one-dimensional; i.e., } y), \text{ where}$$

$\tau_n$  and  $\tau_t$  = normal and tangential stress components at point 1, respectively

$P$  = pressure at point 1

$\mu$  = *absolute dynamic viscosity* of the fluid

$$\text{N} \cdot \text{s} / \text{m}^2 [\text{lbm} / (\text{ft} \cdot \text{sec})]$$

$dv$  = differential velocity

$dy$  = differential distance, normal to boundary

$v$  = velocity at boundary condition

$y$  = normal distance, measured from boundary

$\nu$  = *kinematic viscosity*; m<sup>2</sup>/s (ft<sup>2</sup>/sec)

$$\text{where } \nu = \mu/\rho$$

For a thin Newtonian fluid film and a linear velocity profile,

$$v(y) = vy/\delta; dv/dy = v/\delta, \text{ where}$$

$v$  = velocity of plate on film

$\delta$  = thickness of fluid film

For a power law (non-Newtonian) fluid

$$\tau_t = K (dv/dy)^n, \text{ where}$$

$K$  = consistency index

$n$  = power law index

$n < 1 \equiv$  pseudo plastic

$n > 1 \equiv$  dilatant

### Surface Tension and Capillarity

*Surface tension*  $\sigma$  is the force per unit contact length

$$\sigma = F/L, \text{ where}$$

$\sigma$  = surface tension, force/length

$F$  = surface force at the interface

$L$  = length of interface

The *capillary rise*  $h$  is approximated by

$$h = (4\sigma \cos \beta) / (\gamma d), \text{ where}$$

$h$  = height of the liquid in the vertical tube

$\sigma$  = surface tension

$\beta$  = angle made by the liquid with the wetted tube wall

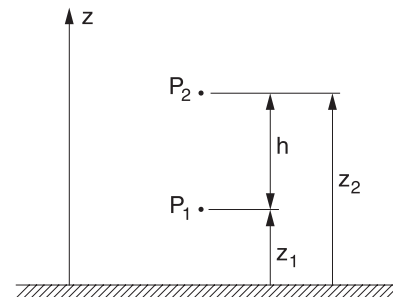
$\gamma$  = specific weight of the liquid

$d$  = the diameter of the capillary tube

## CHARACTERISTICS OF A STATIC LIQUID

### The Pressure Field in a Static Liquid

◆



The difference in pressure between two different points is

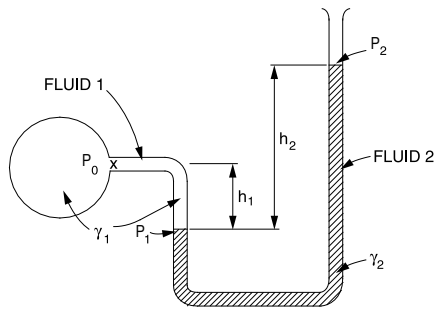
$$P_2 - P_1 = -\gamma (z_2 - z_1) = -\gamma h = -\rho gh$$

Absolute pressure = atmospheric pressure + gage pressure reading

Absolute pressure = atmospheric pressure – vacuum gage pressure reading

◆ Bober, W., and R.A. Kenyon, *Fluid Mechanics*, Wiley, 1980. Diagrams reprinted by permission of William Bober and Richard A. Kenyon.

## Manometers



For a simple manometer,

$$P_0 = P_2 + \gamma_2 h_2 - \gamma_1 h_1 = P_2 + g(\rho_2 h_2 - \rho_1 h_1)$$

$$\text{If } h_1 = h_2 = h$$

$$P_0 = P_2 + (\gamma_2 - \gamma_1)h = P_2 + (\rho_2 - \rho_1)gh$$

Note that the difference between the two densities is used.

$P$  = pressure

$\gamma$  = specific weight of fluid

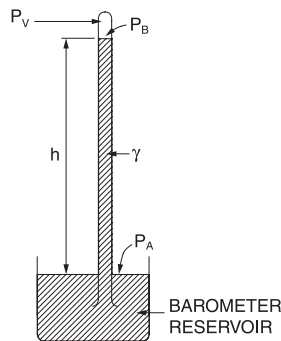
$h$  = height

$g$  = acceleration of gravity

$\rho$  = fluid density

Another device that works on the same principle as the manometer is the simple barometer.

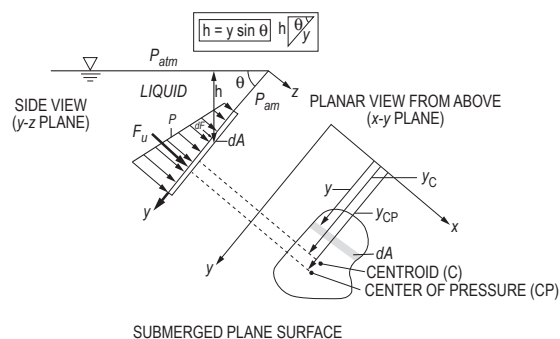
$$P_{\text{atm}} = P_A = P_v + \gamma h = P_B + \gamma h = P_B + \rho g h$$



$P_v$  = vapor pressure of the barometer fluid

## Forces on Submerged Surfaces and the Center of Pressure

•



The pressure on a point at a vertical distance  $h$  below the surface is:

$$P = P_0 + \rho g h, \text{ for } h \geq 0$$

- ♦ Bober, W., and R.A. Kenyon, *Fluid Mechanics*, Wiley, 1980. Diagrams reprinted by permission of William Bober and Richard A. Kenyon.
- Elger, Donald F., et al, *Engineering Fluid Mechanics*, 10th ed., 2012. Reproduced with permission of John Wiley & Sons, Inc.

$P$  = pressure

$P_0$  = atmospheric pressure

$P_C$  = pressure at the centroid of area

$P_{CP}$  = pressure at center of pressure

$y_C$  = slant distance from liquid surface to the centroid of area

$y_C = h_C / \sin \alpha$

$h_C$  = vertical distance from liquid surface to centroid of area

$y_{CP}$  = slant distance from liquid surface to center of pressure

$h_{CP}$  = vertical distance from liquid surface to center of pressure

$\theta$  = angle between liquid surface and edge of submerged surface

$I_{xC}$  = moment of inertia about the centroidal x-axis

If atmospheric pressure acts above the liquid surface and on the non-wetted side of the submerged surface.

$$y_{CP} = y_C + I_{xC} / y_C A$$

$$y_{CP} = y_C + \rho g \sin \alpha I_{xC} / P_C A$$

$$\text{Wetted side: } F_R = (P_0 + \rho g y_C \sin \theta) A$$

$$P_0 \text{ acting both sides: } F_{R_{\text{net}}} = (\rho g y_C \sin \theta) A$$

## Archimedes Principle and Buoyancy

1. The buoyant force exerted on a submerged or floating body is equal to the weight of the fluid displaced by the body.
2. A floating body displaces a weight of fluid equal to its own weight; i.e., a floating body is in equilibrium.

The *center of buoyancy* is located at the centroid of the displaced fluid volume.

In the case of a body lying at the *interface of two immiscible fluids*, the buoyant force equals the sum of the weights of the fluids displaced by the body.

## PRINCIPLES OF ONE-DIMENSIONAL FLUID FLOW

### The Continuity Equation

So long as the flow  $Q$  is continuous, the *continuity equation*, as applied to one-dimensional flows, states that the flow passing two points (1 and 2) in a stream is equal at each point,

$$A_1 v_1 = A_2 v_2.$$

$$Q = Av$$

$$\dot{m} = \rho Q = \rho Av, \text{ where}$$

$Q$  = volumetric flow rate

$\dot{m}$  = mass flow rate

$A$  = cross-sectional area of flow

$v$  = average flow velocity

$\rho$  = the fluid density

For steady, one-dimensional flow,  $\dot{m}$  is a constant. If, in addition, the density is constant, then  $Q$  is constant.

## The Energy Equation

The energy equation for steady incompressible flow with no shaft device is

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g} + h_f \quad \text{or}$$
$$\frac{P_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\rho g} + z_2 + \frac{v_2^2}{2g} + h_f$$

$h_f$  = the head loss, considered a friction effect, and all remaining terms are defined above.

If the cross-sectional area and the elevation of the pipe are the same at both sections (1 and 2), then  $z_1 = z_2$  and  $v_1 = v_2$ .

The pressure drop  $P_1 - P_2$  is given by the following:

$$P_1 - P_2 = \gamma h_f = \rho g h_f$$

## The Field Equation

The field equation is derived when the energy equation is applied to one-dimensional flows. Assuming no friction losses and that no pump or turbine exists between sections 1 and 2 in the system,

$$\frac{P_2}{\gamma} + \frac{v_2^2}{2g} + z_2 = \frac{P_1}{\gamma} + \frac{v_1^2}{2g} + z_1 \quad \text{or}$$
$$\frac{P_2}{\rho} + \frac{v_2^2}{2} + z_2 g = \frac{P_1}{\rho} + \frac{v_1^2}{2} + z_1 g, \quad \text{where}$$

$P_1, P_2$  = pressure at sections 1 and 2

$v_1, v_2$  = average velocity of the fluid at the sections

$z_1, z_2$  = the vertical distance from a datum to the sections (the potential energy)

$\gamma$  = the specific weight of the fluid ( $\rho g$ )

$g$  = the acceleration of gravity

$\rho$  = fluid density

## Hydraulic Gradient (Grade Line)

Hydraulic grade line is the line connecting the sum of pressure and elevation heads at different points in conveyance systems. If a row of piezometers were placed at intervals along the pipe, the grade line would join the water levels in the piezometer water columns.

## Energy Line (Bernoulli Equation)

The Bernoulli equation states that the sum of the pressure, velocity, and elevation heads is constant. The energy line is this sum or the "total head line" above a horizontal datum. The difference between the hydraulic grade line and the energy line is the  $v^2/2g$  term.

## FLUID FLOW CHARACTERIZATION

### Reynolds Number

$$Re = vD\rho/\mu = vD/\nu$$

$$Re' = \frac{v^{(2-n)} D^n \rho}{K \left( \frac{3n+1}{4n} \right)^n 8^{(n-1)}}, \quad \text{where}$$

$\rho$  = the mass density

$D$  = the diameter of the pipe, dimension of the fluid streamline, or characteristic length

$\mu$  = the dynamic viscosity

$\nu$  = the kinematic viscosity

$Re$  = the Reynolds number (Newtonian fluid)

$Re'$  = the Reynolds number (Power law fluid)

$K$  and  $n$  are defined in the Stress, Pressure, and Viscosity section.

The critical Reynolds number  $(Re)_c$  is defined to be the minimum Reynolds number at which a flow will turn turbulent.

Flow through a pipe is generally characterized as laminar for  $Re < 2,100$  and fully turbulent for  $Re > 10,000$ , and transitional flow for  $2,100 < Re < 10,000$ .

The velocity distribution for *laminar flow* in circular tubes or between planes is

$$v(r) = v_{\max} \left[ 1 - \left( \frac{r}{R} \right)^2 \right], \quad \text{where}$$

$r$  = the distance (m) from the centerline

$R$  = the radius (m) of the tube or half the distance between the parallel planes

$v$  = the local velocity (m/s) at  $r$

$v_{\max}$  = the velocity (m/s) at the centerline of the duct

$v_{\max} = 1.18\bar{v}$ , for fully turbulent flow

$v_{\max} = 2\bar{v}$ , for circular tubes in laminar flow and

$v_{\max} = 1.5\bar{v}$ , for parallel planes in laminar flow, where

$\bar{v}$  = the average velocity (m/s) in the duct

The shear stress distribution is

$$\frac{\tau}{\tau_w} = \frac{r}{R}, \quad \text{where}$$

$\tau$  and  $\tau_w$  are the shear stresses at radii  $r$  and  $R$  respectively.

## CONSEQUENCES OF FLUID FLOW

### Head Loss Due to Flow

The *Darcy-Weisbach equation* is

$$h_f = f \frac{L}{D} \frac{v^2}{2g}, \text{ where}$$

$f$  =  $f(\text{Re}, \epsilon/D)$ , the Moody, Darcy, or Stanton friction factor

$D$  = diameter of the pipe

$L$  = length over which the pressure drop occurs

$\epsilon$  = roughness factor for the pipe, and other symbols are defined as before

An alternative formulation employed by chemical engineers is

$$h_f = \left(4f_{\text{Fanning}}\right) \frac{Lv^2}{D2g} = \frac{2f_{\text{Fanning}} Lv^2}{Dg}$$

$$\text{Fanning friction factor, } f_{\text{Fanning}} = \frac{f}{4}$$

A chart that gives  $f$  versus  $\text{Re}$  for various values of  $\epsilon/D$ , known as a *Moody, Darcy, or Stanton diagram*, is available in this section.

### Minor Losses in Pipe Fittings, Contractions, and Expansions

Head losses also occur as the fluid flows through pipe fittings (i.e., elbows, valves, couplings, etc.) and sudden pipe contractions and expansions.

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g} + h_f + h_{f, \text{fitting}}$$

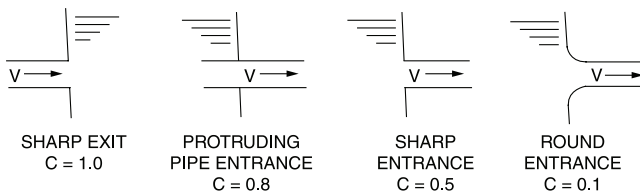
$$\frac{P_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\rho g} + z_2 + \frac{v_2^2}{2g} + h_f + h_{f, \text{fitting}}, \text{ where}$$

$$h_{f, \text{fitting}} = C \frac{v^2}{2g}, \text{ and } \frac{v^2}{2g} = 1 \text{ velocity head}$$

Specific fittings have characteristic values of  $C$ , which will be provided in the problem statement. A generally accepted *nominal value* for head loss in *well-streamlined gradual contractions* is

$$h_{f, \text{fitting}} = 0.04 v^2 / 2g$$

The *head loss* at either an *entrance* or *exit* of a pipe from or to a reservoir is also given by the  $h_{f, \text{fitting}}$  equation. Values for  $C$  for various cases are shown as follows.



Bober, W., and R.A. Kenyon, *Fluid Mechanics*, Wiley, 1980. Diagrams reprinted by permission of William Bober and Richard A. Kenyon.

### Pressure Drop for Laminar Flow

The equation for  $Q$  in terms of the pressure drop  $\Delta P_f$  is the Hagen-Poiseuille equation. This relation is valid only for flow in the laminar region.

$$Q = \frac{\pi R^4 \Delta P_f}{8\mu L} = \frac{\pi D^4 \Delta P_f}{128\mu L}$$

### Flow in Noncircular Conduits

Analysis of flow in conduits having a noncircular cross section uses the *hydraulic radius*  $R_H$ , or the *hydraulic diameter*  $D_H$ , as follows:

$$R_H = \frac{\text{cross-sectional area}}{\text{wetted perimeter}} = \frac{D_H}{4}$$

### Drag Force

The *drag force*  $F_D$  on objects immersed in a large body of flowing fluid or objects moving through a stagnant fluid is

$$F_D = \frac{C_D \rho v^2 A}{2}, \text{ where}$$

$C_D$  = the *drag coefficient*,

$v$  = the velocity (m/s) of the flowing fluid or moving object, and

$A$  = the *projected area* ( $\text{m}^2$ ) of blunt objects such as spheres, ellipsoids, disks, and plates, cylinders, ellipses, and air foils with axes perpendicular to the flow.

$\rho$  = fluid density

For flat plates placed parallel with the flow:

$$C_D = 1.33/\text{Re}^{0.5} \quad (10^4 < \text{Re} < 5 \times 10^5)$$

$$C_D = 0.031/\text{Re}^{1/7} \quad (10^6 < \text{Re} < 10^9)$$

The characteristic length in the Reynolds Number ( $\text{Re}$ ) is the length of the plate parallel with the flow. For blunt objects, the characteristic length is the largest linear dimension (diameter of cylinder, sphere, disk, etc.) that is perpendicular to the flow.

## CHARACTERISTICS OF SELECTED FLOW CONFIGURATIONS

### Open-Channel Flow and/or Pipe Flow of Water

#### Manning's Equation

$$v = (K/n) R_H^{2/3} S^{1/2}, \text{ where}$$

$v$  = velocity (m/s, ft/sec)

$K$  = 1.0 for SI units, 1.486 for USCS units

$n$  = roughness coefficient

$R_H$  = hydraulic radius (m, ft)

$S$  = slope of energy grade line (m/m, ft/ft)

#### Hazen-Williams Equation

$$v = k_1 C R_H^{0.63} S^{0.54}, \text{ where}$$

$k_1$  = 0.849 for SI units, 1.318 for USCS units

$C$  = roughness coefficient, as tabulated in the Civil Engineering section. Other symbols are defined as before.

## Flow Through a Packed Bed

A porous, fixed bed of solid particles can be characterized by

$L$  = length of particle bed (m)

$D_p$  = average particle diameter (m)

$\Phi_s$  = sphericity of particles, dimensionless (0–1)

$\epsilon$  = porosity or void fraction of the particle bed, dimensionless (0–1)

The Ergun equation can be used to estimate pressure loss through a packed bed under laminar and turbulent flow conditions.

$$\frac{\Delta P}{L} = \frac{150 v_o \mu (1 - \epsilon)^2}{\Phi_s^2 D_p^2 \epsilon^3} + \frac{1.75 \rho v_o^2 (1 - \epsilon)}{\Phi_s D_p \epsilon^3}$$

$\Delta P$  = pressure loss across packed bed (Pa)

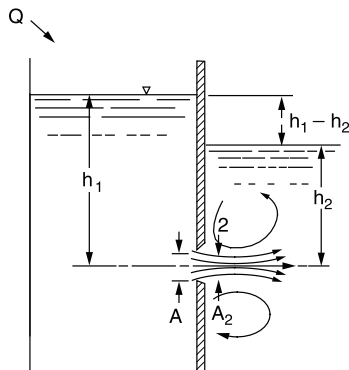
$v_o$  = superficial (flow through empty vessel) fluid velocity (m/s)

$\rho$  = fluid density (kg/m<sup>3</sup>)

$\mu$  = fluid viscosity [kg/(m·s)]

**Submerged Orifice operating under steady-flow conditions:**

•



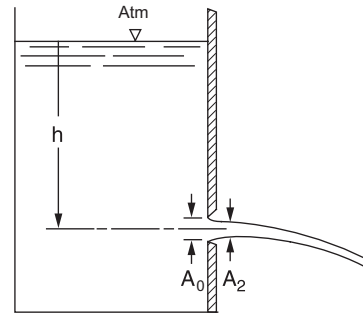
$$Q = A_2 v_2 = C_c C_v A \sqrt{2g(h_1 - h_2)} \\ = CA \sqrt{2g(h_1 - h_2)}$$

in which the product of  $C_c$  and  $C_v$  is defined as the *coefficient of discharge* of the orifice.

$v_2$  = velocity of fluid exiting orifice

## Orifice Discharging Freely into Atmosphere

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$$Q = CA_0 \sqrt{2gh}$$

in which  $h$  is measured from the liquid surface to the centroid of the orifice opening.

$Q$  = volumetric flow

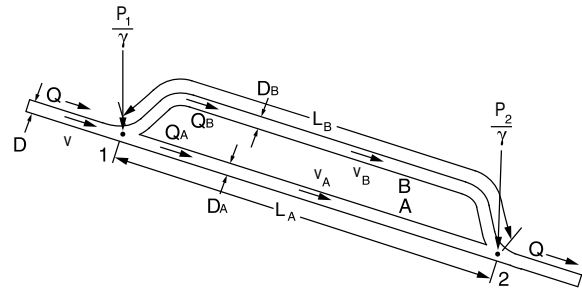
$A_0$  = cross-sectional area of flow

$g$  = acceleration of gravity

$h$  = height of fluid above orifice

## Multipath Pipeline Problems

•



For pipes in parallel, the head loss is the same in each pipe.

$$h_L = f_A \frac{L_A}{D_A} \frac{v_A^2}{2g} = f_B \frac{L_B}{D_B} \frac{v_B^2}{2g} \\ (\pi D^2/4) v = (\pi D_A^2/4) v_A + (\pi D_B^2/4) v_B$$

The total flow rate  $Q$  is the sum of the flow rates in the parallel pipes.

## THE IMPULSE-MOMENTUM PRINCIPLE

The resultant force in a given direction acting on the fluid equals the rate of change of momentum of the fluid.

$$\Sigma F = \Sigma Q_2 \rho_2 v_2 - \Sigma Q_1 \rho_1 v_1, \text{ where}$$

$\Sigma F$  = the resultant of all external forces acting on the control volume

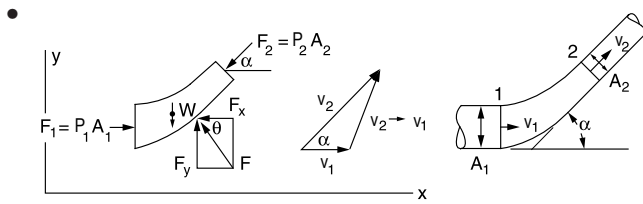
$\Sigma Q_1 \rho_1 v_1$  = the rate of momentum of the fluid flow entering the control volume in the same direction of the force

$\Sigma Q_2 \rho_2 v_2$  = the rate of momentum of the fluid flow leaving the control volume in the same direction of the force

• Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., J.K. Vennard, 1954.

## Pipe Bends, Enlargements, and Contractions

The force exerted by a flowing fluid on a bend, enlargement, or contraction in a pipeline may be computed using the impulse-momentum principle.



$$P_1 A_1 - P_2 A_2 \cos \alpha - F_x = Q\rho (v_2 \cos \alpha - v_1)$$

$$F_y - W - P_2 A_2 \sin \alpha = Q\rho (v_2 \sin \alpha - 0), \text{ where}$$

$F$  = the force exerted by the bend on the fluid (the force exerted by the fluid on the bend is equal in magnitude and opposite in sign),  $F_x$  and  $F_y$  are the  $x$ -component and  $y$ -component of the force  $F = \sqrt{F_x^2 + F_y^2}$  and

$$\theta = \tan^{-1} \left( \frac{F_y}{F_x} \right).$$

$P$  = the internal pressure in the pipe line

$A$  = the cross-sectional area of the pipe line

$W$  = the weight of the fluid

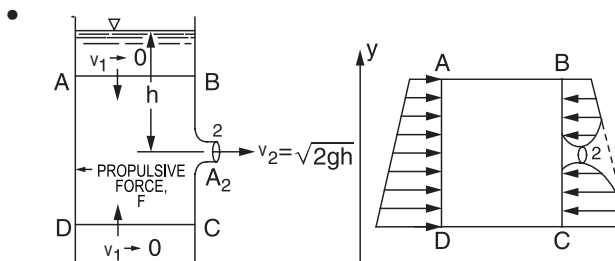
$v$  = the velocity of the fluid flow

$\alpha$  = the angle the pipe bend makes with the horizontal

$\rho$  = the density of the fluid

$Q$  = the quantity of fluid flow

## Jet Propulsion



$$F = Q\rho(v_2 - 0)$$

$$F = 2\gamma h A_2, \text{ where}$$

$F$  = the propulsive force

$\gamma$  = the specific weight of the fluid

$h$  = the height of the fluid above the outlet

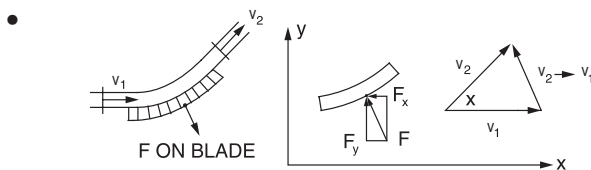
$A_2$  = the area of the nozzle tip

$$Q = A_2 \sqrt{2gh}$$

$$v_2 = \sqrt{2gh}$$

## Deflectors and Blades

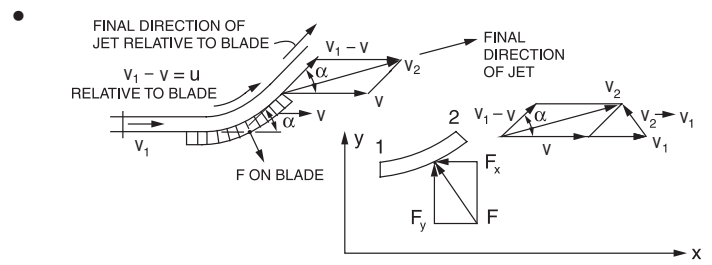
### Fixed Blade



$$-F_x = Q\rho(v_2 \cos \alpha - v_1)$$

$$F_y = Q\rho(v_2 \sin \alpha - 0)$$

## Moving Blade



$$-F_x = Q\rho(v_{2x} - v_{1x})$$

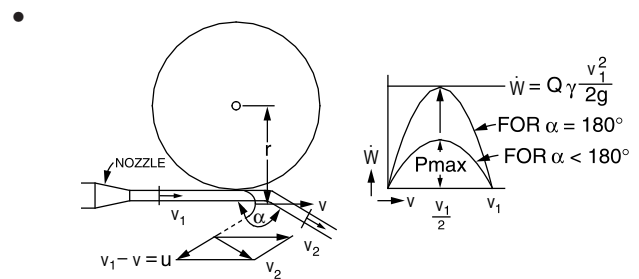
$$= -Q\rho(v_1 - v)(1 - \cos \alpha)$$

$$F_y = Q\rho(v_{2y} - v_{1y})$$

$$= +Q\rho(v_1 - v) \sin \alpha, \text{ where}$$

$v$  = the velocity of the blade.

## Impulse Turbine



$$\dot{W} = Q\rho(v_1 - v)(1 - \cos \alpha)v, \text{ where}$$

$\dot{W}$  = power of the turbine.

$$\dot{W}_{\max} = Q\rho(v_1^2/4)(1 - \cos \alpha)$$

When  $\alpha = 180^\circ$ ,

$$\dot{W}_{\max} = (Q\rho v_1^2)/2 = (Q\gamma v_1^2)/2g$$

## COMPRESSIBLE FLOW

### Mach Number

The local speed of sound in an ideal gas is given by:

$$c = \sqrt{kRT}, \text{ where}$$

$c$   $\equiv$  local speed of sound

$$k \equiv \text{ratio of specific heats} = \frac{c_p}{c_v}$$

$R$   $\equiv$  gas constant

$T$   $\equiv$  absolute temperature

This shows that the acoustic velocity in an ideal gas depends only on its temperature. The *Mach number* (Ma) is the ratio of the fluid velocity to the speed of sound.

$$\text{Ma} \equiv \frac{V}{c}$$

$V$   $\equiv$  mean fluid velocity

• Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., J.K. Vennard, 1954.



## Isentropic Flow Relationships

In an ideal gas for an isentropic process, the following relationships exist between static properties at any two points in the flow.

$$\frac{P_2}{P_1} = \left( \frac{T_2}{T_1} \right)^{\frac{k}{k-1}} = \left( \frac{\rho_2}{\rho_1} \right)^k$$

The stagnation temperature,  $T_0$ , at a point in the flow is related to the static temperature as follows:

$$T_0 = T + \frac{V^2}{2 \cdot c_p}$$

The relationship between the static and stagnation properties ( $T_0$ ,  $P_0$ , and  $\rho_0$ ) at any point in the flow can be expressed as a function of the Mach number as follows:

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} \cdot \text{Ma}^2$$

$$\frac{P_0}{P} = \left( \frac{T_0}{T} \right)^{\frac{k}{k-1}} = \left( 1 + \frac{k-1}{2} \cdot \text{Ma}^2 \right)^{\frac{k}{k-1}}$$

$$\frac{\rho_0}{\rho} = \left( \frac{T_0}{T} \right)^{\frac{1}{k-1}} = \left( 1 + \frac{k-1}{2} \cdot \text{Ma}^2 \right)^{\frac{1}{k-1}}$$

Compressible flows are often accelerated or decelerated through a nozzle or diffuser. For subsonic flows, the velocity decreases as the flow cross-sectional area increases and vice versa. For supersonic flows, the velocity increases as the flow cross-sectional area increases and decreases as the flow cross-sectional area decreases. The point at which the Mach number is sonic is called the throat and its area is represented by the variable,  $A^*$ . The following area ratio holds for any Mach number.

$$\frac{A}{A^*} = \frac{1}{\text{Ma}} \left[ \frac{1 + \frac{1}{2}(k-1)\text{Ma}^2}{\frac{1}{2}(k+1)} \right]^{\frac{(k+1)}{2(k-1)}}$$

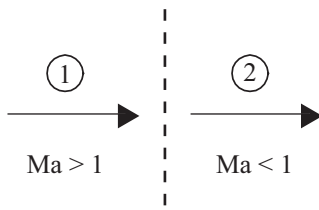
where

$A \equiv \text{area [length}^2\text{]}$

$A^* \equiv \text{area at the sonic point (Ma = 1.0)}$

## Normal Shock Relationships

A normal shock wave is a physical mechanism that slows a flow from supersonic to subsonic. It occurs over an infinitesimal distance. The flow upstream of a normal shock wave is always supersonic and the flow downstream is always subsonic as depicted in the figure.



NORMAL SHOCK

The following equations relate downstream flow conditions to upstream flow conditions for a normal shock wave.

$$\text{Ma}_2 = \sqrt{\frac{(k-1)\text{Ma}_1^2 + 2}{2k\text{Ma}_1^2 - (k-1)}}$$

$$\frac{T_2}{T_1} = \left[ 2 + (k-1)\text{Ma}_1^2 \right] \frac{2k\text{Ma}_1^2 - (k-1)}{(k+1)^2\text{Ma}_1^2}$$

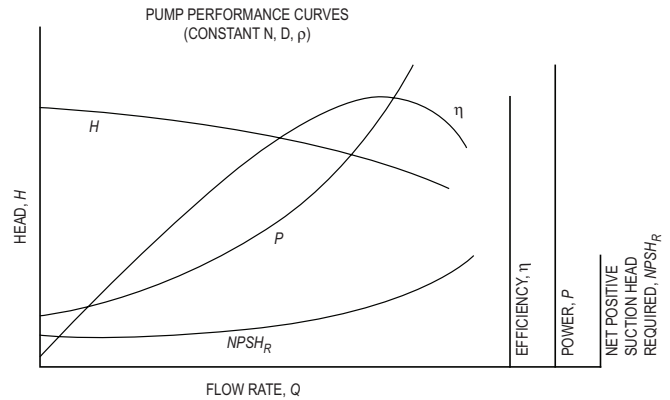
$$\frac{P_2}{P_1} = \frac{1}{k+1} [2k\text{Ma}_1^2 - (k-1)]$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(k+1)\text{Ma}_1^2}{(k-1)\text{Ma}_1^2 + 2}$$

$$T_{01} = T_{02}$$

## FLUID FLOW MACHINERY

### Centrifugal Pump Characteristics



Net Positive Suction Head Available ( $NPSH_A$ )

$$NPSH_A = \frac{P_{\text{atm}}}{\rho g} + H_s - H_f - \frac{V^2}{2g} - \frac{P_{\text{vapor}}}{\rho g}, \text{ where}$$

$P_{\text{atm}}$  = atmospheric pressure at fluid source surface

$H_s$  = static head at pump inlet

$H_f$  = friction losses from fluid source to pump inlet

$V$  = fluid velocity at pump inlet

$P_{\text{vapor}}$  = fluid vapor pressure at pump inlet

$\rho$  = fluid density

$g$  = gravitational constant

Fluid power  $\dot{W}_{\text{fluid}} = \rho g H Q$

Pump (brake) power  $\dot{W} = \frac{\rho g H Q}{\eta_{\text{pump}}}$

Purchased power  $\dot{W}_{\text{purchased}} = \frac{\dot{W}}{\eta_{\text{motor}}}$

$\eta_{\text{pump}}$  = pump efficiency (0 to 1)

$\eta_{\text{motor}}$  = motor efficiency (0 to 1)

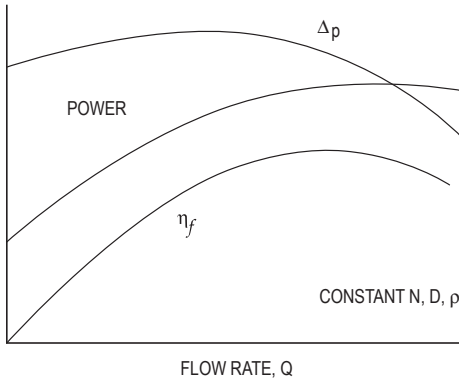
$H$  = head increase provided by pump

## Pump Power Equation

$$\dot{W} = Q\gamma h/\eta = Q\rho gh/\eta_t, \text{ where}$$

- $Q$  = volumetric flow (m<sup>3</sup>/s or cfs)  
 $h$  = head (m or ft) the fluid has to be lifted  
 $\eta_t$  = total efficiency ( $\eta_{\text{pump}} \times \eta_{\text{motor}}$ )  
 $\dot{W}$  = power (kg•m<sup>2</sup>/sec<sup>3</sup> or ft-lbf/sec)

## Fan Characteristics



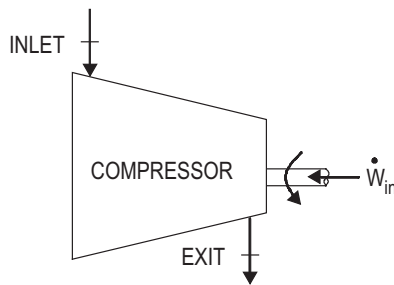
Typical *Backward Curved Fans*

$$\dot{W} = \frac{\Delta P Q}{\eta_f}, \text{ where}$$

- $\dot{W}$  = fan power  
 $\Delta P$  = pressure rise  
 $\eta_f$  = fan efficiency

## Compressors

Compressors consume power to add energy to the working fluid. This energy addition results in an increase in fluid pressure (head).



For an adiabatic compressor with  $\Delta PE = 0$  and negligible  $\Delta KE$ :

$$\dot{W}_{\text{comp}} = -\dot{m}(h_e - h_i)$$

For an ideal gas with constant specific heats:

$$\dot{W}_{\text{comp}} = -\dot{m}c_p(T_e - T_i)$$

Per unit mass:

$$w_{\text{comp}} = -c_p(T_e - T_i)$$

## Compressor Isentropic Efficiency

$$\eta_C = \frac{w_s}{w_a} = \frac{T_{es} - T_i}{T_e - T_i} \text{ where,}$$

- $w_a$   $\equiv$  actual compressor work per unit mass  
 $w_s$   $\equiv$  isentropic compressor work per unit mass  
 $T_{es}$   $\equiv$  isentropic exit temperature

For a compressor where  $\Delta KE$  is included:

$$\begin{aligned} \dot{W}_{\text{comp}} &= -\dot{m} \left( h_e - h_i + \frac{V_e^2 - V_i^2}{2} \right) \\ &= -\dot{m} \left( c_p(T_e - T_i) + \frac{V_e^2 - V_i^2}{2} \right) \end{aligned}$$

## Adiabatic Compression

$$\dot{W}_{\text{comp}} = \frac{\dot{m} P_i k}{(k-1)\rho_i \eta_c} \left[ \left( \frac{P_e}{P_i} \right)^{1-1/k} - 1 \right]$$

- $\dot{W}_{\text{comp}}$  = fluid or gas power (W)  
 $P_i$  = inlet or suction pressure (N/m<sup>2</sup>)  
 $P_e$  = exit or discharge pressure (N/m<sup>2</sup>)  
 $k$  = ratio of specific heats =  $c_p/c_v$   
 $\rho_i$  = inlet gas density (kg/m<sup>3</sup>)  
 $\eta_c$  = isentropic compressor efficiency

## Isothermal Compression

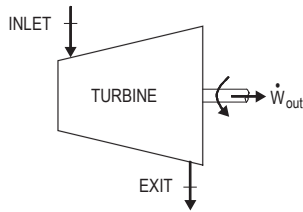
$$\dot{W}_{\text{comp}} = \frac{\bar{R}T_i}{M\eta_c} \ln \frac{P_e}{P_i} (\dot{m})$$

- $\dot{W}_{\text{comp}}$ ,  $P_i$ ,  $P_e$ , and  $\eta_c$  as defined for adiabatic compression  
 $\bar{R}$  = universal gas constant  
 $T_i$  = inlet temperature of gas (K)  
 $M$  = molecular weight of gas (kg/kmol)



## Turbines

Turbines produce power by extracting energy from a working fluid. The energy loss shows up as a decrease in fluid pressure (head).



For an adiabatic turbine with  $\Delta PE = 0$  and negligible  $\Delta KE$ :

$$\dot{W}_{\text{turb}} = \dot{m}(h_i - h_e)$$

For an ideal gas with constant specific heats:

$$\dot{W}_{\text{turb}} = \dot{m}c_p(T_i - T_e)$$

Per unit mass:

$$w_{\text{turb}} = c_p(T_i - T_e)$$

Turbine Isentropic Efficiency

$$\eta_T = \frac{w_a}{w_s} = \frac{T_i - T_e}{T_i - T_{es}}$$

For a turbine where  $\Delta KE$  is included:

$$\begin{aligned}\dot{W}_{\text{turb}} &= \dot{m}\left(h_e - h_i + \frac{V_e^2 - V_i^2}{2}\right) \\ &= \dot{m}\left(c_p(T_e - T_i) + \frac{V_e^2 - V_i^2}{2}\right)\end{aligned}$$

## Performance of Components

### Fans, Pumps, and Compressors

Scaling Laws; Affinity Laws

$$\begin{aligned}\left(\frac{Q}{ND^3}\right)_2 &= \left(\frac{Q}{ND^3}\right)_1 \\ \left(\frac{\dot{m}}{\rho ND^3}\right)_2 &= \left(\frac{\dot{m}}{\rho ND^3}\right)_1 \\ \left(\frac{H}{N^2 D^2}\right)_2 &= \left(\frac{H}{N^2 D^2}\right)_1 \\ \left(\frac{P}{\rho N^2 D^2}\right)_2 &= \left(\frac{P}{\rho N^2 D^2}\right)_1 \\ \left(\frac{\dot{W}}{\rho N^3 D^5}\right)_2 &= \left(\frac{\dot{W}}{\rho N^3 D^5}\right)_1\end{aligned}$$

where

$Q$  = volumetric flow rate

$\dot{m}$  = mass flow rate

$H$  = head

$P$  = pressure rise

$\dot{W}$  = power

$\rho$  = fluid density

$N$  = rotational speed

$D$  = impeller diameter

Subscripts 1 and 2 refer to different but similar machines or to different operating conditions of the same machine.

## FLUID FLOW MEASUREMENT

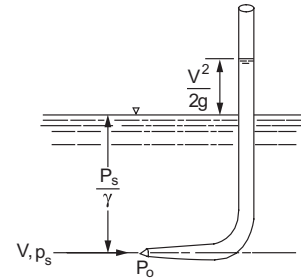
**The Pitot Tube** – From the stagnation pressure equation for an *incompressible fluid*,

$$v = \sqrt{(2/\rho)(P_0 - P_s)} = \sqrt{2g(P_0 - P_s)/\gamma}, \text{ where}$$

$v$  = the velocity of the fluid

$P_0$  = the stagnation pressure

$P_s$  = the static pressure of the fluid at the elevation where the measurement is taken



For a *compressible fluid*, use the above incompressible fluid equation if the Mach number  $\leq 0.3$ .

## Venturi Meters

$$Q = \frac{C_v A_2}{\sqrt{1 - (A_2/A_1)^2}} \sqrt{2g\left(\frac{P_1}{\gamma} + z_1 - \frac{P_2}{\gamma} - z_2\right)}, \text{ where}$$

$Q$  = volumetric flow rate

$C_v$  = the coefficient of velocity

$A$  = cross-sectional area of flow

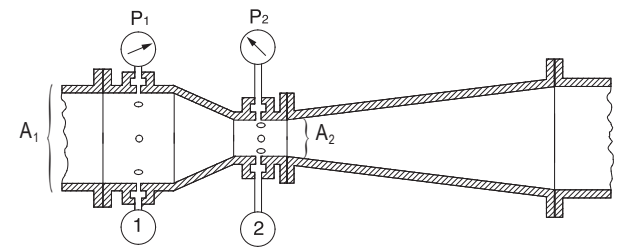
$P$  = pressure

$\gamma = \rho g$

$z_1$  = elevation of venturi entrance

$z_2$  = elevation of venturi throat

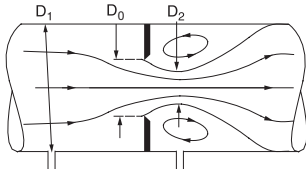
The above equation is for *incompressible fluids*.



- Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., J.K. Vennard, 1954.

## Orifices

The cross-sectional area at the vena contracta  $A_2$  is characterized by a *coefficient of contraction*  $C_c$  and given by  $C_c A$ .



$$Q = C A_0 \sqrt{2g \left( \frac{P_1}{\gamma} + z_1 - \frac{P_2}{\gamma} - z_2 \right)}$$

where  $C$ , the *coefficient of the meter (orifice coefficient)*, is given by

$$C = \frac{C_v C_c}{\sqrt{1 - C_c^2 (A_0/A_1)^2}}$$

ORIFICES AND THEIR NOMINAL COEFFICIENTS				
	SHARP EDGED	ROUNDED	SHORT TUBE	BORDA
$C$	0.61	0.98	0.80	0.51
$C_c$	0.62	1.00	1.00	0.52
$C_v$	0.98	0.98	0.80	0.98

For incompressible flow through a horizontal orifice meter installation

$$Q = C A_0 \sqrt{\frac{2}{\rho} (P_1 - P_2)}$$

## DIMENSIONAL HOMOGENEITY

### Dimensional Analysis

A dimensionally homogeneous equation has the same dimensions on the left and right sides of the equation. Dimensional analysis involves the development of equations that relate dimensionless groups of variables to describe physical phenomena.

**Buckingham Pi Theorem:** The *number of independent dimensionless groups* that may be employed to describe a phenomenon known to involve  $n$  variables is equal to the number  $(n - \bar{r})$ , where  $\bar{r}$  is the number of basic dimensions (e.g., M, L, T) needed to express the variables dimensionally.

## Similitude

In order to use a model to simulate the conditions of the prototype, the model must be *geometrically*, *kinematically*, and *dynamically similar* to the prototype system.

To obtain dynamic similarity between two flow pictures, all independent force ratios that can be written must be the same in both the model and the prototype. Thus, dynamic similarity between two flow pictures (when all possible forces are acting) is expressed in the five simultaneous equations below.

$$\left[ \frac{F_I}{F_P} \right]_p = \left[ \frac{F_I}{F_P} \right]_m = \left[ \frac{\rho v^2}{P} \right]_p = \left[ \frac{\rho v^2}{P} \right]_m$$

$$\left[ \frac{F_I}{F_V} \right]_p = \left[ \frac{F_I}{F_V} \right]_m = \left[ \frac{vl\rho}{\mu} \right]_p = \left[ \frac{vl\rho}{\mu} \right]_m = [Re]_p = [Re]_m$$

$$\left[ \frac{F_I}{F_G} \right]_p = \left[ \frac{F_I}{F_G} \right]_m = \left[ \frac{v^2}{lg} \right]_p = \left[ \frac{v^2}{lg} \right]_m = [Fr]_p = [Fr]_m$$

$$\left[ \frac{F_I}{F_E} \right]_p = \left[ \frac{F_I}{F_E} \right]_m = \left[ \frac{\rho v^2}{E_v} \right]_p = \left[ \frac{\rho v^2}{E_v} \right]_m = [Ca]_p = [Ca]_m$$

$$\left[ \frac{F_I}{F_T} \right]_p = \left[ \frac{F_I}{F_T} \right]_m = \left[ \frac{\rho l v^2}{\sigma} \right]_p = \left[ \frac{\rho l v^2}{\sigma} \right]_m = [We]_p = [We]_m$$

where the subscripts  $p$  and  $m$  stand for *prototype* and *model* respectively, and

$F_I$  = inertia force

$F_P$  = pressure force

$F_V$  = viscous force

$F_G$  = gravity force

$F_E$  = elastic force

$F_T$  = surface tension force

Re = Reynolds number

We = Weber number

Ca = Cauchy number

Fr = Froude number

$l$  = characteristic length

$v$  = velocity

$\rho$  = density

$\sigma$  = surface tension

$E_v$  = bulk modulus

$\mu$  = dynamic viscosity

$P$  = pressure

$g$  = acceleration of gravity

♦ Bober, W., and R.A. Kenyon, *Fluid Mechanics*, Wiley, 1980. Diagrams reprinted by permission of William Bober and Richard A. Kenyon.

• Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., J.K. Vennard, 1954.

## AERODYNAMICS

### Airfoil Theory

The lift force on an airfoil,  $F_L$ , is given by

$$F_L = \frac{C_L \rho v^2 A_p}{2}$$

$C_L$  = the lift coefficient

$\rho$  = fluid density

$v$  = velocity (m/s) of the undisturbed fluid and

$A_p$  = the projected area of the airfoil as seen from above (plan area). This same area is used in defining the drag coefficient for an airfoil.

The lift coefficient,  $C_L$ , can be approximated by the equation

$C_L = 2\pi k_1 \sin(\alpha + \beta)$ , which is valid for small values of  $\alpha$  and  $\beta$

$k_1$  = a constant of proportionality

$\alpha$  = angle of attack (angle between chord of airfoil and direction of flow)

$\beta$  = negative of angle of attack for zero lift

The drag coefficient,  $C_D$ , may be approximated by

$$C_D = C_{D\infty} + \frac{C_L^2}{\pi AR}$$

$C_{D\infty}$  = infinite span drag coefficient

The aspect ratio,  $AR$ , is defined

$$AR = \frac{b^2}{A_p} = \frac{A_p}{c^2}$$

$b$  = span length

$A_p$  = plan area

$c$  = chord length

The aerodynamic moment,  $M$ , is given by

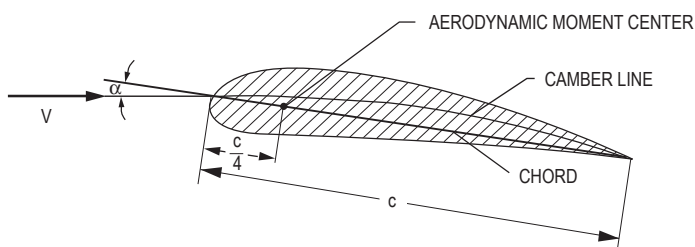
$$M = \frac{C_M \rho v^2 A_p c}{2}$$

where the moment is taken about the front quarter point of the airfoil.

$C_M$  = moment coefficient

$\rho$  = fluid density

$v$  = velocity



◆ **Properties of Water<sup>f</sup> (SI Metric Units)**

Temperature (°C)	Specific Weight <sup>a</sup> $\gamma$ (kN/m <sup>3</sup> )	Density <sup>a</sup> $\rho$ (kg/m <sup>3</sup> )	Absolute Dynamic Viscosity <sup>a</sup> $\mu$ (Pa·s)	Kinematic Viscosity <sup>a</sup> $\nu$ (m <sup>2</sup> /s)	Vapor Pressure <sup>e</sup> $P_v$ (kPa)
0	9.805	999.8	0.001781	0.000001785	0.61
5	9.807	1000.0	0.001518	0.000001518	0.87
10	9.804	999.7	0.001307	0.000001306	1.23
15	9.798	999.1	0.001139	0.000001139	1.70
20	9.789	998.2	0.001002	0.000001003	2.34
25	9.777	997.0	0.000890	0.000000893	3.17
30	9.764	995.7	0.000798	0.000000800	4.24
40	9.730	992.2	0.000653	0.000000658	7.38
50	9.689	988.0	0.000547	0.000000553	12.33
60	9.642	983.2	0.000466	0.000000474	19.92
70	9.589	977.8	0.000404	0.000000413	31.16
80	9.530	971.8	0.000354	0.000000364	47.34
90	9.466	965.3	0.000315	0.000000326	70.10
100	9.399	958.4	0.000282	0.000000294	101.33

◆ **Properties of Water (English Units)**

Temperature (°F)	Specific Weight $\gamma$ (lbf/ft <sup>3</sup> )	Mass Density $\rho$ (lbf-sec <sup>2</sup> /ft <sup>4</sup> )	Absolute Dynamic Viscosity $\mu$ ( $\times 10^{-5}$ lbf-sec/ft <sup>2</sup> )	Kinematic Viscosity $\nu$ ( $\times 10^{-5}$ ft <sup>2</sup> /sec)	Vapor Pressure $P_v$ (psi)
32	62.42	1.940	3.746	1.931	0.09
40	62.43	1.940	3.229	1.664	0.12
50	62.41	1.940	2.735	1.410	0.18
60	62.37	1.938	2.359	1.217	0.26
70	62.30	1.936	2.050	1.059	0.36
80	62.22	1.934	1.799	0.930	0.51
90	62.11	1.931	1.595	0.826	0.70
100	62.00	1.927	1.424	0.739	0.95
110	61.86	1.923	1.284	0.667	1.24
120	61.71	1.918	1.168	0.609	1.69
130	61.55	1.913	1.069	0.558	2.22
140	61.38	1.908	0.981	0.514	2.89
150	61.20	1.902	0.905	0.476	3.72
160	61.00	1.896	0.838	0.442	4.74
170	60.80	1.890	0.780	0.413	5.99
180	60.58	1.883	0.726	0.385	7.51
190	60.36	1.876	0.678	0.362	9.34
200	60.12	1.868	0.637	0.341	11.52
212	59.83	1.860	0.593	0.319	14.70

◆ <sup>a</sup>From "Hydraulic Models," *ASCE Manual of Engineering Practice*, No. 25, ASCE, 1942.

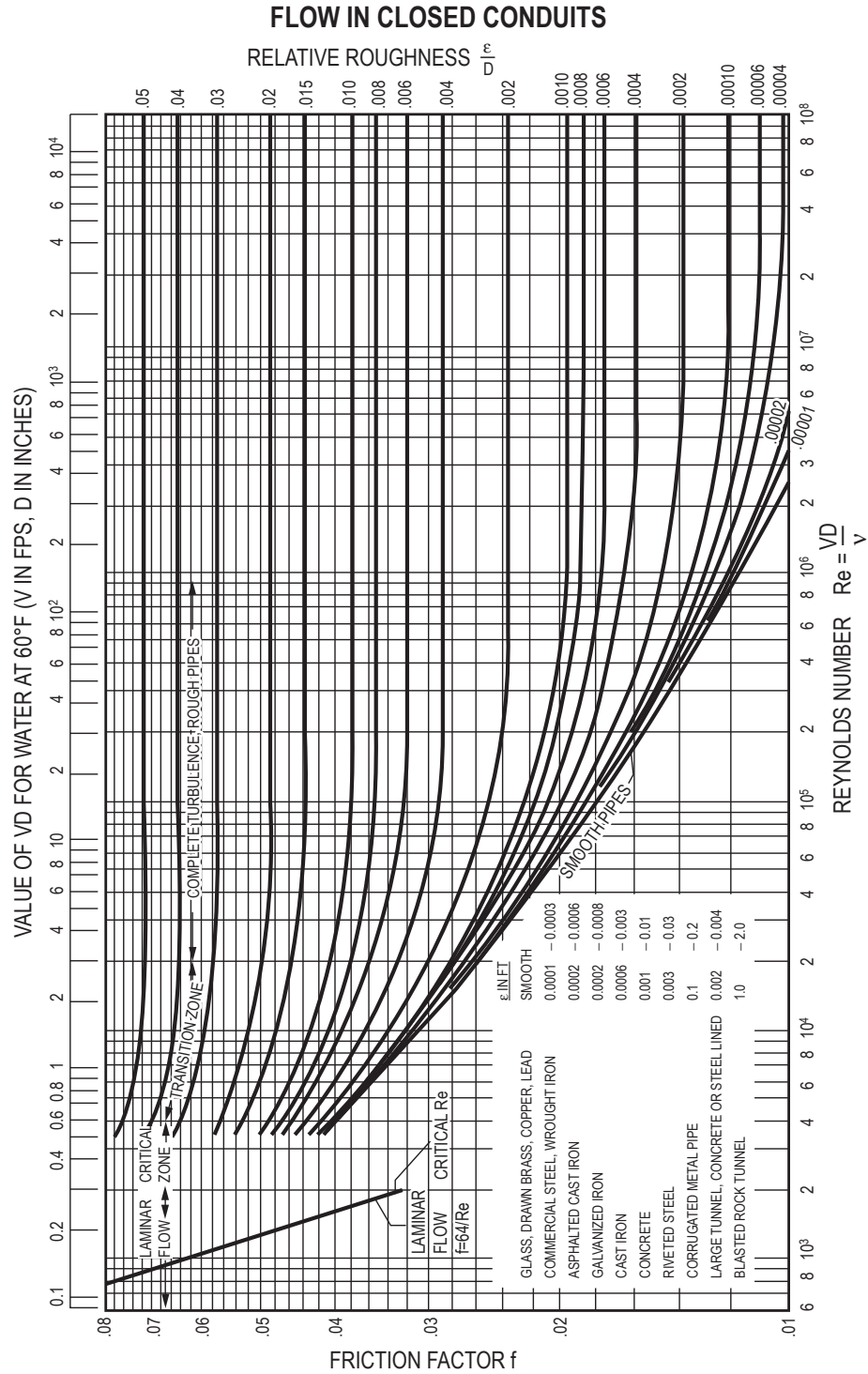
<sup>e</sup>From J.H. Keenan and F.G. Keyes, *Thermodynamic Properties of Steam*, John Wiley & Sons, 1936.

<sup>f</sup>Compiled from many sources including those indicated: *Handbook of Chemistry and Physics*, 54th ed.,

The CRC Press, 1973, and *Handbook of Tables for Applied Engineering Science*, The Chemical Rubber Co., 1970.

Vennard, J.K. and Robert L. Street, *Elementary Fluid Mechanics*, 6th ed., 1982. Reproduced with permission of John Wiley & Sons.

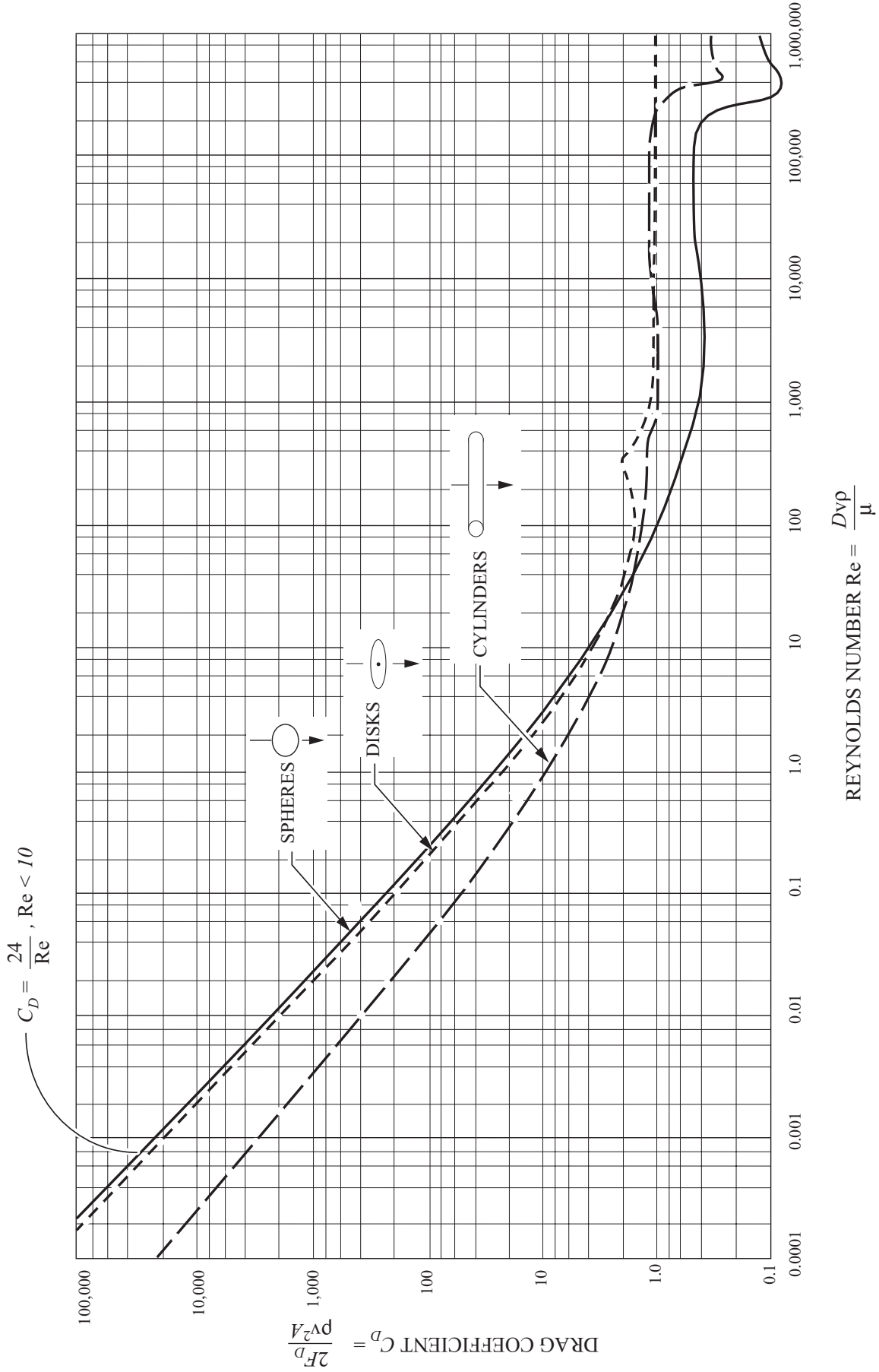
# Moody (Stanton) Diagram



Chow, Ven Te, *Handbook of Applied Hydrology*, McGraw-Hill, 1964.

# Drag Coefficient for Spheres, Disks, and Cylinders

$$C_D = \frac{24}{Re}, Re < 10$$



Note: Intermediate divisions are 2, 4, 6, and 8

# HEAT TRANSFER

There are three modes of heat transfer: conduction, convection, and radiation.

## BASIC HEAT TRANSFER RATE EQUATIONS

### Conduction

Fourier's Law of Conduction

$$\dot{Q} = -kA \frac{dT}{dx}, \text{ where}$$

$\dot{Q}$  = rate of heat transfer (W)

$k$  = the thermal conductivity [W/(m•K)]

$A$  = the surface area perpendicular to direction of heat transfer (m<sup>2</sup>)

### Convection

Newton's Law of Cooling

$$\dot{Q} = hA(T_w - T_\infty), \text{ where}$$

$h$  = the convection heat transfer coefficient of the fluid [W/(m<sup>2</sup>•K)]

$A$  = the convection surface area (m<sup>2</sup>)

$T_w$  = the wall surface temperature (K)

$T_\infty$  = the bulk fluid temperature (K)

### Radiation

The radiation emitted by a body is given by

$$\dot{Q} = \varepsilon \sigma AT^4, \text{ where}$$

$\varepsilon$  = the emissivity of the body

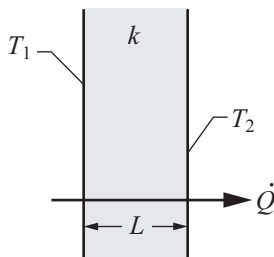
$\sigma$  = the Stefan-Boltzmann constant  
=  $5.67 \times 10^{-8}$  W/(m<sup>2</sup>•K<sup>4</sup>)

$A$  = the body surface area (m<sup>2</sup>)

$T$  = the absolute temperature (K)

## CONDUCTION

### Conduction Through a Plane Wall



$$\dot{Q} = \frac{-kA(T_2 - T_1)}{L}, \text{ where}$$

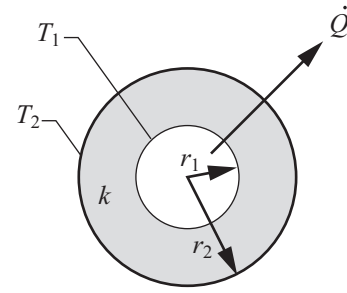
$A$  = wall surface area normal to heat flow (m<sup>2</sup>)

$L$  = wall thickness (m)

$T_1$  = temperature of one surface of the wall (K)

$T_2$  = temperature of the other surface of the wall (K)

### Conduction Through a Cylindrical Wall

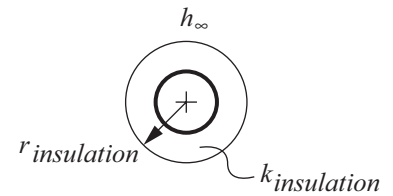


Cylinder (Length =  $L$ )

$$\dot{Q} = \frac{2\pi kL(T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$

### Critical Insulation Radius

$$r_{cr} = \frac{k_{insulation}}{h_\infty}$$



### Thermal Resistance (R)

$$\dot{Q} = \frac{\Delta T}{R_{total}}$$

Resistances in series are added:  $R_{total} = \sum R$ , where

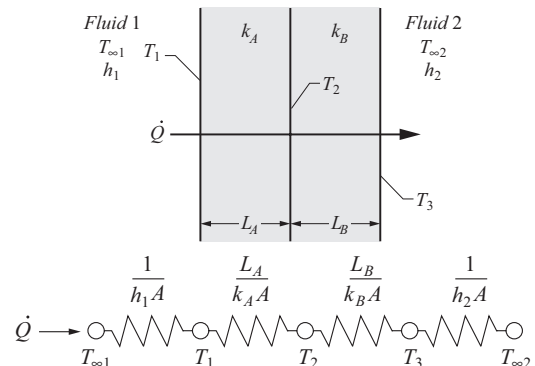
Plane Wall Conduction Resistance (K/W):  $R = \frac{L}{kA}$ , where  
 $L$  = wall thickness

Cylindrical Wall Conduction Resistance (K/W):  $R = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi kL}$ , where  
 $L$  = cylinder length

$L$  = cylinder length

Convection Resistance (K/W) :  $R = \frac{1}{hA}$

### Composite Plane Wall



To evaluate surface or intermediate temperatures:

$$\dot{Q} = \frac{T_1 - T_2}{R_A} = \frac{T_2 - T_3}{R_B}$$

## Transient Conduction Using the Lumped Capacitance Model

The lumped capacitance model is valid if

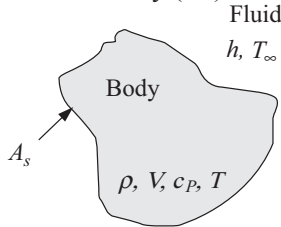
$$\text{Biot number, } Bi = \frac{hV}{kA_s} \ll 1, \text{ where}$$

$h$  = the convection heat transfer coefficient of the fluid  
[W/(m<sup>2</sup>•K)]

$V$  = the volume of the body (m<sup>3</sup>)

$k$  = thermal conductivity of the body [W/(m•K)]

$A_s$  = the surface area of the body (m<sup>2</sup>)



### Constant Fluid Temperature

If the temperature may be considered uniform within the body at any time, the heat transfer rate at the body surface is given by

$$\dot{Q} = hA_s(T - T_\infty) = -\rho V(c_P)\left(\frac{dT}{dt}\right), \text{ where}$$

$T$  = the body temperature (K)

$T_\infty$  = the fluid temperature (K)

$\rho$  = the density of the body (kg/m<sup>3</sup>)

$c_P$  = the heat capacity of the body [J/(kg•K)]

$t$  = time (s)

The temperature variation of the body with time is

$$T - T_\infty = (T_i - T_\infty)e^{-\beta t}, \text{ where}$$

$$\beta = \frac{hA_s}{\rho V c_P} \quad \text{where } \beta = \frac{1}{\tau} \text{ and } \tau = \text{time constant (s)}$$

The total heat transferred ( $Q_{total}$ ) up to time  $t$  is

$$Q_{total} = \rho V c_P (T_i - T), \text{ where}$$

$T_i$  = initial body temperature (K)

### Fins

For a straight fin with uniform cross section (assuming negligible heat transfer from tip),

$$\dot{Q} = \sqrt{hPkA_c}(T_b - T_\infty)\tanh(mL_c), \text{ where}$$

$h$  = the convection heat transfer coefficient of the fluid  
[W/(m<sup>2</sup>•K)]

$P$  = perimeter of exposed fin cross section (m)

$k$  = fin thermal conductivity [W/(m•K)]

$A_c$  = fin cross-sectional area (m<sup>2</sup>)

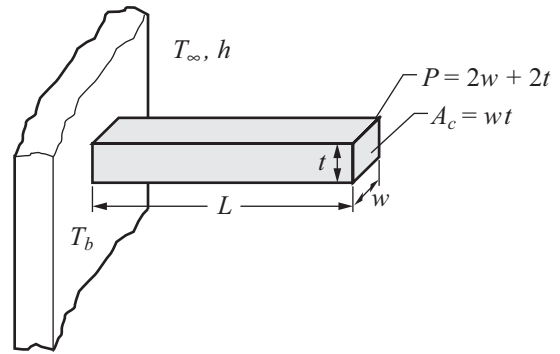
$T_b$  = temperature at base of fin (K)

$T_\infty$  = fluid temperature (K)

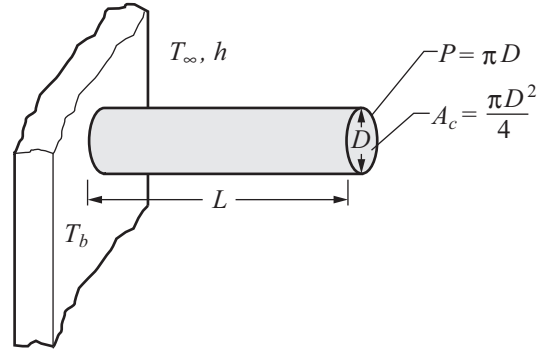
$$m = \sqrt{\frac{hP}{kA_c}}$$

$$L_c = L + \frac{A_c}{P}, \text{ corrected length of fin (m)}$$

### Rectangular Fin



### Pin Fin



## CONVECTION

### Terms

$D$  = diameter (m)

$\bar{h}$  = average convection heat transfer coefficient of the fluid  
[W/(m<sup>2</sup>•K)]

$L$  = length (m)

$\overline{Nu}$  = average Nusselt number

$Pr$  = Prandtl number =  $\frac{c_P \mu}{k}$

$u_m$  = mean velocity of fluid (m/s)

$u_\infty$  = free stream velocity of fluid (m/s)

$\mu$  = dynamic viscosity of fluid [kg/(s•m)]

$\rho$  = density of fluid (kg/m<sup>3</sup>)

### External Flow

In all cases, evaluate fluid properties at average temperature between that of the body and that of the flowing fluid.

### Flat Plate of Length $L$ in Parallel Flow

$$Re_L = \frac{\rho u_\infty L}{\mu}$$

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.6640 Re_L^{1/2} Pr^{1/3} \quad (Re_L < 10^5)$$

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.0366 Re_L^{0.8} Pr^{1/3} \quad (Re_L > 10^5)$$



### Cylinder of Diameter $D$ in Cross Flow

$$\text{Re}_D = \frac{\rho u_\infty D}{\mu}$$

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = C \text{Re}_D^n \text{Pr}^{1/3}, \text{ where}$$

$\text{Re}_D$	$C$	$n$
1 – 4	0.989	0.330
4 – 40	0.911	0.385
40 – 4,000	0.683	0.466
4,000 – 40,000	0.193	0.618
40,000 – 250,000	0.0266	0.805

### Flow Over a Sphere of Diameter, $D$

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = 2.0 + 0.60 \text{Re}_D^{1/2} \text{Pr}^{1/3},$$

$$(1 < \text{Re}_D < 70,000; 0.6 < \text{Pr} < 400)$$

### Internal Flow

$$\text{Re}_D = \frac{\rho u_m D}{\mu}$$

### Laminar Flow in Circular Tubes

For laminar flow ( $\text{Re}_D < 2300$ ), fully developed conditions

$$\text{Nu}_D = 4.36 \quad (\text{uniform heat flux})$$

$$\text{Nu}_D = 3.66 \quad (\text{constant surface temperature})$$

For laminar flow ( $\text{Re}_D < 2300$ ), combined entry length with constant surface temperature

$$\text{Nu}_D = 1.86 \left( \frac{\text{Re}_D \text{Pr}}{L/D} \right)^{1/3} \left( \frac{\mu_b}{\mu_s} \right)^{0.14}, \text{ where}$$

$L$  = length of tube (m)

$D$  = tube diameter (m)

$\mu_b$  = dynamic viscosity of fluid [kg/(s·m)] at bulk temperature of fluid,  $T_b$

$\mu_s$  = dynamic viscosity of fluid [kg/(s·m)] at inside surface temperature of the tube,  $T_s$

### Turbulent Flow in Circular Tubes

For turbulent flow ( $\text{Re}_D > 10^4$ ,  $\text{Pr} > 0.7$ ) for either uniform surface temperature or uniform heat flux condition, Sieder-Tate equation offers good approximation:

$$\text{Nu}_D = 0.023 \text{Re}_D^{0.8} \text{Pr}^{1/3} \left( \frac{\mu_b}{\mu_s} \right)^{0.14}$$

### Noncircular Ducts

In place of the diameter,  $D$ , use the equivalent (hydraulic) diameter ( $D_H$ ) defined as

$$D_H = \frac{4 \times \text{cross-sectional area}}{\text{wetted perimeter}}$$

### Circular Annulus ( $D_o > D_i$ )

In place of the diameter,  $D$ , use the equivalent (hydraulic) diameter ( $D_H$ ) defined as

$$D_H = D_o - D_i$$

### Liquid Metals ( $0.003 < \text{Pr} < 0.05$ )

$$\text{Nu}_D = 6.3 + 0.0167 \text{Re}_D^{0.85} \text{Pr}^{0.93} \quad (\text{uniform heat flux})$$

$$\text{Nu}_D = 7.0 + 0.025 \text{Re}_D^{0.8} \text{Pr}^{0.8} \quad (\text{constant wall temperature})$$

### Boiling

Evaporation occurring at a solid-liquid interface when

$$T_{\text{solid}} > T_{\text{sat, liquid}}$$

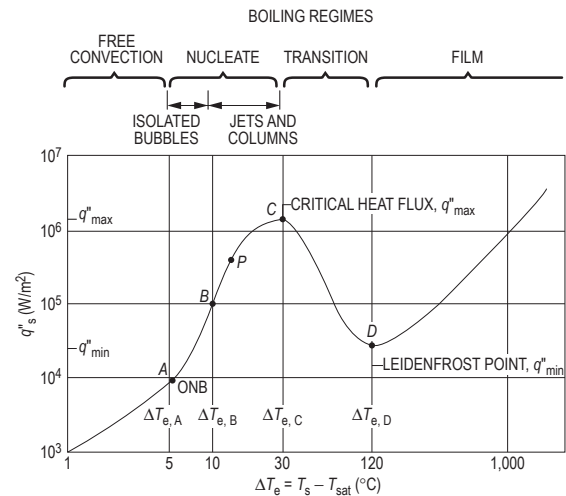
$$q'' = h(T_s - T_{\text{sat}}) = h\Delta T_e, \text{ where } \Delta T_e = \text{excess temperature}$$

**Pool Boiling** – Liquid is quiescent; motion near solid surface is due to free convection and mixing induced by bubble growth and detachment.

**Forced Convection Boiling** – Fluid motion is induced by external means in addition to free convection and bubble-induced mixing.

**Sub-Cooled Boiling** – Temperature of liquid is below saturation temperature; bubbles forming at surface may condense in the liquid.

**Saturated Boiling** – Liquid temperature slightly exceeds the saturation temperature; bubbles forming at the surface are propelled through liquid by buoyancy forces.



Typical boiling curve for water at one atmosphere: surface heat flux  $q''_s$  as a function of excess temperature,  $\Delta T_e = T_s - T_{\text{sat}}$

**Free Convection Boiling** – Insufficient vapor is in contact with the liquid phase to cause boiling at the saturation temperature.

**Nucleate Boiling** – Isolated bubbles form at nucleation sites and separate from surface; vapor escapes as jets or columns.

◆ Incropera, Frank P. and David P. DeWitt, *Fundamentals of Heat and Mass Transfer*, 3rd ed., Wiley, 1990. Reproduced with permission of John Wiley & Sons, Inc.

- ◆ For nucleate boiling a widely used correlation was proposed in 1952 by Rohsenow:

$$\dot{q}_{\text{nucleate}} = \mu_l h_{fg} \left[ \frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/2} \left[ \frac{c_{pl}(T_s - T_{\text{sat}})}{C_{sf} h_{fg} \text{Pr}_l^n} \right]^3$$

- $\dot{q}_{\text{nucleate}}$  = nucleate boiling heat flux, W/m<sup>2</sup>  
 $\mu_l$  = viscosity of the liquid, kg/(m·s)  
 $h_{fg}$  = enthalpy of vaporization, J/kg  
 $g$  = gravitational acceleration, m/s<sup>2</sup>  
 $\rho_l$  = density of the liquid, kg/m<sup>3</sup>  
 $\rho_v$  = density of the vapor, kg/m<sup>3</sup>  
 $\sigma$  = surface tension of liquid–vapor interface, N/m  
 $c_{pl}$  = specific heat of the liquid, J/(kg·°C)  
 $T_s$  = surface temperature of the heater, °C  
 $T_{\text{sat}}$  = saturation temperature of the fluid, °C  
 $C_{sf}$  = experimental constant that depends on surface–fluid combination  
 $\text{Pr}_l$  = Prandtl number of the liquid  
 $n$  = experimental constant that depends on the fluid

#### ◆ Peak Heat Flux

The maximum (or critical) heat flux (CHF) in nucleate pool boiling:

$$\dot{q}_{\text{max}} = C_{cr} h_{fg} [\sigma g \rho_v^2 (\rho_l - \rho_v)]^{1/4}$$

$C_{cr}$  is a constant whose value depends on the heater geometry, but generally is about 0.15.

The CHF is independent of the fluid–heating surface combination, as well as the viscosity, thermal conductivity, and specific heat of the liquid.

The CHF increases with pressure up to about one-third of the critical pressure, and then starts to decrease and becomes zero at the critical pressure.

The CHF is proportional to  $h_{fg}$ , and large maximum heat fluxes can be obtained using fluids with a large enthalpy of vaporization, such as water.



Values of the coefficient  $C_{cr}$  for maximum heat flux (dimensionless parameter  $L^* = L[g(\rho_l - \rho_v)/\sigma]^{1/2}$ )

Heater Geometry	$C_{cr}$	Charac. Dimension of Heater, $L$	Range of $L^*$
Large horizontal flat heater	0.149	Width or diameter	$L^* > 27$
Small horizontal flat heater <sup>1</sup>	$18.9 K_1$	Width or diameter	$9 < L^* < 20$
Large horizontal cylinder	0.12	Radius	$L^* > 1.2$
Small horizontal cylinder	$0.12 L^{*-0.25}$	Radius	$0.15 < L^* < 1.2$
Large sphere	0.11	Radius	$L^* > 4.26$
Small sphere	$0.227 L^{*-0.5}$	Radius	$0.15 < L^* < 4.26$

<sup>1</sup> $K_1 = \sigma/[g(\rho_l - \rho_v)A_{\text{heater}}]$

#### ◆ Minimum Heat Flux

Minimum heat flux, which occurs at the Leidenfrost point, is of practical interest since it represents the lower limit for the heat flux in the film boiling regime.

Zuber derived the following expression for the minimum heat flux for a large horizontal plate

$$\dot{q}_{\text{min}} = 0.09 \rho_v h_{fg} \left[ \frac{\sigma g (\rho_l - \rho_v)}{(\rho_l + \rho_v)^2} \right]^{1/4}$$

The relation above can be in error by 50% or more.

*Transition Boiling* – Rapid bubble formation results in vapor film on surface and oscillation between film and nucleate boiling.

*Film Boiling* – Surface completely covered by vapor blanket; includes significant radiation through vapor film.

#### ◆ Film Boiling

The heat flux for film boiling on a horizontal cylinder or sphere of diameter  $D$  is given by

$$\dot{q}_{\text{film}} = C_{\text{film}} \left[ \frac{g k_v^3 \rho_v (\rho_l - \rho_v) [h_{fg} + 0.4 c_{pv} (T_s - T_{\text{sat}})]}{\mu_v D (T_s - T_{\text{sat}})} \right]^{1/4} (T_s - T_{\text{sat}})$$

$$C_{\text{film}} = \begin{cases} 0.62 & \text{for horizontal cylinders} \\ 0.67 & \text{for spheres} \end{cases}$$

#### Film Condensation of a Pure Vapor On a Vertical Surface

$$\overline{Nu}_L = \frac{\bar{h}L}{k_l} = 0.943 \left[ \frac{\rho_l^2 g h_{fg} L^3}{\mu_l k_l (T_{\text{sat}} - T_s)} \right]^{0.25}, \text{ where}$$

- $\rho_l$  = density of liquid phase of fluid (kg/m<sup>3</sup>)  
 $g$  = gravitational acceleration (9.81 m/s<sup>2</sup>)  
 $h_{fg}$  = latent heat of vaporization [J/kg]  
 $L$  = length of surface [m]  
 $\mu_l$  = dynamic viscosity of liquid phase of fluid [kg/(s·m)]  
 $k_l$  = thermal conductivity of liquid phase of fluid [W/(m·K)]  
 $T_{\text{sat}}$  = saturation temperature of fluid [K]  
 $T_s$  = temperature of vertical surface [K]

Note: Evaluate all liquid properties at the average temperature between the saturated temperature,  $T_{\text{sat}}$ , and the surface temperature,  $T_s$ .

#### Outside Horizontal Tubes

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = 0.729 \left[ \frac{\rho_l^2 g h_{fg} D^3}{\mu_l k_l (T_{\text{sat}} - T_s)} \right]^{0.25}, \text{ where}$$

$D$  = tube outside diameter (m)

Note: Evaluate all liquid properties at the average temperature between the saturated temperature,  $T_{\text{sat}}$ , and the surface temperature,  $T_s$ .

◆ Cengel, Yunus A., and Afshin J. Ghajar, *Heat and Mass Transfer*, 4th ed., McGraw-Hill, 2011.

## Natural (Free) Convection

### Vertical Flat Plate in Large Body of Stationary Fluid

Equation also can apply to vertical cylinder of sufficiently large diameter in large body of stationary fluid.

$$\bar{h} = C \left( \frac{k}{L} \right) Ra_L^n, \text{ where}$$

$L$  = the length of the plate (cylinder) in the vertical direction

$$Ra_L = \text{Rayleigh Number} = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} \text{Pr}$$

$T_s$  = surface temperature (K)

$T_\infty$  = fluid temperature (K)

$\beta$  = coefficient of thermal expansion (1/K)

(For an ideal gas:  $\beta = \frac{2}{T_s + T_\infty}$  with  $T$  in absolute temperature)

$\nu$  = kinematic viscosity ( $\text{m}^2/\text{s}$ )

Range of $Ra_L$	$C$	$n$
$10^4 - 10^9$	0.59	1/4
$10^9 - 10^{13}$	0.10	1/3

### Long Horizontal Cylinder in Large Body of Stationary Fluid

$$\bar{h} = C \left( \frac{k}{D} \right) Ra_D^n, \text{ where}$$

$$Ra_D = \frac{g\beta(T_s - T_\infty)D^3}{\nu^2} \text{Pr}$$

$Ra_D$	$C$	$n$
$10^{-3} - 10^2$	1.02	0.148
$10^2 - 10^4$	0.850	0.188
$10^4 - 10^7$	0.480	0.250
$10^7 - 10^{12}$	0.125	0.333

## Heat Exchangers

The rate of heat transfer in a heat exchanger is

$$\dot{Q} = UAF\Delta T_{lm}, \text{ where}$$

$A$  = any convenient reference area ( $\text{m}^2$ )

$F$  = heat exchanger configuration correction factor  
( $F = 1$  if temperature change of one fluid is negligible)

$U$  = overall heat transfer coefficient based on area  $A$  and the log mean temperature difference [ $\text{W}/(\text{m}^2 \cdot \text{K})$ ]

$\Delta T_{lm}$  = log mean temperature difference (K)

## Overall Heat Transfer Coefficient for Concentric Tube and Shell-and-Tube Heat Exchangers

$$\frac{1}{UA} = \frac{1}{h_i A_i} + \frac{R_{fi}}{A_i} + \frac{\ln\left(\frac{D_o}{D_i}\right)}{2\pi k L} + \frac{R_{fo}}{A_o} + \frac{1}{h_o A_o}, \text{ where}$$

$A_i$  = inside area of tubes ( $\text{m}^2$ )

$A_o$  = outside area of tubes ( $\text{m}^2$ )

$D_i$  = inside diameter of tubes (m)

$D_o$  = outside diameter of tubes (m)

$h_i$  = convection heat transfer coefficient for inside of tubes [ $\text{W}/(\text{m}^2 \cdot \text{K})$ ]

$h_o$  = convection heat transfer coefficient for outside of tubes [ $\text{W}/(\text{m}^2 \cdot \text{K})$ ]

$k$  = thermal conductivity of tube material [ $\text{W}/(\text{m} \cdot \text{K})$ ]

$R_{fi}$  = fouling factor for inside of tube [ $(\text{m}^2 \cdot \text{K})/\text{W}$ ]

$R_{fo}$  = fouling factor for outside of tube [ $(\text{m}^2 \cdot \text{K})/\text{W}$ ]

## Log Mean Temperature Difference (LMTD)

For *counterflow* in tubular heat exchangers

$$\Delta T_{lm} = \frac{(T_{Ho} - T_{Ci}) - (T_{Hi} - T_{Co})}{\ln\left(\frac{T_{Ho} - T_{Ci}}{T_{Hi} - T_{Co}}\right)}$$

For *parallel flow* in tubular heat exchangers

$$\Delta T_{lm} = \frac{(T_{Ho} - T_{Co}) - (T_{Hi} - T_{Ci})}{\ln\left(\frac{T_{Ho} - T_{Co}}{T_{Hi} - T_{Ci}}\right)}, \text{ where}$$

$\Delta T_{lm}$  = log mean temperature difference (K)

$T_{Hi}$  = inlet temperature of the hot fluid (K)

$T_{Ho}$  = outlet temperature of the hot fluid (K)

$T_{Ci}$  = inlet temperature of the cold fluid (K)

$T_{Co}$  = outlet temperature of the cold fluid (K)

## Heat Exchanger Effectiveness, $\epsilon$

$$\epsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{\text{actual heat transfer rate}}{\text{maximum possible heat transfer rate}}$$

$$\epsilon = \frac{C_H (T_{Hi} - T_{Ho})}{C_{\min} (T_{Hi} - T_{Ci})} \quad \text{or} \quad \epsilon = \frac{C_C (T_{Co} - T_{Ci})}{C_{\min} (T_{Hi} - T_{Ci})}$$

where

$C = \dot{m}c_p$  = heat capacity rate (W/K)

$C_{\min}$  = smaller of  $C_C$  or  $C_H$

### Number of Transfer Units (NTU)

$$NTU = \frac{UA}{C_{\min}}$$

### Effectiveness-NTU Relations

$$C_r = \frac{C_{\min}}{C_{\max}} = \text{heat capacity ratio}$$

For *parallel flow concentric tube* heat exchanger

$$\varepsilon = \frac{1 - \exp[-NTU(1 + C_r)]}{1 + C_r}$$

$$NTU = -\frac{\ln[1 - \varepsilon(1 + C_r)]}{1 + C_r}$$

For *counterflow concentric tube* heat exchanger

$$\varepsilon = \frac{1 - \exp[-NTU(1 - C_r)]}{1 - C_r \exp[-NTU(1 - C_r)]} \quad (C_r < 1)$$

$$\varepsilon = \frac{NTU}{1 + NTU} \quad (C_r = 1)$$

$$NTU = \frac{1}{C_r - 1} \ln\left(\frac{\varepsilon - 1}{\varepsilon C_r - 1}\right) \quad (C_r < 1)$$

$$NTU = \frac{\varepsilon}{1 - \varepsilon} \quad (C_r = 1)$$

## **RADIATION**

### **Types of Bodies**

#### Any Body

For any body,  $\alpha + \rho + \tau = 1$ , where

$\alpha$  = absorptivity (ratio of energy absorbed to incident energy)

$\rho$  = reflectivity (ratio of energy reflected to incident energy)

$\tau$  = transmissivity (ratio of energy transmitted to incident energy)

#### Opaque Body

For an opaque body:  $\alpha + \rho = 1$

#### Gray Body

A gray body is one for which

$$\alpha = \varepsilon, (0 < \alpha < 1; 0 < \varepsilon < 1), \text{ where}$$

$\varepsilon$  = the emissivity of the body

For a gray body:  $\varepsilon + \rho = 1$

*Real bodies* are frequently approximated as gray bodies.

### Black body

A black body is defined as one that absorbs all energy incident upon it. It also emits radiation at the maximum rate for a body of a particular size at a particular temperature. For such a body

$$\alpha = \varepsilon = 1$$

### **Shape Factor (View Factor, Configuration Factor)**

#### **Relations**

#### Reciprocity Relations

$$A_i F_{ij} = A_j F_{ji}, \text{ where}$$

$A_i$  = surface area ( $m^2$ ) of surface  $i$

$F_{ij}$  = shape factor (view factor, configuration factor); fraction of the radiation leaving surface  $i$  that is intercepted by surface  $j$ ;  $0 \leq F_{ij} \leq 1$

#### Summation Rule for $N$ Surfaces

$$\sum_{j=1}^N F_{ij} = 1$$

### **Net Energy Exchange by Radiation between Two Bodies**

#### Body Small Compared to its Surroundings

$$\dot{Q}_{12} = \varepsilon \sigma A (T_1^4 - T_2^4), \text{ where}$$

$\dot{Q}_{12}$  = the net heat transfer rate from the body (W)

$\varepsilon$  = the emissivity of the body

$\sigma$  = the Stefan-Boltzmann constant

$$[\sigma = 5.67 \times 10^{-8} \text{ W}/(\text{m}^2 \cdot \text{K}^4)]$$

$A$  = the body surface area ( $m^2$ )

$T_1$  = the absolute temperature (K) of the body surface

$T_2$  = the absolute temperature (K) of the surroundings

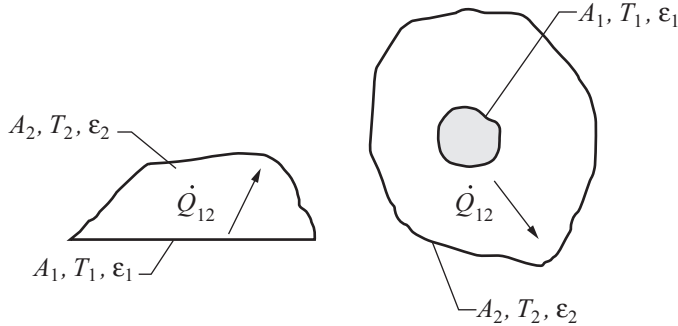
### Net Energy Exchange by Radiation between Two Black Bodies

The net energy exchange by radiation between two black bodies that see each other is given by

$$\dot{Q}_{12} = A_1 F_{12} \sigma (T_1^4 - T_2^4)$$

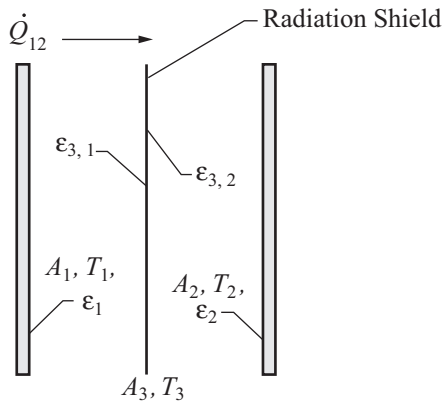
### Net Energy Exchange by Radiation between Two Diffuse-Gray Surfaces that Form an Enclosure

#### *Generalized Cases*



$$\dot{Q}_{12} = \frac{\sigma (T_1^4 - T_2^4)}{\frac{1 - \epsilon_1}{\epsilon_1 A_1} + \frac{1}{A_1 F_{12}} + \frac{1 - \epsilon_2}{\epsilon_2 A_2}}$$

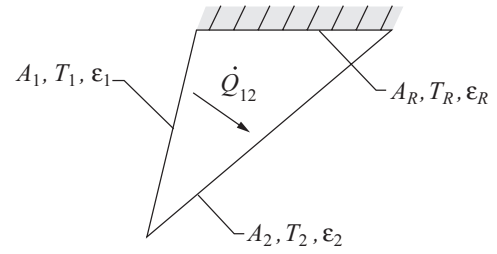
### One-Dimensional Geometry with Thin Low-Emissivity Shield Inserted between Two Parallel Plates



$$\dot{Q}_{12} = \frac{\sigma (T_1^4 - T_2^4)}{\frac{1 - \epsilon_1}{\epsilon_1 A_1} + \frac{1}{A_1 F_{13}} + \frac{1 - \epsilon_{3,1}}{\epsilon_{3,1} A_3} + \frac{1 - \epsilon_{3,2}}{\epsilon_{3,2} A_3} + \frac{1}{A_3 F_{32}} + \frac{1 - \epsilon_2}{\epsilon_2 A_2}}$$

### Reradiating Surface

Reradiating Surfaces are considered to be insulated or adiabatic ( $\dot{Q}_R = 0$ ).



$$\dot{Q}_{12} = \frac{\sigma (T_1^4 - T_2^4)}{\frac{1 - \epsilon_1}{\epsilon_1 A_1} + \frac{1}{A_1 F_{12} + \left[ \left( \frac{1}{A_1 F_{1R}} \right) + \left( \frac{1}{A_2 F_{2R}} \right) \right]^{-1}} + \frac{1 - \epsilon_2}{\epsilon_2 A_2}}$$

# INSTRUMENTATION, MEASUREMENT, AND CONTROLS

## MEASUREMENT

### Definitions

**Transducer** – a device used to convert a physical parameter such as temperature, pressure, flow, light intensity, etc. into an electrical signal (also called a *sensor*).

**Transducer sensitivity** – the ratio of change in electrical signal magnitude to the change in magnitude of the physical parameter being measured.

**Resistance Temperature Detector (RTD)** – a device used to relate change in resistance to change in temperature. Typically made from platinum, the controlling equation for an RTD is given by:

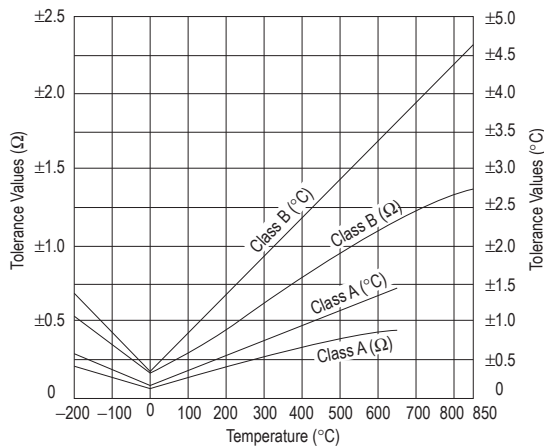
$$R_T = R_0[1 + \alpha(T - T_0)], \text{ where}$$

$R_T$  = resistance of the RTD at temperature  $T$  (in °C)

$R_0$  = resistance of the RTD at the reference temperature  $T_0$  (usually 0°C)

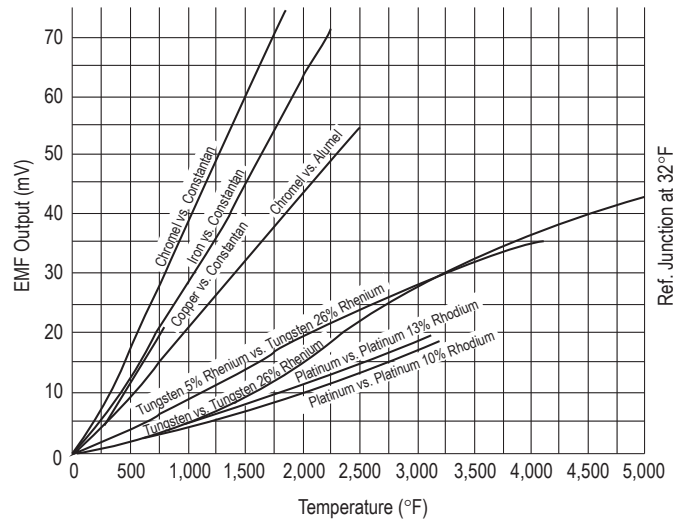
$\alpha$  = temperature coefficient of the RTD

The following graph shows tolerance values as a function of temperature for 100-Ω RTDs.

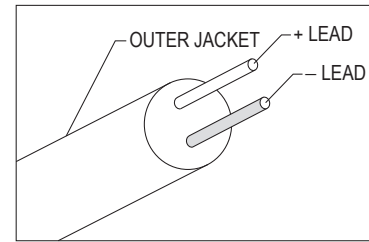


From Tempco Manufactured Products, as posted on [www.tempco.com](http://www.tempco.com), July 2013.

**Thermocouple (TC)** – a device used to relate change in voltage to change in temperature. A thermocouple consists of two dissimilar conductors in contact, which produce a voltage when heated.



From Convectronics Inc., as posted on [www.convectronics.com](http://www.convectronics.com), July 2013.



Typical Thermocouple (TC) Cable

From Convectronics Inc., as posted on [www.convectronics.com](http://www.convectronics.com), July 2013.

ANSI Code	Alloy Combination and Color		Outer Jacket Color		Maximum Thermocouple Temperature Range	Environment
	+ Lead	- Lead	Thermocouple Leads	Extension Cable		
J	IRON Fe (magnetic) White	CONSTANTAN COPPER-NICKEL Cu-Ni Red	Brown	Black	-346 to 2,193°F -210 to 1,200°C	Reducing, Vacuum, Inert. Limited use in Oxidizing at High Temperatures. Not Recommended for Low Temperatures
K	NICKELCHROMIUM Ni-Cr Yellow	NICKEL-ALUMINUM Ni-Al (magnetic) Red	Brown	Yellow	-454 to 2,501°F -270 to 1,372°C	Clean Oxidizing and Inert. Limited Use in Vacuum or Reducing.
T	COPPER Cu Blue	CONSTANTAN COPPER-NICKEL Cu-Ni Red	Brown	Blue	-454 to 752°F -270 to 400°C	Mild Oxidizing, Reducing Vacuum or Inert. Good where moisture is present.
E	NICKELCHROMIUM Ni-Cr Purple	CONSTANTAN COPPER-NICKEL Cu-Ni Red	Brown	Purple	-454 to 1,832°F -270 to 1,000°C	Oxidizing or Inert. Limited Use in Vacuum or Reducing.

From Omega Engineering, as posted on [www.omega.com](http://www.omega.com), July 2013.

*Strain Gage* – a device whose electrical resistance varies in proportion to the amount of strain in the device.

*Gage factor (GF)* – the ratio of fractional change in electrical resistance to the fractional change in length (strain):

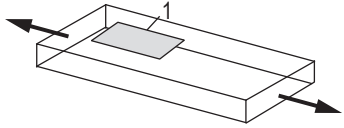
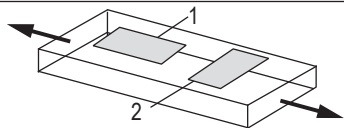
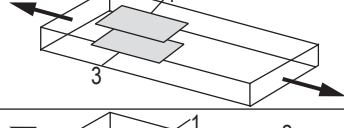
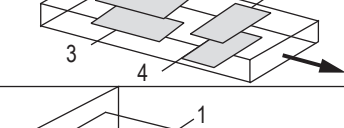
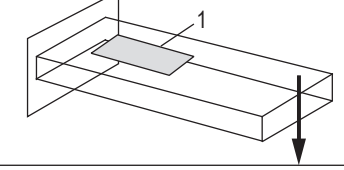
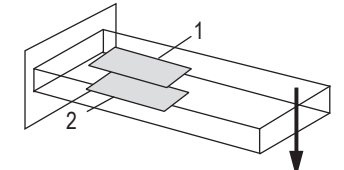
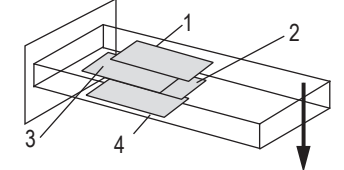
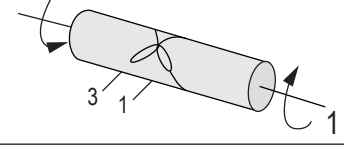
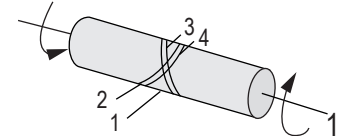
$$GF = \frac{\Delta R/R}{\Delta L/L} = \frac{\Delta R/R}{\epsilon}, \text{ where}$$

$R$  = nominal resistance of the strain gage at nominal length  $L$

$\Delta R$  = change in resistance due the change in length  $\Delta L$

$\epsilon$  = normal strain sensed by the gage

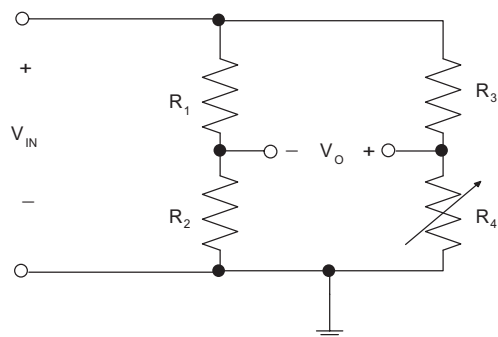
The gage factor for metallic strain gages is typically around 2.

Strain	Gage Setup	Bridge Type	Sensitivity mV/V @ 100 $\mu\epsilon$	Details
Axial		1/4	0.5	Good: Simplest to implement, but must use a dummy gage if compensating for temperature. Also responds to bending strain.
		1/2	0.65	Better: Temperature compensated, but it is sensitive to bending strain.
		1/2	1.0	Better: Rejects bending strain, but not temperature. Must use dummy gages if compensating for temperature
		Full	1.3	Best: More sensitive and compensates for both temperature and bending strain.
Bending		1/4	0.5	Good: Simplest to implement, but must use a dummy gage if compensating for temperature. Responds equally to axial strain.
		1/2	1.0	Better: Rejects axial strain and is temperature compensated.
		Full	2.0	Best: Rejects axial strain and is temperature compensated. Most sensitive to bending strain.
Torsional and Shear		1/2	1.0	Good: Gages must be mounted at 45 degrees from centerline.
		Full	2.0	Best: Most sensitive full-bridge version of previous setup. Rejects both axial and bending strains.

From National Instruments Corporation, as posted on [www.ni.com](http://www.ni.com), July 2013.



**Wheatstone Bridge** – an electrical circuit used to measure changes in resistance.



WHEATSTONE BRIDGE

If  $\frac{R_1}{R_2} = \frac{R_3}{R_4}$  then  $V_0 = 0$  V and the bridge is said to be *balanced*.

If  $R_1 = R_2 = R_3 = R$  and  $R_4 = R + \Delta R$ , where  $\Delta R \ll R$ , then

$$V_0 \approx \frac{\Delta R}{4R} \cdot V_{IN}$$

◆ **Pressure sensors** – can alternatively be called pressure transducers, pressure transmitters, pressure senders, pressure indicators, piezometers, and manometers.

◆

Pressure Relative Measurement Types	Comparison
Absolute	Relative to 0 Pa, the pressure in a vacuum
Gage	Relative to local atmospheric pressure
Vacuum	Relative to either absolute vacuum (0 Pa) or local atmospheric pressure
Differential	Relative to another pressurized container
Sealed	Relative to sea level pressure

• **pH sensor** – a typical pH meter consists of a special measuring probe connected to an electronic meter that measures and displays the pH reading.

$$E_{el} = E^0 - S(\text{pH}_a - \text{pH}_i)$$

$E_{el}$  = electrode potential

$E^0$  = zero potential

$S$  = slope (mV per pH unit)

$\text{pH}_a$  = pH value of the measured solution

$\text{pH}_i$  = pH value of the internal buffer

◆ From National Instruments Corporation, as posted on [www.ni.com](http://www.ni.com), July 2013.

◆ From Alliance Technical Sales, Inc., as posted on [www.alliancets.com](http://www.alliancets.com), July 2013.

### Examples of Common Chemical Sensors

Sensor Type	Principle	Materials	Analyte
Semiconducting oxide sensor	Conductivity impedance	SnO <sub>2</sub> , TiO <sub>2</sub> , ZnO <sub>2</sub> , WO <sub>3</sub> , polymers	O <sub>2</sub> , H <sub>2</sub> , CO, SO <sub>x</sub> , NO <sub>x</sub> , combustible hydrocarbons, alcohol, H <sub>2</sub> S, NH <sub>3</sub>
Electrochemical sensor (liquid electrolyte)	Amperimetric	composite Pt, Au catalyst	H <sub>2</sub> , O <sub>2</sub> , O <sub>3</sub> , CO, H <sub>2</sub> S, SO <sub>2</sub> , NO <sub>x</sub> , NH <sub>3</sub> , glucose, hydrazine
Ion-selective electrode (ISE)	Potentiometric	glass, LaF <sub>3</sub> , CaF <sub>2</sub>	pH, K <sup>+</sup> , Na <sup>+</sup> , Cl <sup>-</sup> , Ca <sup>2+</sup> , Mg <sup>2+</sup> , F <sup>-</sup> , Ag <sup>+</sup>
Solid electrode sensor	Amperimetric Potentiometric	YSZ, H <sup>+</sup> -conductor YSZ, β-alumina, Nasicon, Nafion	O <sub>2</sub> , H <sub>2</sub> , CO, combustible hydrocarbons, O <sub>2</sub> , H <sub>2</sub> , CO <sub>2</sub> , CO, NO <sub>x</sub> , SO <sub>x</sub> , H <sub>2</sub> S, Cl <sub>2</sub> H <sub>2</sub> O, combustible hydrocarbons
Piezoelectric sensor	Mechanical w/ polymer film	quartz	combustible hydrocarbons, VOCs
Catalytic combustion sensor	Calorimetric	Pt/Al <sub>2</sub> O <sub>3</sub> , Pt-wire	H <sub>2</sub> , CO, combustible hydrocarbons
Pyroelectric sensor	Calorimetric	Pyroelectric + film	Vapors
Optical sensors	Colorimetric fluorescence	optical fiber/indicator dye	Acids, bases, combustible hydrocarbons, biologicals

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## SAMPLING

When a continuous-time or analog signal is sampled using a discrete-time method, certain basic concepts should be considered. The sampling rate or frequency is given by

$$f_s = \frac{1}{\Delta t}$$

Nyquist's (Shannon's) sampling theorem states that in order to accurately reconstruct the analog signal from the discrete sample points, the sample rate must be larger than twice the highest frequency contained in the measured signal. Denoting this frequency, which is called the Nyquist frequency, as  $f_N$ , the sampling theorem requires that

$$f_s > 2f_N$$

When the above condition is not met, the higher frequencies in the measured signal will not be accurately represented and will appear as lower frequencies in the sampled data. These are known as alias frequencies.

### Analog-to-Digital Conversion

When converting an analog signal to digital form, the resolution of the conversion is an important factor. For a measured analog signal over the nominal range  $[V_L, V_H]$ , where  $V_L$  is the low end of the voltage range and  $V_H$  is the nominal high end of the voltage range, the voltage resolution is given by

$$\varepsilon_V = \frac{V_H - V_L}{2^n}$$

where  $n$  is the number of conversion bits of the A/D converter with typical values of 4, 8, 10, 12, or 16. This number is a key design parameter. After converting an analog signal, the A/D converter produces an integer number of  $n$  bits. Call this number  $N$ . Note that the range of  $N$  is  $[0, 2^n - 1]$ . When calculating the discrete voltage,  $V$ , using the reading,  $N$ , from the A/D converter the following equation is used.

$$V = \varepsilon_V N + V_L$$

Note that with this strategy, the highest measurable voltage is one voltage resolution less than  $V_H$ , or  $V_H - \varepsilon_V$ .

### Signal Conditioning

Signal conditioning of the measured analog signal is often required to prevent alias frequencies from being measured, and to reduce measurement errors.

## MEASUREMENT UNCERTAINTY

Suppose that a calculated result  $R$  depends on measurements whose values are  $x_1 \pm w_1$ ,  $x_2 \pm w_2$ ,  $x_3 \pm w_3$ , etc., where  $R = f(x_1, x_2, x_3, \dots, x_n)$ ,  $x_i$  is the measured value, and  $w_i$  is the uncertainty in that value. The uncertainty in  $R$ ,  $w_R$ , can be estimated using the Kline-McClintock equation:

$$w_R = \sqrt{\left(w_1 \frac{\partial f}{\partial x_1}\right)^2 + \left(w_2 \frac{\partial f}{\partial x_2}\right)^2 + \dots + \left(w_n \frac{\partial f}{\partial x_n}\right)^2}$$

## CONTROL SYSTEMS

The linear time-invariant transfer function model represented by the block diagram

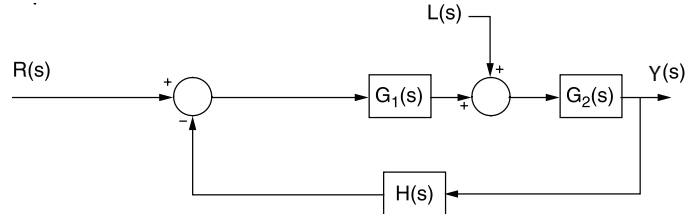


can be expressed as the ratio of two polynomials in the form

$$\frac{Y(s)}{X(s)} = G(s) = \frac{N(s)}{D(s)} = K \frac{\prod_{m=1}^M (s - z_m)}{\prod_{n=1}^N (s - p_n)}$$

where the  $M$  zeros,  $z_m$ , and the  $N$  poles,  $p_n$ , are the roots of the numerator polynomial,  $N(s)$ , and the denominator polynomial,  $D(s)$ , respectively.

One classical negative feedback control system model block diagram is



where  $G_1(s)$  is a controller or compensator,  $G_2(s)$  represents a plant model, and  $H(s)$  represents the measurement dynamics.  $Y(s)$  represents the controlled variable,  $R(s)$  represents the reference input, and  $L(s)$  represents a disturbance.  $Y(s)$  is related to  $R(s)$  and  $L(s)$  by

$$Y(s) = \frac{G_1(s) G_2(s)}{1 + G_1(s) G_2(s) H(s)} R(s) + \frac{G_2(s)}{1 + G_1(s) G_2(s) H(s)} L(s)$$

$G_1(s) G_2(s) H(s)$  is the open-loop transfer function. The closed-loop characteristic equation is

$$1 + G_1(s) G_2(s) H(s) = 0$$

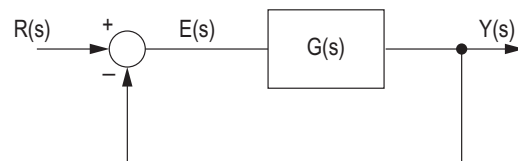
System performance studies normally include

1. Steady-state analysis using constant inputs based on the Final Value Theorem. If all poles of a  $G(s)$  function have negative real parts, then

$$\text{dc gain} = \lim_{s \rightarrow 0} G(s)$$

Note that  $G(s)$  could refer to either an open-loop or a closed-loop transfer function.

For the unity feedback control system model



with the open-loop transfer function defined by

$$G(s) = \frac{K_B}{s^T} \times \frac{\prod_{m=1}^M (1 + s/\omega_m)}{\prod_{n=1}^N (1 + s/\omega_n)}$$

The following steady-state error analysis table can be constructed where  $T$  denotes the type of system, i.e., type 0, type 1, etc.

Steady-State Error $e_{ss}$			
Input \ Type	$T = 0$	$T = 1$	$T = 2$
Unit Step	$1/(K_B + 1)$	0	0
Ramp	$\infty$	$1/K_B$	0
Acceleration	$\infty$	$\infty$	$1/K_B$

2. Frequency response evaluations to determine dynamic performance and stability. For example, relative stability can be quantified in terms of
  - a. Gain margin (GM), which is the additional gain required to produce instability in the unity gain feedback control system. If at  $\omega = \omega_{180}$ ,  $\angle G(j\omega_{180}) = -180^\circ$ ; then
 
$$GM = -20 \log_{10} (|G(j\omega_{180})|)$$
  - b. Phase margin (PM), which is the additional phase required to produce instability. Thus,
 
$$PM = 180^\circ + \angle G(j\omega_{0dB})$$
 where  $\omega_{0dB}$  is the  $\omega$  that satisfies  $|G(j\omega)| = 1$ .
3. Transient responses are obtained by using Laplace transforms or computer solutions with numerical integration.

Common Compensator/Controller forms are

PID Controller  $G_C(s) = K \left( 1 + \frac{1}{T_I s} + T_D s \right)$

Lag or Lead Compensator  $G_C(s) = K \left( \frac{1 + sT_1}{1 + sT_2} \right)$  depending on the ratio of  $T_1/T_2$ .

## Routh Test

For the characteristic equation

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0 = 0$$

the coefficients are arranged into the first two rows of an array. Additional rows are computed. The array and coefficient computations are defined by:

$$\begin{array}{cccccc} a_n & a_{n-2} & a_{n-4} & \dots & \dots & \dots \\ a_{n-1} & a_{n-3} & a_{n-5} & \dots & \dots & \dots \\ b_1 & b_2 & b_3 & \dots & \dots & \dots \\ c_1 & c_2 & c_3 & \dots & \dots & \dots \end{array}$$

where

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \quad c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \quad c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$$

The necessary and sufficient conditions for all the roots of the equation to have negative real parts is that all the elements in the first column be of the same sign and nonzero.

## First-Order Control System Models

The transfer function model for a first-order system is

$$\frac{Y(s)}{R(s)} = \frac{K}{\tau s + 1}, \text{ where}$$

$K$  = steady-state gain

$\tau$  = time constant

The step response of a first-order system to a step input of magnitude  $M$  is

$$y(t) = y_0 e^{-t/\tau} + KM(1 - e^{-t/\tau})$$

In the chemical process industry,  $y_0$  is typically taken to be zero, and  $y(t)$  is referred to as a deviation variable.

For systems with time delay (dead time or transport lag)  $\theta$ , the transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K e^{-\theta s}}{\tau s + 1}$$

The step response for  $t \geq \theta$  to a step of magnitude  $M$  is

$$y(t) = \left[ y_0 e^{-(t-\theta)/\tau} + KM(1 - e^{-(t-\theta)/\tau}) \right] u(t - \theta), \text{ where}$$

$u(t)$  is the unit step function.

## Second-Order Control System Models

One standard second-order control system model is

$$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \text{ where}$$

$K$  = steady-state gain

$\zeta$  = the damping ratio

$\omega_n$  = the undamped natural ( $\zeta = 0$ ) frequency

$\omega_d = \omega_n \sqrt{1 - \zeta^2}$ , the damped natural frequency

$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$ , the damped resonant frequency

If the damping ratio  $\zeta$  is less than unity, the system is said to be underdamped; if  $\zeta$  is equal to unity, it is said to be critically damped; and if  $\zeta$  is greater than unity, the system is said to be overdamped.

For a unit step input to a normalized underdamped second-order control system, the time required to reach a peak value  $t_p$  and the value of that peak  $M_p$  are given by

$$t_p = \pi / (\omega_n \sqrt{1 - \zeta^2})$$
$$M_p = 1 + e^{-\pi\zeta / \sqrt{1 - \zeta^2}}$$

The percent overshoot (% OS) of the response is given by

$$\% \text{ OS} = 100e^{-\pi\zeta / \sqrt{1 - \zeta^2}}$$

For an underdamped second-order system, the logarithmic decrement is

$$\delta = \frac{1}{m} \ln \left( \frac{x_k}{x_{k+m}} \right) = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}}$$

where  $x_k$  and  $x_{k+m}$  are the amplitudes of oscillation at cycles  $k$  and  $k + m$ , respectively. The period of oscillation  $\tau$  is related to  $\omega_d$  by

$$\omega_d \tau = 2\pi$$

The time required for the output of a second-order system to settle to within 2% of its final value (2% settling time) is defined to be

$$T_s = \frac{4}{\zeta\omega_n}$$

An alternative form commonly employed in the chemical process industry is

$$\frac{Y(s)}{R(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}, \text{ where}$$

$K$  = steady-state gain

$\zeta$  = the damping ratio

$\tau$  = the inverse natural frequency

## Root Locus

The root locus is the locus of points in the complex  $s$ -plane satisfying

$$1 + K \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)} = 0 \quad m \leq n$$

as  $K$  is varied. The  $p_i$  and  $z_j$  are the open-loop poles and zeros, respectively. When  $K$  is increased from zero, the locus has the following properties.

1. Locus branches exist on the real axis to the left of an odd number of open-loop poles and/or zeros.
2. The locus originates at the open-loop poles  $p_1, \dots, p_n$  and terminates at the zeros  $z_1, \dots, z_m$ . If  $m < n$  then  $(n - m)$  branches terminate at infinity at asymptote angles

$$\alpha = \frac{(2k + 1)180^\circ}{n - m} \quad k = 0, \pm 1, \pm 2, \pm 3, \dots$$

with the real axis.

3. The intersection of the real axis with the asymptotes is called the asymptote centroid and is given by

$$\sigma_A = \frac{\sum_{i=1}^n \text{Re}(p_i) - \sum_{i=1}^m \text{Re}(z_i)}{n - m}$$

4. If the locus crosses the imaginary ( $\omega$ ) axis, the values of  $K$  and  $\omega$  are given by letting  $s = j\omega$  in the defining equation.

## State-Variable Control System Models

One common state-variable model for dynamic systems has the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (\text{state equation})$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (\text{output equation})$$

where

$\mathbf{x}(t) = N$  by 1 state vector ( $N$  state variables)

$\mathbf{u}(t) = R$  by 1 input vector ( $R$  inputs)

$\mathbf{y}(t) = M$  by 1 output vector ( $M$  outputs)

$\mathbf{A}$  = system matrix

$\mathbf{B}$  = input distribution matrix

$\mathbf{C}$  = output matrix

$\mathbf{D}$  = feed-through matrix

The orders of the matrices are defined via variable definitions.

State-variable models are used to handle multiple inputs and multiple outputs. Furthermore, state-variable models can be formulated for open-loop system components or the complete closed-loop system.

The Laplace transform of the time-invariant state equation is

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

from which

$$\mathbf{X}(s) = \Phi(s) \mathbf{x}(0) + \Phi(s) \mathbf{B}\mathbf{U}(s)$$

where the Laplace transform of the state transition matrix is

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1}.$$

The state-transition matrix

$$\Phi(t) = L^{-1} \{ \Phi(s) \}$$

(also defined as  $e^{\mathbf{A}t}$ ) can be used to write

$$\mathbf{x}(t) = \Phi(t) \mathbf{x}(0) + \int_0^t \Phi(t - \tau) \mathbf{B}\mathbf{u}(\tau) d\tau$$

The output can be obtained with the output equation; e.g., the Laplace transform output is

$$\mathbf{Y}(s) = \{ \mathbf{C}\Phi(s) \mathbf{B} + \mathbf{D} \} \mathbf{U}(s) + \mathbf{C}\Phi(s) \mathbf{x}(0)$$

The latter term represents the output(s) due to initial conditions, whereas the former term represents the output(s) due to the  $\mathbf{U}(s)$  inputs and gives rise to transfer function definitions.

# ENGINEERING ECONOMICS

Factor Name	Converts	Symbol	Formula
Single Payment Compound Amount	to $F$ given $P$	$(F/P, i\%, n)$	$(1 + i)^n$
Single Payment Present Worth	to $P$ given $F$	$(P/F, i\%, n)$	$(1 + i)^{-n}$
Uniform Series Sinking Fund	to $A$ given $F$	$(A/F, i\%, n)$	$\frac{i}{(1 + i)^n - 1}$
Capital Recovery	to $A$ given $P$	$(A/P, i\%, n)$	$\frac{i(1 + i)^n}{(1 + i)^n - 1}$
Uniform Series Compound Amount	to $F$ given $A$	$(F/A, i\%, n)$	$\frac{(1 + i)^n - 1}{i}$
Uniform Series Present Worth	to $P$ given $A$	$(P/A, i\%, n)$	$\frac{(1 + i)^n - 1}{i(1 + i)^n}$
Uniform Gradient Present Worth	to $P$ given $G$	$(P/G, i\%, n)$	$\frac{(1 + i)^n - 1}{i^2(1 + i)^n} - \frac{n}{i(1 + i)^n}$
Uniform Gradient † Future Worth	to $F$ given $G$	$(F/G, i\%, n)$	$\frac{(1 + i)^n - 1}{i^2} - \frac{n}{i}$
Uniform Gradient Uniform Series	to $A$ given $G$	$(A/G, i\%, n)$	$\frac{1}{i} - \frac{n}{(1 + i)^n - 1}$

## NOMENCLATURE AND DEFINITIONS

$A$  ..... Uniform amount per interest period  
 $B$  ..... Benefit  
 $BV$  ..... Book value  
 $C$  ..... Cost  
 $d$  ..... Inflation adjusted interest rate per interest period  
 $D_j$  ..... Depreciation in year  $j$   
 $F$  ..... Future worth, value, or amount  
 $f$  ..... General inflation rate per interest period  
 $G$  ..... Uniform gradient amount per interest period  
 $i$  ..... Interest rate per interest period  
 $i_e$  ..... Annual effective interest rate  
 $m$  ..... Number of compounding periods per year  
 $n$  ..... Number of compounding periods; or the expected life of an asset  
 $P$  ..... Present worth, value, or amount  
 $r$  ..... Nominal annual interest rate  
 $S_n$  ..... Expected salvage value in year  $n$

## Subscripts

$j$  ..... at time  $j$   
 $n$  ..... at time  $n$   
 $\dagger$  .....  $F/G = (F/A - n)/i = (F/A) \times (A/G)$

## Risk

Risk is the chance of an outcome other than what is planned to occur or expected in the analysis.

## NON-ANNUAL COMPOUNDING

$$i_e = \left(1 + \frac{r}{m}\right)^m - 1$$

## BREAK-EVEN ANALYSIS

By altering the value of any one of the variables in a situation, holding all of the other values constant, it is possible to find a value for that variable that makes the two alternatives equally economical. This value is the break-even point.

Break-even analysis is used to describe the percentage of capacity of operation for a manufacturing plant at which income will just cover expenses.

The payback period is the period of time required for the profit or other benefits of an investment to equal the cost of the investment.

## INFLATION

To account for inflation, the dollars are deflated by the general inflation rate per interest period  $f$ , and then they are shifted over the time scale using the interest rate per interest period  $i$ . Use an inflation adjusted interest rate per interest period  $d$  for computing present worth values  $P$ .

The formula for  $d$  is  $d = i + f + (i \times f)$

## DEPRECIATION

### Straight Line

$$D_j = \frac{C - S_n}{n}$$

### Modified Accelerated Cost Recovery System (MACRS)

$$D_j = (\text{factor}) C$$

A table of MACRS factors is provided below.

## BOOK VALUE

$$BV = \text{initial cost} - \sum D_j$$

## TAXATION

Income taxes are paid at a specific rate on taxable income. Taxable income is total income less depreciation and ordinary expenses. Expenses do not include capital items, which should be depreciated.

## CAPITALIZED COSTS

Capitalized costs are present worth values using an assumed perpetual period of time.

$$\text{Capitalized Costs} = P = \frac{A}{i}$$

## BONDS

Bond value equals the present worth of the payments the purchaser (or holder of the bond) receives during the life of the bond at some interest rate  $i$ .

Bond yield equals the computed interest rate of the bond value when compared with the bond cost.

## RATE-OF-RETURN

The minimum acceptable rate-of-return (MARR) is that interest rate that one is willing to accept, or the rate one desires to earn on investments. The rate-of-return on an investment is the interest rate that makes the benefits and costs equal.

## BENEFIT-COST ANALYSIS

In a benefit-cost analysis, the benefits  $B$  of a project should exceed the estimated costs  $C$ .

$$B - C \geq 0, \text{ or } B/C \geq 1$$

MACRS FACTORS				
Year	Recovery Period (Years)			
	3	5	7	10
	Recovery Rate (Percent)			
1	33.33	20.00	14.29	10.00
2	44.45	32.00	24.49	18.00
3	14.81	19.20	17.49	14.40
4	7.41	11.52	12.49	11.52
5		11.52	8.93	9.22
6		5.76	8.92	7.37
7			8.93	6.55
8			4.46	6.55
9				6.56
10				6.55
11				3.28

Factor Table -  $i = 0.50\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.9950	0.9950	0.0000	1.0050	1.0000	1.0050	1.0000	0.0000
2	0.9901	1.9851	0.9901	1.0100	2.0050	0.5038	0.4988	0.4988
3	0.9851	2.9702	2.9604	1.0151	3.0150	0.3367	0.3317	0.9967
4	0.9802	3.9505	5.9011	1.0202	4.0301	0.2531	0.2481	1.4938
5	<b>0.9754</b>	<b>4.9259</b>	<b>9.8026</b>	<b>1.0253</b>	<b>5.0503</b>	<b>0.2030</b>	<b>0.1980</b>	<b>1.9900</b>
6	0.9705	5.8964	14.6552	1.0304	6.0755	0.1696	0.1646	2.4855
7	0.9657	6.8621	20.4493	1.0355	7.1059	0.1457	0.1407	2.9801
8	0.9609	7.8230	27.1755	1.0407	8.1414	0.1278	0.1228	3.4738
9	0.9561	8.7791	34.8244	1.0459	9.1821	0.1139	0.1089	3.9668
10	<b>0.9513</b>	<b>9.7304</b>	<b>43.3865</b>	<b>1.0511</b>	<b>10.2280</b>	<b>0.1028</b>	<b>0.0978</b>	<b>4.4589</b>
11	0.9466	10.6770	52.8526	1.0564	11.2792	0.0937	0.0887	4.9501
12	0.9419	11.6189	63.2136	1.0617	12.3356	0.0861	0.0811	5.4406
13	0.9372	12.5562	74.4602	1.0670	13.3972	0.0796	0.0746	5.9302
14	0.9326	13.4887	86.5835	1.0723	14.4642	0.0741	0.0691	6.4190
15	<b>0.9279</b>	<b>14.4166</b>	<b>99.5743</b>	<b>1.0777</b>	<b>15.5365</b>	<b>0.0694</b>	<b>0.0644</b>	<b>6.9069</b>
16	0.9233	15.3399	113.4238	1.0831	16.6142	0.0652	0.0602	7.3940
17	0.9187	16.2586	128.1231	1.0885	17.6973	0.0615	0.0565	7.8803
18	0.9141	17.1728	143.6634	1.0939	18.7858	0.0582	0.0532	8.3658
19	0.9096	18.0824	160.0360	1.0994	19.8797	0.0553	0.0503	8.8504
20	<b>0.9051</b>	<b>18.9874</b>	<b>177.2322</b>	<b>1.1049</b>	<b>20.9791</b>	<b>0.0527</b>	<b>0.0477</b>	<b>9.3342</b>
21	0.9006	19.8880	195.2434	1.1104	22.0840	0.0503	0.0453	9.8172
22	0.8961	20.7841	214.0611	1.1160	23.1944	0.0481	0.0431	10.2993
23	0.8916	21.6757	233.6768	1.1216	24.3104	0.0461	0.0411	10.7806
24	0.8872	22.5629	254.0820	1.1272	25.4320	0.0443	0.0393	11.2611
25	<b>0.8828</b>	<b>23.4456</b>	<b>275.2686</b>	<b>1.1328</b>	<b>26.5591</b>	<b>0.0427</b>	<b>0.0377</b>	<b>11.7407</b>
30	0.8610	27.7941	392.6324	1.1614	32.2800	0.0360	0.0310	14.1265
40	0.8191	36.1722	681.3347	1.2208	44.1588	0.0276	0.0226	18.8359
50	0.7793	44.1428	1,035.6966	1.2832	56.6452	0.0227	0.0177	23.4624
60	0.7414	51.7256	1,448.6458	1.3489	69.7700	0.0193	0.0143	28.0064
100	<b>0.6073</b>	<b>78.5426</b>	<b>3,562.7934</b>	<b>1.6467</b>	<b>129.3337</b>	<b>0.0127</b>	<b>0.0077</b>	<b>45.3613</b>

Factor Table -  $i = 1.00\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.9901	0.9901	0.0000	1.0100	1.0000	1.0100	1.0000	0.0000
2	0.9803	1.9704	0.9803	1.0201	2.0100	0.5075	0.4975	0.4975
3	0.9706	2.9410	2.9215	1.0303	3.0301	0.3400	0.3300	0.9934
4	0.9610	3.9020	5.8044	1.0406	4.0604	0.2563	0.2463	1.4876
5	<b>0.9515</b>	<b>4.8534</b>	<b>9.6103</b>	<b>1.0510</b>	<b>5.1010</b>	<b>0.2060</b>	<b>0.1960</b>	<b>1.9801</b>
6	0.9420	5.7955	14.3205	1.0615	6.1520	0.1725	0.1625	2.4710
7	0.9327	6.7282	19.9168	1.0721	7.2135	0.1486	0.1386	2.9602
8	0.9235	7.6517	26.3812	1.0829	8.2857	0.1307	0.1207	3.4478
9	0.9143	8.5650	33.6959	1.0937	9.3685	0.1167	0.1067	3.9337
10	<b>0.9053</b>	<b>9.4713</b>	<b>41.8435</b>	<b>1.1046</b>	<b>10.4622</b>	<b>0.1056</b>	<b>0.0956</b>	<b>4.4179</b>
11	0.8963	10.3676	50.8067	1.1157	11.5668	0.0965	0.0865	4.9005
12	0.8874	11.2551	60.5687	1.1268	12.6825	0.0888	0.0788	5.3815
13	0.8787	12.1337	71.1126	1.1381	13.8093	0.0824	0.0724	5.8607
14	0.8700	13.0037	82.4221	1.1495	14.9474	0.0769	0.0669	6.3384
15	<b>0.8613</b>	<b>13.8651</b>	<b>94.4810</b>	<b>1.1610</b>	<b>16.0969</b>	<b>0.0721</b>	<b>0.0621</b>	<b>6.8143</b>
16	0.8528	14.7179	107.2734	1.1726	17.2579	0.0679	0.0579	7.2886
17	0.8444	15.5623	120.7834	1.1843	18.4304	0.0643	0.0543	7.7613
18	0.8360	16.3983	134.9957	1.1961	19.6147	0.0610	0.0510	8.2323
19	0.8277	17.2260	149.8950	1.2081	20.8109	0.0581	0.0481	8.7017
20	<b>0.8195</b>	<b>18.0456</b>	<b>165.4664</b>	<b>1.2202</b>	<b>22.0190</b>	<b>0.0554</b>	<b>0.0454</b>	<b>9.1694</b>
21	0.8114	18.8570	181.6950	1.2324	23.2392	0.0530	0.0430	9.6354
22	0.8034	19.6604	198.5663	1.2447	24.4716	0.0509	0.0409	10.0998
23	0.7954	20.4558	216.0660	1.2572	25.7163	0.0489	0.0389	10.5626
24	0.7876	21.2434	234.1800	1.2697	26.9735	0.0471	0.0371	11.0237
25	<b>0.7798</b>	<b>22.0232</b>	<b>252.8945</b>	<b>1.2824</b>	<b>28.2432</b>	<b>0.0454</b>	<b>0.0354</b>	<b>11.4831</b>
30	0.7419	25.8077	355.0021	1.3478	34.7849	0.0387	0.0277	13.7557
40	0.6717	32.8347	596.8561	1.4889	48.8864	0.0305	0.0205	18.1776
50	0.6080	39.1961	879.4176	1.6446	64.4632	0.0255	0.0155	22.4363
60	0.5504	44.9550	1,192.8061	1.8167	81.6697	0.0222	0.0122	26.5333
100	<b>0.3697</b>	<b>63.0289</b>	<b>2,605.7758</b>	<b>2.7048</b>	<b>170.4814</b>	<b>0.0159</b>	<b>0.0059</b>	<b>41.3426</b>



Factor Table -  $i = 1.50\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.9852	0.9852	0.0000	1.0150	1.0000	1.0150	1.0000	0.0000
2	0.9707	1.9559	0.9707	1.0302	2.0150	0.5113	0.4963	0.4963
3	0.9563	2.9122	2.8833	1.0457	3.0452	0.3434	0.3284	0.9901
4	0.9422	3.8544	5.7098	1.0614	4.0909	0.2594	0.2444	1.4814
5	<b>0.9283</b>	<b>4.7826</b>	<b>9.4229</b>	<b>1.0773</b>	<b>5.1523</b>	<b>0.2091</b>	<b>0.1941</b>	<b>1.9702</b>
6	0.9145	5.6972	13.9956	1.0934	6.2296	0.1755	0.1605	2.4566
7	0.9010	6.5982	19.4018	1.1098	7.3230	0.1516	0.1366	2.9405
8	0.8877	7.4859	26.6157	1.1265	8.4328	0.1336	0.1186	3.4219
9	0.8746	8.3605	32.6125	1.1434	9.5593	0.1196	0.1046	3.9008
10	<b>0.8617</b>	<b>9.2222</b>	<b>40.3675</b>	<b>1.1605</b>	<b>10.7027</b>	<b>0.1084</b>	<b>0.0934</b>	<b>4.3772</b>
11	0.8489	10.0711	48.8568	1.1779	11.8633	0.0993	0.0843	4.8512
12	0.8364	10.9075	58.0571	1.1956	13.0412	0.0917	0.0767	5.3227
13	0.8240	11.7315	67.9454	1.2136	14.2368	0.0852	0.0702	5.7917
14	0.8118	12.5434	78.4994	1.2318	15.4504	0.0797	0.0647	6.2582
15	<b>0.7999</b>	<b>13.3432</b>	<b>89.6974</b>	<b>1.2502</b>	<b>16.6821</b>	<b>0.0749</b>	<b>0.0599</b>	<b>6.7223</b>
16	0.7880	14.1313	101.5178	1.2690	17.9324	0.0708	0.0558	7.1839
17	0.7764	14.9076	113.9400	1.2880	19.2014	0.0671	0.0521	7.6431
18	0.7649	15.6726	126.9435	1.3073	20.4894	0.0638	0.0488	8.0997
19	0.7536	16.4262	140.5084	1.3270	21.7967	0.0609	0.0459	8.5539
20	<b>0.7425</b>	<b>17.1686</b>	<b>154.6154</b>	<b>1.3469</b>	<b>23.1237</b>	<b>0.0582</b>	<b>0.0432</b>	<b>9.0057</b>
21	0.7315	17.9001	169.2453	1.3671	24.4705	0.0559	0.0409	9.4550
22	0.7207	18.6208	184.3798	1.3876	25.8376	0.0537	0.0387	9.9018
23	0.7100	19.3309	200.0006	1.4084	27.2251	0.0517	0.0367	10.3462
24	0.6995	20.0304	216.0901	1.4295	28.6335	0.0499	0.0349	10.7881
25	<b>0.6892</b>	<b>20.7196</b>	<b>232.6310</b>	<b>1.4509</b>	<b>30.0630</b>	<b>0.0483</b>	<b>0.0333</b>	<b>11.2276</b>
30	0.6398	24.0158	321.5310	1.5631	37.5387	0.0416	0.0266	13.3883
40	0.5513	29.9158	524.3568	1.8140	54.2679	0.0334	0.0184	17.5277
50	0.4750	34.9997	749.9636	2.1052	73.6828	0.0286	0.0136	21.4277
60	0.4093	39.3803	988.1674	2.4432	96.2147	0.0254	0.0104	25.0930
100	<b>0.2256</b>	<b>51.6247</b>	<b>1,937.4506</b>	<b>4.4320</b>	<b>228.8030</b>	<b>0.0194</b>	<b>0.0044</b>	<b>37.5295</b>

Factor Table -  $i = 2.00\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.9804	0.9804	0.0000	1.0200	1.0000	1.0200	1.0000	0.0000
2	0.9612	1.9416	0.9612	1.0404	2.0200	0.5150	0.4950	0.4950
3	0.9423	2.8839	2.8458	1.0612	3.0604	0.3468	0.3268	0.9868
4	0.9238	3.8077	5.6173	1.0824	4.1216	0.2626	0.2426	1.4752
5	<b>0.9057</b>	<b>4.7135</b>	<b>9.2403</b>	<b>1.1041</b>	<b>5.2040</b>	<b>0.2122</b>	<b>0.1922</b>	<b>1.9604</b>
6	0.8880	5.6014	13.6801	1.1262	6.3081	0.1785	0.1585	2.4423
7	0.8706	6.4720	18.9035	1.1487	7.4343	0.1545	0.1345	2.9208
8	0.8535	7.3255	24.8779	1.1717	8.5830	0.1365	0.1165	3.3961
9	0.8368	8.1622	31.5720	1.1951	9.7546	0.1225	0.1025	3.8681
10	<b>0.8203</b>	<b>8.9826</b>	<b>38.9551</b>	<b>1.2190</b>	<b>10.9497</b>	<b>0.1113</b>	<b>0.0913</b>	<b>4.3367</b>
11	0.8043	9.7868	46.9977	1.2434	12.1687	0.1022	0.0822	4.8021
12	0.7885	10.5753	55.6712	1.2682	13.4121	0.0946	0.0746	5.2642
13	0.7730	11.3484	64.9475	1.2936	14.6803	0.0881	0.0681	5.7231
14	0.7579	12.1062	74.7999	1.3195	15.9739	0.0826	0.0626	6.1786
15	<b>0.7430</b>	<b>12.8493</b>	<b>85.2021</b>	<b>1.3459</b>	<b>17.2934</b>	<b>0.0778</b>	<b>0.0578</b>	<b>6.6309</b>
16	0.7284	13.5777	96.1288	1.3728	18.6393	0.0737	0.0537	7.0799
17	0.7142	14.2919	107.5554	1.4002	20.0121	0.0700	0.0500	7.5256
18	0.7002	14.9920	119.4581	1.4282	21.4123	0.0667	0.0467	7.9681
19	0.6864	15.6785	131.8139	1.4568	22.8406	0.0638	0.0438	8.4073
20	<b>0.6730</b>	<b>16.3514</b>	<b>144.6003</b>	<b>1.4859</b>	<b>24.2974</b>	<b>0.0612</b>	<b>0.0412</b>	<b>8.8433</b>
21	0.6598	17.0112	157.7959	1.5157	25.7833	0.0588	0.0388	9.2760
22	0.6468	17.6580	171.3795	1.5460	27.2990	0.0566	0.0366	9.7055
23	0.6342	18.2922	185.3309	1.5769	28.8450	0.0547	0.0347	10.1317
24	0.6217	18.9139	199.6305	1.6084	30.4219	0.0529	0.0329	10.5547
25	<b>0.6095</b>	<b>19.5235</b>	<b>214.2592</b>	<b>1.6406</b>	<b>32.0303</b>	<b>0.0512</b>	<b>0.0312</b>	<b>10.9745</b>
30	0.5521	22.3965	291.7164	1.8114	40.5681	0.0446	0.0246	13.0251
40	0.4529	27.3555	461.9931	2.2080	60.4020	0.0366	0.0166	16.8885
50	0.3715	31.4236	642.3606	2.6916	84.5794	0.0318	0.0118	20.4420
60	0.3048	34.7609	823.6975	3.2810	114.0515	0.0288	0.0088	23.6961
100	<b>0.1380</b>	<b>43.0984</b>	<b>1,464.7527</b>	<b>7.2446</b>	<b>312.2323</b>	<b>0.0232</b>	<b>0.0032</b>	<b>33.9863</b>



Factor Table -  $i = 4.00\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.9615	0.9615	0.0000	1.0400	1.0000	1.0400	1.0000	0.0000
2	0.9246	1.8861	0.9246	1.0816	2.0400	0.5302	0.4902	0.4902
3	0.8890	2.7751	2.7025	1.1249	3.1216	0.3603	0.3203	0.9739
4	0.8548	3.6299	5.2670	1.1699	4.2465	0.2755	0.2355	1.4510
5	<b>0.8219</b>	<b>4.4518</b>	<b>8.5547</b>	<b>1.2167</b>	<b>5.4163</b>	<b>0.2246</b>	<b>0.1846</b>	<b>1.9216</b>
6	0.7903	5.2421	12.5062	1.2653	6.6330	0.1908	0.1508	2.3857
7	0.7599	6.0021	17.0657	1.3159	7.8983	0.1666	0.1266	2.8433
8	0.7307	6.7327	22.1806	1.3686	9.2142	0.1485	0.1085	3.2944
9	0.7026	7.4353	27.8013	1.4233	10.5828	0.1345	0.0945	3.7391
10	<b>0.6756</b>	<b>8.1109</b>	<b>33.8814</b>	<b>1.4802</b>	<b>12.0061</b>	<b>0.1233</b>	<b>0.0833</b>	<b>4.1773</b>
11	0.6496	8.7605	40.3772	1.5395	13.4864	0.1141	0.0741	4.6090
12	0.6246	9.3851	47.2477	1.6010	15.0258	0.1066	0.0666	5.0343
13	0.6006	9.9856	54.4546	1.6651	16.6268	0.1001	0.0601	5.4533
14	0.5775	10.5631	61.9618	1.7317	18.2919	0.0947	0.0547	5.8659
15	<b>0.5553</b>	<b>11.1184</b>	<b>69.7355</b>	<b>1.8009</b>	<b>20.0236</b>	<b>0.0899</b>	<b>0.0499</b>	<b>6.2721</b>
16	0.5339	11.6523	77.7441	1.8730	21.8245	0.0858	0.0458	6.6720
17	0.5134	12.1657	85.9581	1.9479	23.6975	0.0822	0.0422	7.0656
18	0.4936	12.6593	94.3498	2.0258	25.6454	0.0790	0.0390	7.4530
19	0.4746	13.1339	102.8933	2.1068	27.6712	0.0761	0.0361	7.8342
20	<b>0.4564</b>	<b>13.5903</b>	<b>111.5647</b>	<b>2.1911</b>	<b>29.7781</b>	<b>0.0736</b>	<b>0.0336</b>	<b>8.2091</b>
21	0.4388	14.0292	120.3414	2.2788	31.9692	0.0713	0.0313	8.5779
22	0.4220	14.4511	129.2024	2.3699	34.2480	0.0692	0.0292	8.9407
23	0.4057	14.8568	138.1284	2.4647	36.6179	0.0673	0.0273	9.2973
24	0.3901	15.2470	147.1012	2.5633	39.0826	0.0656	0.0256	9.6479
25	<b>0.3751</b>	<b>15.6221</b>	<b>156.1040</b>	<b>2.6658</b>	<b>41.6459</b>	<b>0.0640</b>	<b>0.0240</b>	<b>9.9925</b>
30	0.3083	17.2920	201.0618	3.2434	56.0849	0.0578	0.0178	11.6274
40	0.2083	19.7928	286.5303	4.8010	95.0255	0.0505	0.0105	14.4765
50	0.1407	21.4822	361.1638	7.1067	152.6671	0.0466	0.0066	16.8122
60	0.0951	22.6235	422.9966	10.5196	237.9907	0.0442	0.0042	18.6972
100	<b>0.0198</b>	<b>24.5050</b>	<b>563.1249</b>	<b>50.5049</b>	<b>1,237.6237</b>	<b>0.0408</b>	<b>0.0008</b>	<b>22.9800</b>

Factor Table -  $i = 6.00\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.9434	0.9434	0.0000	1.0600	1.0000	1.0600	1.0000	0.0000
2	0.8900	1.8334	0.8900	1.1236	2.0600	0.5454	0.4854	0.4854
3	0.8396	2.6730	2.5692	1.1910	3.1836	0.3741	0.3141	0.9612
4	0.7921	3.4651	4.9455	1.2625	4.3746	0.2886	0.2286	1.4272
5	<b>0.7473</b>	<b>4.2124</b>	<b>7.9345</b>	<b>1.3382</b>	<b>5.6371</b>	<b>0.2374</b>	<b>0.1774</b>	<b>1.8836</b>
6	0.7050	4.9173	11.4594	1.4185	6.9753	0.2034	0.1434	2.3304
7	0.6651	5.5824	15.4497	1.5036	8.3938	0.1791	0.1191	2.7676
8	0.6274	6.2098	19.8416	1.5938	9.8975	0.1610	0.1010	3.1952
9	0.5919	6.8017	24.5768	1.6895	11.4913	0.1470	0.0870	3.6133
10	<b>0.5584</b>	<b>7.3601</b>	<b>29.6023</b>	<b>1.7908</b>	<b>13.1808</b>	<b>0.1359</b>	<b>0.0759</b>	<b>4.0220</b>
11	0.5268	7.8869	34.8702	1.8983	14.9716	0.1268	0.0668	4.4213
12	0.4970	8.3838	40.3369	2.0122	16.8699	0.1193	0.0593	4.8113
13	0.4688	8.8527	45.9629	2.1329	18.8821	0.1130	0.0530	5.1920
14	0.4423	9.2950	51.7128	2.2609	21.0151	0.1076	0.0476	5.5635
15	<b>0.4173</b>	<b>9.7122</b>	<b>57.5546</b>	<b>2.3966</b>	<b>23.2760</b>	<b>0.1030</b>	<b>0.0430</b>	<b>5.9260</b>
16	0.3936	10.1059	63.4592	2.5404	25.6725	0.0990	0.0390	6.2794
17	0.3714	10.4773	69.4011	2.6928	28.2129	0.0954	0.0354	6.6240
18	0.3505	10.8276	75.3569	2.8543	30.9057	0.0924	0.0324	6.9597
19	0.3305	11.1581	81.3062	3.0256	33.7600	0.0896	0.0296	7.2867
20	<b>0.3118</b>	<b>11.4699</b>	<b>87.2304</b>	<b>3.2071</b>	<b>36.7856</b>	<b>0.0872</b>	<b>0.0272</b>	<b>7.6051</b>
21	0.2942	11.7641	93.1136	3.3996	39.9927	0.0850	0.0250	7.9151
22	0.2775	12.0416	98.9412	3.6035	43.3923	0.0830	0.0230	8.2166
23	0.2618	12.3034	104.7007	3.8197	46.9958	0.0813	0.0213	8.5099
24	0.2470	12.5504	110.3812	4.0489	50.8156	0.0797	0.0197	8.7951
25	<b>0.2330</b>	<b>12.7834</b>	<b>115.9732</b>	<b>4.2919</b>	<b>54.8645</b>	<b>0.0782</b>	<b>0.0182</b>	<b>9.0722</b>
30	0.1741	13.7648	142.3588	5.7435	79.0582	0.0726	0.0126	10.3422
40	0.0972	15.0463	185.9568	10.2857	154.7620	0.0665	0.0065	12.3590
50	0.0543	15.7619	217.4574	18.4202	290.3359	0.0634	0.0034	13.7964
60	0.0303	16.1614	239.0428	32.9877	533.1282	0.0619	0.0019	14.7909
100	<b>0.0029</b>	<b>16.6175</b>	<b>272.0471</b>	<b>339.3021</b>	<b>5,638.3681</b>	<b>0.0602</b>	<b>0.0002</b>	<b>16.3711</b>

Factor Table -  $i = 8.00\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.9259	0.9259	0.0000	1.0800	1.0000	1.0800	1.0000	0.0000
2	0.8573	1.7833	0.8573	1.1664	2.0800	0.5608	0.4808	0.4808
3	0.7938	2.5771	2.4450	1.2597	3.2464	0.3880	0.3080	0.9487
4	0.7350	3.3121	4.6501	1.3605	4.5061	0.3019	0.2219	1.4040
5	<b>0.6806</b>	<b>3.9927</b>	<b>7.3724</b>	<b>1.4693</b>	<b>5.8666</b>	<b>0.2505</b>	<b>0.1705</b>	<b>1.8465</b>
6	0.6302	4.6229	10.5233	1.5869	7.3359	0.2163	0.1363	2.2763
7	0.5835	5.2064	14.0242	1.7138	8.9228	0.1921	0.1121	2.6937
8	0.5403	5.7466	17.8061	1.8509	10.6366	0.1740	0.0940	3.0985
9	0.5002	6.2469	21.8081	1.9990	12.4876	0.1601	0.0801	3.4910
10	<b>0.4632</b>	<b>6.7101</b>	<b>25.9768</b>	<b>2.1589</b>	<b>14.4866</b>	<b>0.1490</b>	<b>0.0690</b>	<b>3.8713</b>
11	0.4289	7.1390	30.2657	2.3316	16.6455	0.1401	0.0601	4.2395
12	0.3971	7.5361	34.6339	2.5182	18.9771	0.1327	0.0527	4.5957
13	0.3677	7.9038	39.0463	2.7196	21.4953	0.1265	0.0465	4.9402
14	0.3405	8.2442	43.4723	2.9372	24.2149	0.1213	0.0413	5.2731
15	<b>0.3152</b>	<b>8.5595</b>	<b>47.8857</b>	<b>3.1722</b>	<b>27.1521</b>	<b>0.1168</b>	<b>0.0368</b>	<b>5.5945</b>
16	0.2919	8.8514	52.2640	3.4259	30.3243	0.1130	0.0330	5.9046
17	0.2703	9.1216	56.5883	3.7000	33.7502	0.1096	0.0296	6.2037
18	0.2502	9.3719	60.8426	3.9960	37.4502	0.1067	0.0267	6.4920
19	0.2317	9.6036	65.0134	4.3157	41.4463	0.1041	0.0241	6.7697
20	<b>0.2145</b>	<b>9.8181</b>	<b>69.0898</b>	<b>4.6610</b>	<b>45.7620</b>	<b>0.1019</b>	<b>0.0219</b>	<b>7.0369</b>
21	0.1987	10.0168	73.0629	5.0338	50.4229	0.0998	0.0198	7.2940
22	0.1839	10.2007	76.9257	5.4365	55.4568	0.0980	0.0180	7.5412
23	0.1703	10.3711	80.6726	5.8715	60.8933	0.0964	0.0164	7.7786
24	0.1577	10.5288	84.2997	6.3412	66.7648	0.0950	0.0150	8.0066
25	<b>0.1460</b>	<b>10.6748</b>	<b>87.8041</b>	<b>6.8485</b>	<b>73.1059</b>	<b>0.0937</b>	<b>0.0137</b>	<b>8.2254</b>
30	0.0994	11.2578	103.4558	10.0627	113.2832	0.0888	0.0088	9.1897
40	0.0460	11.9246	126.0422	21.7245	259.0565	0.0839	0.0039	10.5699
50	0.0213	12.2335	139.5928	46.9016	573.7702	0.0817	0.0017	11.4107
60	0.0099	12.3766	147.3000	101.2571	1,253.2133	0.0808	0.0008	11.9015
100	<b>0.0005</b>	<b>12.4943</b>	<b>155.6107</b>	<b>2,199.7613</b>	<b>27,484.5157</b>	<b>0.0800</b>		<b>12.4545</b>

Factor Table -  $i = 10.00\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.9091	0.9091	0.0000	1.1000	1.0000	1.1000	1.0000	0.0000
2	0.8264	1.7355	0.8264	1.2100	2.1000	0.5762	0.4762	0.4762
3	0.7513	2.4869	2.3291	1.3310	3.3100	0.4021	0.3021	0.9366
4	0.6830	3.1699	4.3781	1.4641	4.6410	0.3155	0.2155	1.3812
5	<b>0.6209</b>	<b>3.7908</b>	<b>6.8618</b>	<b>1.6105</b>	<b>6.1051</b>	<b>0.2638</b>	<b>0.1638</b>	<b>1.8101</b>
6	0.5645	4.3553	9.6842	1.7716	7.7156	0.2296	0.1296	2.2236
7	0.5132	4.8684	12.7631	1.9487	9.4872	0.2054	0.1054	2.6216
8	0.4665	5.3349	16.0287	2.1436	11.4359	0.1874	0.0874	3.0045
9	0.4241	5.7590	19.4215	2.3579	13.5735	0.1736	0.0736	3.3724
10	<b>0.3855</b>	<b>6.1446</b>	<b>22.8913</b>	<b>2.5937</b>	<b>15.9374</b>	<b>0.1627</b>	<b>0.0627</b>	<b>3.7255</b>
11	0.3505	6.4951	26.3962	2.8531	18.5312	0.1540	0.0540	4.0641
12	0.3186	6.8137	29.9012	3.1384	21.3843	0.1468	0.0468	4.3884
13	0.2897	7.1034	33.3772	3.4523	24.5227	0.1408	0.0408	4.6988
14	0.2633	7.3667	36.8005	3.7975	27.9750	0.1357	0.0357	4.9955
15	<b>0.2394</b>	<b>7.6061</b>	<b>40.1520</b>	<b>4.1772</b>	<b>31.7725</b>	<b>0.1315</b>	<b>0.0315</b>	<b>5.2789</b>
16	0.2176	7.8237	43.4164	4.5950	35.9497	0.1278	0.0278	5.5493
17	0.1978	8.0216	46.5819	5.0545	40.5447	0.1247	0.0247	5.8071
18	0.1799	8.2014	49.6395	5.5599	45.5992	0.1219	0.0219	6.0526
19	0.1635	8.3649	52.5827	6.1159	51.1591	0.1195	0.0195	6.2861
20	<b>0.1486</b>	<b>8.5136</b>	<b>55.4069</b>	<b>6.7275</b>	<b>57.2750</b>	<b>0.1175</b>	<b>0.0175</b>	<b>6.5081</b>
21	0.1351	8.6487	58.1095	7.4002	64.0025	0.1156	0.0156	6.7189
22	0.1228	8.7715	60.6893	8.1403	71.4027	0.1140	0.0140	6.9189
23	0.1117	8.8832	63.1462	8.9543	79.5430	0.1126	0.0126	7.1085
24	0.1015	8.9847	65.4813	9.8497	88.4973	0.1113	0.0113	7.2881
25	<b>0.0923</b>	<b>9.0770</b>	<b>67.6964</b>	<b>10.8347</b>	<b>98.3471</b>	<b>0.1102</b>	<b>0.0102</b>	<b>7.4580</b>
30	0.0573	9.4269	77.0766	17.4494	164.4940	0.1061	0.0061	8.1762
40	0.0221	9.7791	88.9525	45.2593	442.5926	0.1023	0.0023	9.0962
50	0.0085	9.9148	94.8889	117.3909	1,163.9085	0.1009	0.0009	9.5704
60	0.0033	9.9672	97.7010	304.4816	3,034.8164	0.1003	0.0003	9.8023
100	<b>0.0001</b>	<b>9.9993</b>	<b>99.9202</b>	<b>13,780.6123</b>	<b>137,796.1234</b>	<b>0.1000</b>		<b>9.9927</b>

Factor Table -  $i = 12.00\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.8929	0.8929	0.0000	1.1200	1.0000	1.1200	1.0000	0.0000
2	0.7972	1.6901	0.7972	1.2544	2.1200	0.5917	0.4717	0.4717
3	0.7118	2.4018	2.2208	1.4049	3.3744	0.4163	0.2963	0.9246
4	0.6355	3.0373	4.1273	1.5735	4.7793	0.3292	0.2092	1.3589
5	<b>0.5674</b>	<b>3.6048</b>	<b>6.3970</b>	<b>1.7623</b>	<b>6.3528</b>	<b>0.2774</b>	<b>0.1574</b>	<b>1.7746</b>
6	0.5066	4.1114	8.9302	1.9738	8.1152	0.2432	0.1232	2.1720
7	0.4523	4.5638	11.6443	2.2107	10.0890	0.2191	0.0991	2.5515
8	0.4039	4.9676	14.4714	2.4760	12.2997	0.2013	0.0813	2.9131
9	0.3606	5.3282	17.3563	2.7731	14.7757	0.1877	0.0677	3.2574
10	<b>0.3220</b>	<b>5.6502</b>	<b>20.2541</b>	<b>3.1058</b>	<b>17.5487</b>	<b>0.1770</b>	<b>0.0570</b>	<b>3.5847</b>
11	0.2875	5.9377	23.1288	3.4785	20.6546	0.1684	0.0484	3.8953
12	0.2567	6.1944	25.9523	3.8960	24.1331	0.1614	0.0414	4.1897
13	0.2292	6.4235	28.7024	4.3635	28.0291	0.1557	0.0357	4.4683
14	0.2046	6.6282	31.3624	4.8871	32.3926	0.1509	0.0309	4.7317
15	<b>0.1827</b>	<b>6.8109</b>	<b>33.9202</b>	<b>5.4736</b>	<b>37.2797</b>	<b>0.1468</b>	<b>0.0268</b>	<b>4.9803</b>
16	0.1631	6.9740	36.3670	6.1304	42.7533	0.1434	0.0234	5.2147
17	0.1456	7.1196	38.6973	6.8660	48.8837	0.1405	0.0205	5.4353
18	0.1300	7.2497	40.9080	7.6900	55.7497	0.1379	0.0179	5.6427
19	0.1161	7.3658	42.9979	8.6128	63.4397	0.1358	0.0158	5.8375
20	<b>0.1037</b>	<b>7.4694</b>	<b>44.9676</b>	<b>9.6463</b>	<b>72.0524</b>	<b>0.1339</b>	<b>0.0139</b>	<b>6.0202</b>
21	0.0926	7.5620	46.8188	10.8038	81.6987	0.1322	0.0122	6.1913
22	0.0826	7.6446	48.5543	12.1003	92.5026	0.1308	0.0108	6.3514
23	0.0738	7.7184	50.1776	13.5523	104.6029	0.1296	0.0096	6.5010
24	0.0659	7.7843	51.6929	15.1786	118.1552	0.1285	0.0085	6.6406
25	<b>0.0588</b>	<b>7.8431</b>	<b>53.1046</b>	<b>17.0001</b>	<b>133.3339</b>	<b>0.1275</b>	<b>0.0075</b>	<b>6.7708</b>
30	0.0334	8.0552	58.7821	29.9599	241.3327	0.1241	0.0041	7.2974
40	0.0107	8.2438	65.1159	93.0510	767.0914	0.1213	0.0013	7.8988
50	0.0035	8.3045	67.7624	289.0022	2,400.0182	0.1204	0.0004	8.1597
60	0.0011	8.3240	68.8100	897.5969	7,471.6411	0.1201	0.0001	8.2664
100		<b>8.3332</b>	<b>69.4336</b>	<b>83,522.2657</b>	<b>696,010.5477</b>	<b>0.1200</b>		<b>8.3321</b>

Factor Table -  $i = 18.00\%$ 

$n$	$P/F$	$P/A$	$P/G$	$F/P$	$F/A$	$A/P$	$A/F$	$A/G$
1	0.8475	0.8475	0.0000	1.1800	1.0000	1.1800	1.0000	0.0000
2	0.7182	1.5656	0.7182	1.3924	2.1800	0.6387	0.4587	0.4587
3	0.6086	2.1743	1.9354	1.6430	3.5724	0.4599	0.2799	0.8902
4	0.5158	2.6901	3.4828	1.9388	5.2154	0.3717	0.1917	1.2947
5	<b>0.4371</b>	<b>3.1272</b>	<b>5.2312</b>	<b>2.2878</b>	<b>7.1542</b>	<b>0.3198</b>	<b>0.1398</b>	<b>1.6728</b>
6	0.3704	3.4976	7.0834	2.6996	9.4423	0.2859	0.1059	2.0252
7	0.3139	3.8115	8.9670	3.1855	12.1415	0.2624	0.0824	2.3526
8	0.2660	4.0776	10.8292	3.7589	15.3270	0.2452	0.0652	2.6558
9	0.2255	4.3030	12.6329	4.4355	19.0859	0.2324	0.0524	2.9358
10	<b>0.1911</b>	<b>4.4941</b>	<b>14.3525</b>	<b>5.2338</b>	<b>23.5213</b>	<b>0.2225</b>	<b>0.0425</b>	<b>3.1936</b>
11	0.1619	4.6560	15.9716	6.1759	28.7551	0.2148	0.0348	3.4303
12	0.1372	4.7932	17.4811	7.2876	34.9311	0.2086	0.0286	3.6470
13	0.1163	4.9095	18.8765	8.5994	42.2187	0.2037	0.0237	3.8449
14	0.0985	5.0081	20.1576	10.1472	50.8180	0.1997	0.0197	4.0250
15	<b>0.0835</b>	<b>5.0916</b>	<b>21.3269</b>	<b>11.9737</b>	<b>60.9653</b>	<b>0.1964</b>	<b>0.0164</b>	<b>4.1887</b>
16	0.0708	5.1624	22.3885	14.1290	72.9390	0.1937	0.0137	4.3369
17	0.0600	5.2223	23.3482	16.6722	87.0680	0.1915	0.0115	4.4708
18	0.0508	5.2732	24.2123	19.6731	103.7403	0.1896	0.0096	4.5916
19	0.0431	5.3162	24.9877	23.2144	123.4135	0.1881	0.0081	4.7003
20	<b>0.0365</b>	<b>5.3527</b>	<b>25.6813</b>	<b>27.3930</b>	<b>146.6280</b>	<b>0.1868</b>	<b>0.0068</b>	<b>4.7978</b>
21	0.0309	5.3837	26.3000	32.3238	174.0210	0.1857	0.0057	4.8851
22	0.0262	5.4099	26.8506	38.1421	206.3448	0.1848	0.0048	4.9632
23	0.0222	5.4321	27.3394	45.0076	244.4868	0.1841	0.0041	5.0329
24	0.0188	5.4509	27.7725	53.1090	289.4944	0.1835	0.0035	5.0950
25	<b>0.0159</b>	<b>5.4669</b>	<b>28.1555</b>	<b>62.6686</b>	<b>342.6035</b>	<b>0.1829</b>	<b>0.0029</b>	<b>5.1502</b>
30	0.0070	5.5168	29.4864	143.3706	790.9480	0.1813	0.0013	5.3448
40	0.0013	5.5482	30.5269	750.3783	4,163.2130	0.1802	0.0002	5.5022
50	0.0003	5.5541	30.7856	3,927.3569	21,813.0937	0.1800		5.5428
60	0.0001	5.5553	30.8465	20,555.1400	114,189.6665	0.1800		5.5526
100		<b>5.5556</b>	<b>30.8642</b>	<b>15,424,131.91</b>	<b>85,689,616.17</b>	<b>0.1800</b>		<b>5.5555</b>

# CHEMICAL ENGINEERING

## CHEMICAL REACTION ENGINEERING

### Nomenclature

A chemical reaction may be expressed by the general equation



The rate of reaction of any component is defined as the moles of that component formed per unit time per unit volume.

$$-r_A = -\frac{1}{V} \frac{dN_A}{dt} \quad (\text{negative because A disappears})$$

$$-r_A = \frac{-dC_A}{dt} \quad \text{if } V \text{ is constant}$$

The rate of reaction is frequently expressed by

$$-r_A = k f_r(C_A, C_B, \dots), \text{ where}$$

$k$  = reaction rate constant

$C_I$  = concentration of component  $I$

In the conversion of  $A$ , the fractional conversion  $X_A$  is defined as the moles of  $A$  reacted per mole of  $A$  fed.

$$X_A = (C_{A0} - C_A)/C_{A0} \quad \text{if } V \text{ is constant}$$

The Arrhenius equation gives the dependence of  $k$  on temperature

$$k = A e^{-E_a/\bar{R}T}, \text{ where}$$

$A$  = pre-exponential or frequency factor

$E_a$  = activation energy (J/mol, cal/mol)

$T$  = temperature (K)

$\bar{R}$  = gas law constant = 8.314 J/(mol•K)

For values of rate constant ( $k_i$ ) at two temperatures ( $T_i$ ),

$$E_a = \frac{RT_1T_2}{(T_1 - T_2)} \ln \left( \frac{k_1}{k_2} \right)$$

### Reaction Order

$$\text{If } -r_A = k C_A^x C_B^y$$

the reaction is  $x$  order with respect to reactant  $A$  and  $y$  order with respect to reactant  $B$ . The overall order is

$$n = x + y$$

### Batch Reactor, Constant Volume

For a well-mixed, constant-volume batch reactor

$$-r_A = -dC_A/dt$$

$$t = -C_{A0} \int_0^{X_A} dX_A / (-r_A)$$

### Zero-Order Irreversible Reaction

$$-r_A = k C_A^0 = k(1)$$

$$-dC_A/dt = k \quad \text{or}$$

$$C_A = C_{A0} - kt$$

$$dX_A/dt = k/C_{A0} \quad \text{or}$$

$$C_{A0} X_A = kt$$

### First-Order Irreversible Reaction

$$-r_A = k C_A$$

$$-dC_A/dt = k C_A \quad \text{or}$$

$$\ln(C_A/C_{A0}) = -kt$$

$$dX_A/dt = k(1 - X_A) \quad \text{or}$$

$$\ln(1 - X_A) = -kt$$

### Second-Order Irreversible Reaction

$$-r_A = k C_A^2$$

$$-dC_A/dt = k C_A^2 \quad \text{or}$$

$$1/C_A - 1/C_{A0} = kt$$

$$dX_A/dt = k C_{A0} (1 - X_A)^2 \quad \text{or}$$

$$X_A [C_{A0} (1 - X_A)] = kt$$

### First-Order Reversible Reactions



$$-r_A = -\frac{dC_A}{dt} = k_1 C_A - k_2 C_R$$

$$K_c = k_1/k_2 = \hat{C}_R/\hat{C}_A$$

$$M = C_{R0}/C_{A0}$$

$$\frac{d\hat{X}_A}{dt} = \frac{k_1(M+1)}{M+\hat{X}_A} (\hat{X}_A - X_A)$$

$$-\ln \left( 1 - \frac{X_A}{\hat{X}_A} \right) = -\ln \frac{C_A - \hat{C}_A}{C_{A0} - \hat{C}_A}$$

$$= \frac{(M+1)}{(M+\hat{X}_A)} k_1 t$$

### Reactions of Shifting Order

$$-r_A = \frac{k_1 C_A}{1 + k_2 C_A}$$

$$\ln\left(\frac{C_{A0}}{C_A}\right) + k_2(C_{A0} - C_A) = k_1 t$$

$$\frac{\ln(C_{A0}/C_A)}{C_{A0} - C_A} = -k_2 + \frac{k_1 t}{C_{A0} - C_A}$$

This form of the rate equation is used for elementary enzyme-catalyzed reactions and for elementary surfaced-catalyzed reactions.

### Batch Reactor, Variable Volume

If the volume of the reacting mass varies with the conversion (such as a variable-volume batch reactor) according to

$$V = V_{X_A=0}(1 + \epsilon_A X_A)$$

(i.e., at constant pressure), where

$$\epsilon_A = \frac{V_{X_A=1} - V_{X_A=0}}{V_{X_A=0}} = \frac{\Delta V}{V_{X_A=0}}$$

then at any time

$$C_A = C_{A0} \left[ \frac{1 - X_A}{1 + \epsilon_A X_A} \right]$$

and

$$t = -C_{A0} \int_0^{X_A} dX_A / [(1 + \epsilon_A X_A)(-r_A)]$$

For a first-order irreversible reaction,

$$kt = -\ln(1 - X_A) = -\ln\left(1 - \frac{\Delta V}{\epsilon_A V_{X_A=0}}\right)$$

### Flow Reactors, Steady State

Space-time  $\tau$  is defined as the reactor volume divided by the inlet volumetric feed rate. Space-velocity  $SV$  is the reciprocal of space-time,  $SV = 1/\tau$ .

#### Plug-Flow Reactor (PFR)

$$\tau = \frac{C_{A0} V_{PFR}}{F_{A0}} = C_{A0} \int_0^{X_A} \frac{dX_A}{(-r_A)}, \text{ where}$$

$F_{A0}$  = moles of  $A$  fed per unit time.

#### Continuous-Stirred Tank Reactor (CSTR)

For a constant volume, well-mixed CSTR

$$\frac{\tau}{C_{A0}} = \frac{V_{CSTR}}{F_{A0}} = \frac{X_A}{-r_A}, \text{ where}$$

$-r_A$  is evaluated at exit stream conditions.

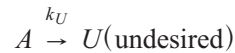
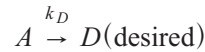
### Continuous-Stirred Tank Reactors in Series

With a first-order reaction  $A \rightarrow R$ , no change in volume.

$$\tau_{N\text{-reactors}} = N\tau_{\text{individual}} = \frac{N}{k} \left[ \left( \frac{C_{A0}}{C_{AN}} \right)^{1/N} - 1 \right], \text{ where}$$

$N$  = number of CSTRs (equal volume) in series, and  
 $C_{AN}$  = concentration of  $A$  leaving the  $N$ th CSTR.

### Two Irreversible Reactions in Parallel



$$-r_A = -dc_A/dt = k_D C_A^x + k_U C_A^y$$

$$r_D = dc_D/dt = k_D C_A^x$$

$$r_U = dc_U/dt = k_U C_A^y$$

$Y_D$  = instantaneous fractional yield of  $D$

$$= dC_D / (-dC_A)$$

$\bar{Y}_D$  = overall fractional yield of  $D$

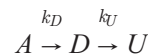
$$= N_{Df} / (N_{A0} - N_{Af})$$

where  $N_{Af}$  and  $N_{Df}$  are measured at the outlet of the flow reactor.

$\bar{S}_{DU}$  = overall selectivity to  $D$

$$= N_{Df} / N_{Uf}$$

### Two First-Order Irreversible Reactions in Series



$$r_A = -dC_A/dt = k_D C_A$$

$$r_D = dC_D/dt = k_D C_A - k_U C_D$$

$$r_U = dC_U/dt = k_U C_D$$

Yield and selectivity definitions are identical to those for two irreversible reactions in parallel. The optimal yield of  $D$  in a PFR is

$$\frac{C_{D,\max}}{C_{A0}} = \left( \frac{k_D}{k_U} \right)^{k_U / (k_U - k_D)}$$

at time

$$\tau_{\max} = \frac{1}{k_{\log \text{ mean}}} = \frac{\ln(k_U/k_D)}{(k_U - k_D)}$$

The optimal yield of  $D$  in a CSTR is

$$\frac{C_{D,\max}}{C_{A0}} = \frac{1}{[(k_U/k_D)^{1/2} + 1]^2}$$

at time

$$\tau_{\max} = 1 / \sqrt{k_D k_U}$$

## MASS TRANSFER

### Diffusion

#### Molecular Diffusion

$$\text{Gas: } N_A = \frac{p_A}{P}(N_A + N_B) - \frac{D_m}{RT} \frac{\partial p_A}{\partial z}$$

$$\text{Liquid: } N_A = x_A(N_A + N_B) - CD_m \frac{\partial x_A}{\partial z}$$

where,

$N_i$  = molar flux of component  $i$

$P$  = pressure

$p_i$  = partial pressure of component  $i$

$D_m$  = mass diffusivity

$R$  = universal gas constant

$T$  = temperature

$z$  = length

#### Unidirectional Diffusion of a Gas A Through a Second Stagnant Gas B ( $N_B = 0$ )

$$N_A = \frac{D_m P}{RT(p_B)_{lm}} \times \frac{(p_{A2} - p_{A1})}{z_2 - z_1}$$

in which  $(p_B)_{lm}$  is the log mean of  $p_{B2}$  and  $p_{B1}$

$$(p_{BM})_{lm} = \frac{p_{B2} - p_{B1}}{\ln\left(\frac{p_{B2}}{p_{B1}}\right)}$$

$N_i$  = diffusive flux [mole/(time  $\times$  area)] of component  $i$  through area  $A$ , in  $z$  direction

$D_m$  = mass diffusivity

$p_i$  = partial pressure of species  $i$

$C$  = concentration (mole/volume)

$(z_2 - z_1)$  = diffusion flow path length

#### Equimolar Counter-Diffusion (Gases)

( $N_B = -N_A$ )

$$N_A = D_m / (RT) \times [(p_{A1} - p_{A2}) / (\Delta z)]$$

$$N_A = D_m (C_{A1} - C_{A2}) / \Delta z$$

### Convection

#### Two-Film Theory (for Equimolar Counter-Diffusion)

$$\begin{aligned} N_A &= k'_G (p_{AG} - p_{Ai}) \\ &= k'_L (C_{Ai} - C_{AL}) \\ &= K'_G (p_{AG} - p_A^*) \\ &= K'_L (C_A^* - C_{AL}) \end{aligned}$$

where,

$N_A$  = molar flux of component A

$k'_G$  = gas phase mass transfer coefficient

$k'_L$  = liquid phase mass transfer coefficient

$K'_G$  = overall gas phase mass transfer coefficient

$K'_L$  = overall liquid phase mass transfer coefficient

$p_{AG}$  = partial pressure in component A in the bulk gas phase

$p_{Ai}$  = partial pressure at component A at the gas-liquid interface

$C_{Ai}$  = concentration (mole/volume) of component A in the liquid phase at the gas-liquid interface

$C_{AL}$  = concentration of component A in the bulk liquid phase

$p_A^*$  = partial pressure of component A in equilibrium with  $C_{AL}$

$C_A^*$  = concentration of component A in equilibrium with the bulk gas vapor composition of A

#### Overall Coefficients

$$1/K'_G = 1/k'_G + H/k'_L$$

$$1/K'_L = 1/Hk'_G + 1/k'_L$$

$H$  = Henry's Law constant where  $p_A^* = H C_{AL}$  and  $C_A^* = p_{AG}/H$

#### Dimensionless Group Equation (Sherwood)

For the turbulent flow inside a tube the Sherwood number

$$\text{Sh} = \left( \frac{k_m D}{D_m} \right) = 0.023 \left( \frac{DV\rho}{\mu} \right)^{0.8} \left( \frac{\mu}{\rho D_m} \right)^{1/3}$$

where,

$D$  = inside diameter

$D_m$  = diffusion coefficient

$V$  = average velocity in the tube

$\rho$  = fluid density

$\mu$  = fluid viscosity

$k_m$  = mass transfer coefficient

## Distillation

Definitions:

$\alpha$  = relative volatility

$B$  = molar bottoms-product rate

$D$  = molar overhead-product rate

$F$  = molar feed rate

$L$  = molar liquid downflow rate

$R_D$  = ratio of reflux to overhead product

$V$  = molar vapor upflow rate

$W$  = total moles in still pot

$x$  = mole fraction of the more volatile component in the liquid phase

$y$  = mole fraction of the more volatile component in the vapor phase

Subscripts:

$B$  = bottoms product

$D$  = overhead product

$F$  = feed

$m$  = any plate in stripping section of column

$m+1$  = plate below plate  $m$

$n$  = any plate in rectifying section of column

$n+1$  = plate below plate  $n$

$o$  = original charge in still pot

## Flash (or equilibrium) Distillation

Component material balance:

$$Fz_F = yV + xL$$

Overall material balance:

$$F = V + L$$

## Differential (Simple or Rayleigh) Distillation

$$\ln\left(\frac{W}{W_o}\right) = \int_{x_o}^x \frac{dx}{y-x}$$

When the relative volatility  $\alpha$  is constant,

$$y = \alpha x / [1 + (\alpha - 1)x]$$

can be substituted to give

$$\ln\left(\frac{W}{W_o}\right) = \frac{1}{(\alpha - 1)} \ln\left[\frac{x(1 - x_o)}{x_o(1 - x)}\right] + \ln\left[\frac{1 - x_o}{1 - x}\right]$$

For binary system following Raoult's Law

$$\alpha = (y/x)_a / (y/x)_b = p_a / p_b, \text{ where}$$

$p_i$  = partial pressure of component  $i$ .

## Continuous Distillation (Binary System)

Constant molal overflow is assumed.

Equilibrium stages numbered from top.

## Overall Material Balances

Total Material:

$$F = D + B$$

Component  $A$ :

$$Fz_F = Dx_D + Bx_B$$

## Operating Lines

### Rectifying section

Total Material:

$$V_{n+1} = L_n + D$$

Component  $A$ :

$$V_{n+1}y_{n+1} = L_nx_n + Dx_D$$

$$y_{n+1} = [L_n / (L_n + D)] x_n + Dx_D / (L_n + D)$$

### Stripping section

Total Material:

$$L_m = V_{m+1} + B$$

Component  $A$ :

$$L_mx_m = V_{m+1}y_{m+1} + Bx_B$$

$$y_{m+1} = [L_m / (L_m - B)] x_m - Bx_B / (L_m - B)$$

## Reflux ratio

Ratio of reflux to overhead product

$$R_D = L_R / D = (V_R - D) / D$$

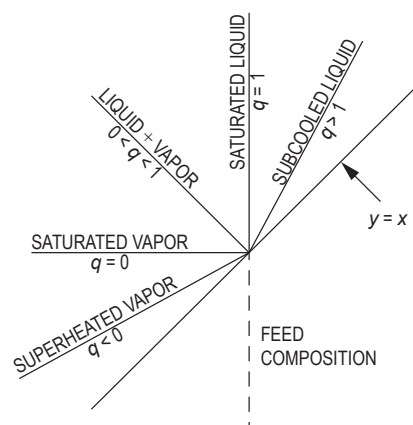
Minimum reflux ratio is defined as that value which results in an infinite number of contact stages. For a binary system the equation of the operating line is

$$y = \frac{R_{\min}}{R_{\min} + 1} x + \frac{x_D}{R_{\min} + 1}$$

## Feed condition line

slope =  $q / (q - 1)$ , where

$$q = \frac{\text{heat to convert one mol of feed to saturated vapor}}{\text{molar heat of vaporization}}$$



q-LINE SLOPES

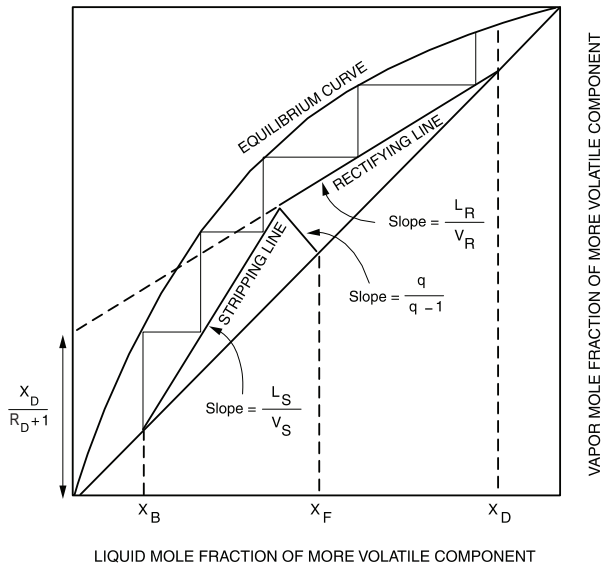
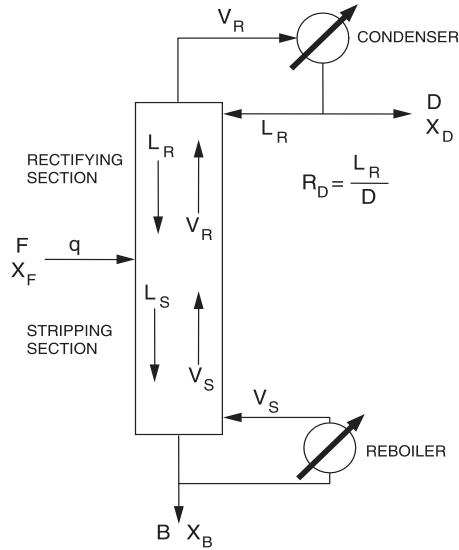
### Murphree plate efficiency

$$E_{ME} = (y_n - y_{n+1}) / (y_n^* - y_{n+1}), \text{ where}$$

$y_n$  = concentration of vapor above equilibrium stage  $n$

$y_{n+1}$  = concentration of vapor entering from equilibrium stage below  $n$

$y_n^*$  = concentration of vapor in equilibrium with liquid leaving equilibrium stage  $n$



### Absorption (packed columns)

#### Continuous Contact Columns

$$Z = NTU_G \cdot HTU_G = NTU_L \cdot HTU_L = N_{EQ} \cdot HETP$$

$Z$  = column height

$NTU_G$  = number of transfer units (gas phase)

$NTU_L$  = number of transfer units (liquid phase)

$N_{EQ}$  = number of equilibrium stages

$HTU_G$  = height of transfer unit (gas phase)

$HTU_L$  = height of transfer unit (liquid phase)

$HETP$  = height equivalent to theoretical plate (stage)

$$HTU_G = \frac{G}{K'_G a} \quad HTU_L = \frac{L}{K'_L a}$$

$G$  = gas phase mass velocity (mass or moles/flow area • time)

$L$  = liquid phase mass velocity (mass or moles/flow area • time)

$K'_G$  = overall gas phase mass transfer coefficient (mass or moles/mass transfer area • time)

$K'_L$  = overall liquid phase mass transfer coefficient (mass or moles/mass transfer area • time)

$a$  = mass transfer area/volume of column (length<sup>-1</sup>)

$$NTU_G = \int_{y_1}^{y_2} \frac{dy}{(y - y^*)} \quad NTU_L = \int_{x_1}^{x_2} \frac{dx}{(x^* - x)}$$

$y$  = gas phase solute mole fraction

$x$  = liquid phase solute mole fraction

$y^*$  =  $K \cdot x$ , where  $K$  = equilibrium constant

$x^*$  =  $y/K$ , where  $K$  = equilibrium constant

$y_2, x_2$  = mole fractions at the lean end of column

$y_1, x_1$  = mole fractions at the rich end of column

For dilute solutions (constant  $G/L$  and constant  $K$  value for entire column):

$$NTU_G = \frac{y_1 - y_2}{(y - y^*)_{LM}}$$

$$(y - y^*)_{LM} = \frac{(y_1 - y_1^*) - (y_2 - y_2^*)}{\ln \left( \frac{y_1 - y_1^*}{y_2 - y_2^*} \right)}$$

For a chemically reacting system—absorbed solute reacts in the liquid phase—the preceding relation simplifies to:

$$NTU_G = \ln \left( \frac{y_1}{y_2} \right)$$



## TRANSPORT PHENOMENA—MOMENTUM, HEAT, AND MASS TRANSFER ANALOGY

For the equations which apply to *turbulent flow in circular tubes*, the following definitions apply:

$$\text{Nu} = \text{Nusselt Number} \left[ \frac{hD}{k} \right]$$

$$\text{Pr} = \text{Prandtl Number} (c_p \mu / k)$$

$$\text{Re} = \text{Reynolds Number} (DV\rho/\mu)$$

$$\text{Sc} = \text{Schmidt Number} [\mu/(\rho D_m)]$$

$$\text{Sh} = \text{Sherwood Number} (k_m D/D_m)$$

$$\text{St} = \text{Stanton Number} [h/(c_p G)]$$

$$c_m = \text{concentration (mol/m}^3\text{)}$$

$$c_p = \text{heat capacity of fluid [J/(kg}\cdot\text{K)]}$$

$$D = \text{tube inside diameter (m)}$$

$$D_m = \text{diffusion coefficient (m}^2\text{/s)}$$

$$(dc_m/dy)_w = \text{concentration gradient at the wall (mol/m}^4\text{)}$$

$$(dT/dy)_w = \text{temperature gradient at the wall (K/m)}$$

$$(dv/dy)_w = \text{velocity gradient at the wall (s}^{-1}\text{)}$$

$$f = \text{Moody friction factor}$$

$$G = \text{mass velocity [kg/(m}^2\cdot\text{s)]}$$

$$h = \text{heat-transfer coefficient at the wall [W/(m}^2\cdot\text{K)]}$$

$$k = \text{thermal conductivity of fluid [W/(m}\cdot\text{K)]}$$

$$k_m = \text{mass-transfer coefficient (m/s)}$$

$$L = \text{length over which pressure drop occurs (m)}$$

$$(N/A)_w = \text{inward mass-transfer flux at the wall [mol/(m}^2\cdot\text{s)]}$$

$$(\dot{Q}/A)_w = \text{inward heat-transfer flux at the wall (W/m}^2\text{)}$$

$$y = \text{distance measured from inner wall toward centerline (m)}$$

$$\Delta c_m = \text{concentration difference between wall and bulk fluid (mol/m}^3\text{)}$$

$$\Delta T = \text{temperature difference between wall and bulk fluid (K)}$$

$$\mu = \text{absolute dynamic viscosity (N}\cdot\text{s/m}^2\text{)}$$

$$\tau_w = \text{shear stress (momentum flux) at the tube wall (N/m}^2\text{)}$$

Definitions already introduced also apply.

## Rate of Transfer as a Function of Gradients at the Wall

*Momentum Transfer:*

$$\tau_w = -\mu \left( \frac{dv}{dy} \right)_w = -\frac{f\rho V^2}{8} = \left( \frac{D}{4} \right) \left( -\frac{\Delta p}{L} \right)_f$$

*Heat Transfer*

$$\left( \frac{\dot{Q}}{A} \right)_w = -k \left( \frac{dT}{dy} \right)_w$$

*Mass Transfer in Dilute Solutions*

$$\left( \frac{N}{A} \right)_w = -D_m \left( \frac{dc_m}{dy} \right)_w$$

## Rate of Transfer in Terms of Coefficients

*Momentum Transfer*

$$\tau_w = \frac{f\rho V^2}{8}$$

*Heat Transfer*

$$\left( \frac{\dot{Q}}{A} \right)_w = h\Delta T$$

*Mass Transfer*

$$\left( \frac{N}{A} \right)_w = k_m \Delta c_m$$

## Use of Friction Factor ( $f$ ) to Predict Heat-Transfer and Mass Transfer Coefficients (Turbulent Flow)

*Heat Transfer*

$$j_H = \left( \frac{\text{Nu}}{\text{RePr}} \right) \text{Pr}^{2/3} = \frac{f}{8}$$

*Mass Transfer*

$$j_M = \left( \frac{\text{Sh}}{\text{ReSc}} \right) \text{Sc}^{2/3} = \frac{f}{8}$$

## COST ESTIMATION

### Cost Indexes

Cost indexes are used to update historical cost data to the present. If a purchase cost is available for an item of equipment in year  $M$ , the equivalent current cost would be found by:

$$\text{Current \$} = (\text{Cost in year } M) \left( \frac{\text{Current Index}}{\text{Index in year } M} \right)$$

Component	Range
<i>Direct costs</i>	
Purchased equipment-delivered (including fabricated equipment and process machinery such as pumps and compressors)	100
Purchased-equipment installation	39–47
Instrumentation and controls (installed)	9–18
Piping (installed)	16–66
Electrical (installed)	10–11
Buildings (including services)	18–29
Yard improvements	10–13
Service facilities (installed)	40–70
Land (if purchase is required)	6
Total direct plant cost	264–346
<i>Indirect costs</i>	
Engineering and supervision	32–33
Construction expenses	34–41
Total direct and indirect plant costs	336–420
Contractor's fee (about 5% of direct and indirect plant costs)	17–21
Contingency (about 10% of direct and indirect plant costs)	36–42
Fixed-capital investment	387–483
Working capital (about 15% of total capital investment)	68–86
Total capital investment	455–569

### Scaling of Equipment Costs

The cost of Unit A at one capacity related to the cost of a similar Unit B with  $X$  times the capacity of Unit A is approximately  $X^n$  times the cost of Unit B.

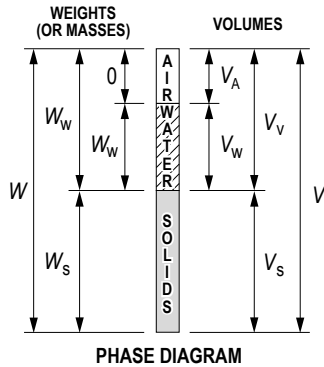
$$\text{Cost of Unit A} = \text{cost of Unit B} \left( \frac{\text{capacity of Unit A}}{\text{capacity of Unit B}} \right)^n$$

**Typical Exponents (*n*) for Equipment Cost vs. Capacity**

<b>Equipment</b>	<b>Size range</b>	<b>Exponent</b>
Dryer, drum, single vacuum	10 to 10 <sup>2</sup> ft <sup>2</sup>	0.76
Dryer, drum, single atmospheric	10 to 10 <sup>2</sup> ft <sup>2</sup>	0.40
Fan, centrifugal	10 <sup>3</sup> to 10 <sup>4</sup> ft <sup>3</sup> /min	0.44
Fan, centrifugal	2 × 10 <sup>4</sup> to 7 × 10 <sup>4</sup> ft <sup>3</sup> /min	1.17
Heat exchanger, shell and tube, floating head, c.s.	100 to 400 ft <sup>2</sup>	0.60
Heat exchanger, shell and tube, fixed sheet, c.s.	100 to 400 ft <sup>2</sup>	0.44
Motor, squirrel cage, induction, 440 volts, explosion proof	5 to 20 hp	0.69
Motor, squirrel cage, induction, 440 volts, explosion proof	20 to 200 hp	0.99
Tray, bubble cup, c.s.	3- to 10-ft diameter	1.20
Tray, sieve, c.s.	3- to 10-ft diameter	0.86

# CIVIL ENGINEERING

## GEOTECHNICAL Phase Relationships



Volume of voids

$$V_v = V_A + V_w$$

Total unit weight

$$\gamma = W/V$$

Saturated unit weight

$$\gamma_{sat} = (G_s + e) \gamma_w / (1 + e) = \gamma(G_s + e) / (1 + \omega)$$

$$\gamma_w = 62.4 \text{ lb/ft}^3 \text{ or } 9.81 \text{ kN/m}^3$$

Effective (submerged) unit weight

$$\gamma' = \gamma_{sat} - \gamma_w$$

Unit weight of solids

$$\gamma_s = W_s / V_s$$

Dry unit weight

$$\gamma_D = W_s / V$$

Water content (%)

$$\omega = (W_w / W_s) \times 100$$

Specific gravity of soil solids

$$G_s = (W_s / V_s) / \gamma_w$$

Void ratio

$$e = V_v / V_s$$

Porosity

$$n = V_v / V = e / (1 + e)$$

Degree of saturation (%)

$$S = (V_w / V_v) \times 100$$

Relative density

$$D_r = [(e_{max} - e) / (e_{max} - e_{min})] \times 100$$

$$= [(\gamma_{D \text{ field}} - \gamma_{D \text{ min}}) / (\gamma_{D \text{ max}} - \gamma_{D \text{ min}})] [\gamma_{D \text{ max}} / \gamma_{D \text{ field}}] \times 100$$

Relative compaction (%)

$$RC = (\gamma_{D \text{ field}} / \gamma_{D \text{ max}}) \times 100$$

Plasticity index

$$PI = LL - PL$$

LL = liquid limit

PL = Plastic limit

Coefficient of uniformity

$$C_u = D_{60} / D_{10}$$

Coefficient of concavity (or curvature)

$$C_c = (D_{30})^2 / (D_{10} \times D_{60})$$

Hydraulic conductivity (also coefficient of permeability)

From constant head test:  $k = Q / (iAt_e)$

$$i = dh/dL$$

$Q$  = total quantity of water

From falling head test:  $k = 2.303[(aL)/(At_e)] \log_{10}(h_1/h_2)$

$A$  = cross-sectional area of test specimen perpendicular to flow

$a$  = cross-sectional area of reservoir tube

$t_e$  = elapsed time

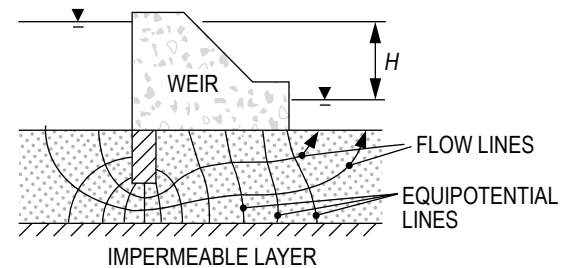
$h_1$  = head at time  $t = 0$

$h_2$  = head at time  $t = t_e$

$L$  = length of soil column

Discharge velocity,  $v = ki$

Flow Nets



FLOW NET

$$Q = kH (N_f / N_d)$$

$N_f$  = number of flow channels

$N_d$  = number of equipotential drops

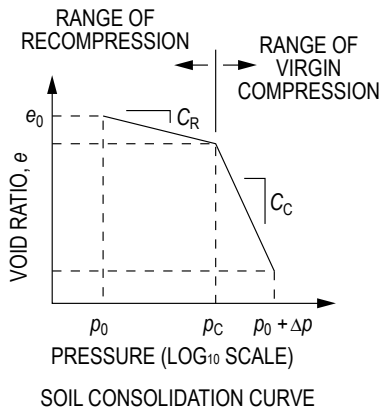
$H$  = total hydraulic head differential

Factor of safety against seepage liquefaction

$$FS_s = i_c / i_e$$

$$i_c = (\gamma_{sat} - \gamma_w) / \gamma_w$$

$i_e$  = seepage exit gradient



$e_0$  = initial void ratio (prior to consolidation)

$\Delta e$  = change in void ratio

$p_0$  = initial effective consolidation stress,  $\sigma'_0$

$p_c$  = past maximum consolidation stress,  $\sigma'_c$

$\Delta p$  = induced change in consolidation stress at center of consolidating stratum

$$\Delta p = I q_s$$

$I$  = Stress influence value at center of consolidating stratum

$q_s$  = applied surface stress causing consolidation

$$\text{If } (p_0 \text{ and } p_0 + \Delta p) < p_c, \text{ then } \Delta H = \frac{H_0}{1 + e_0} \left[ C_R \log \frac{p_0 + \Delta p}{p_0} \right]$$

$$\text{If } (p_0 \text{ and } p_0 + \Delta p) > p_c, \text{ then } \Delta H = \frac{H_0}{1 + e_0} \left[ C_C \log \frac{p_0 + \Delta p}{p_0} \right]$$

$$\text{If } p_0 < p_c < (p_0 + \Delta p), \text{ then } \Delta H = \frac{H_0}{1 + e_0} \left[ C_R \log \frac{p_c}{p_0} + C_C \log \frac{p_0 + \Delta p}{p_c} \right]$$

where:  $\Delta H$  = change in thickness of soil layer

Compression index

In virgin compression range:  $C_C = \Delta e / \Delta \log p$

By correlation to liquid limit:  $C_C = 0.009 (LL - 10)$

Recompression index

In recompression range:  $C_R = \Delta e / \Delta \log p$

By correlation to compression index,  $C_C$ :  $C_R = C_C / 6$

Ultimate consolidation settlement in soil layer

$$S_{ULT} = \epsilon_v H_s$$

$H_s$  = thickness of soil layer

$$\epsilon_v = \Delta e_{TOT} / (1 + e_0)$$

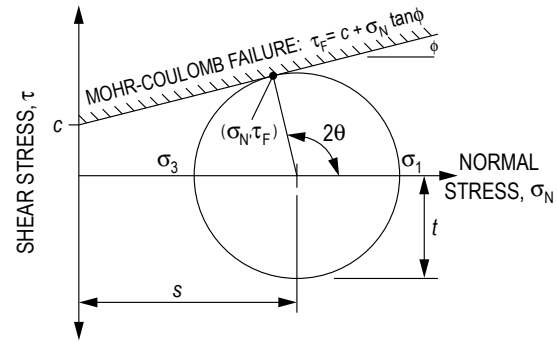
$\Delta e_{TOT}$  = total change in void ratio due to recompression and virgin compression

Approximate settlement (at time  $t = t_c$ )

$$S_T = U_{AV} S_{ULT}$$

$U_{AV}$  = average degree of consolidation

$t_c$  = elapsed time since application of consolidation load



$s$  = mean normal stress

$t$  = maximum shear stress

$\sigma_1$  = major principal stress

$\sigma_3$  = minor principal stress

$\theta$  = orientation angle between plane of existing normal stress and plane of major principal stress

Total normal stress

$$\sigma_N = P/A$$

$P$  = normal force

$A$  = cross-sectional area over which force acts

Effective stress

$$\sigma' = \sigma - u$$

$$u = h_u \gamma_w$$

$h_u$  = uplift or pressure head

Shear stress

$$\tau = T/A$$

$T$  = shearing force

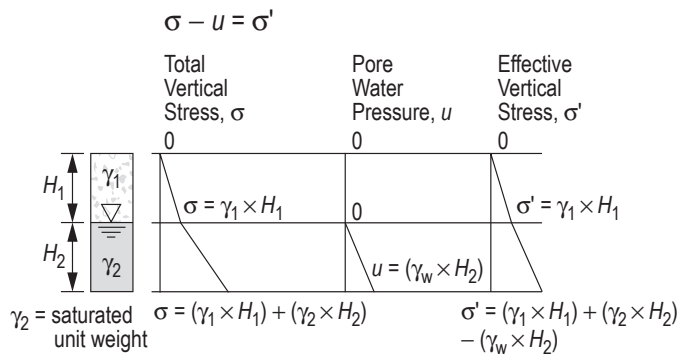
Shear stress at failure

$$\tau_F = c + \sigma_N \tan \phi$$

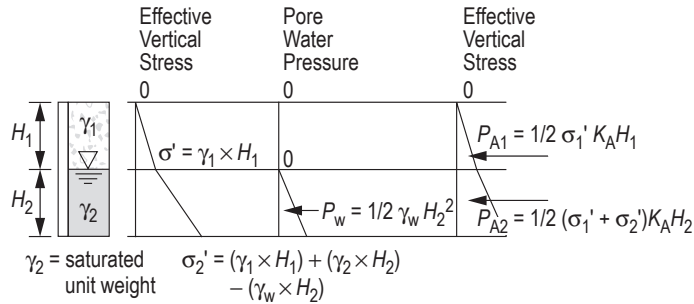
$c$  = cohesion

$\phi$  = angle of internal friction

## Vertical Stress Profiles



## Horizontal Stress Profiles and Forces



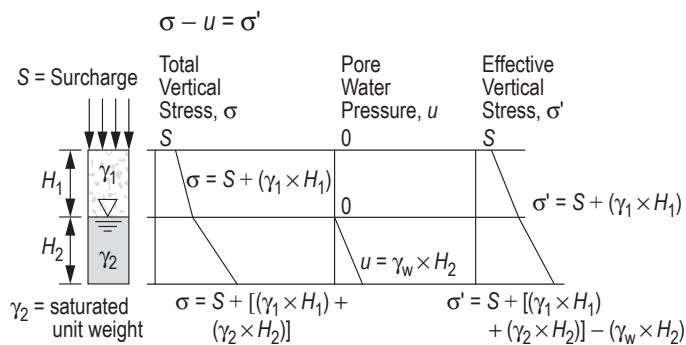
Active forces on retaining wall per unit wall length (as shown):

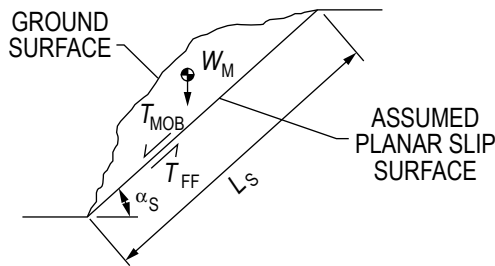
$K_A$  = Rankine active earth pressure coefficient (smooth wall,  $c = 0$ , level backfill) =  $\tan^2 (45^\circ - \phi/2)$

Passive forces on retaining wall per unit wall length (similar to the active forces shown):

$K_p$  = Rankine passive earth pressure coefficient (smooth wall,  $c = 0$ , level backfill) =  $\tan^2 (45^\circ + \phi/2)$

## Vertical Stress Profiles with Surcharge





### SLOPE FAILURE ALONG PLANAR SURFACE

- FS = factor of safety against slope instability  
 =  $T_{FF}/T_{MOB}$   
 $T_{FF}$  = available shearing resistance along slip surface  
 =  $cL_S + W_M \cos\alpha_S \tan\phi$   
 $T_{MOB}$  = mobilized shear force along slip surface  
 =  $W_M \sin\alpha_S$   
 $L_S$  = length of assumed planar slip surface  
 $W_M$  = weight of soil above slip surface  
 $\alpha_S$  = angle of assumed slip surface with respect to horizontal

### AASHTO Soil Classification

GENERAL CLASSIFICATION	GRANULAR MATERIALS ( 35% OR LESS PASSING 0.075 SIEVE )							SILT-CLAY MATERIALS ( MORE THAN 35% PASSING 0.075 SIEVE )			
GROUP CLASSIFICATION	A-1		A-3	A-2				A-4	A-5	A-6	A-7-5 A-7-6
	A-1-a	A-1-b		A-2-4	A-2-5	A-2-6	A-2-7				
SIEVE ANALYSIS, PERCENT PASSING: 2.00 mm (No. 10) 0.425 mm (No. 40) 0.075 mm (No. 200)	≤ 50 ≤ 30 ≤ 15	— ≤ 50 ≤ 25	— ≥ 51 ≤ 10	— — ≤ 35	— — ≤ 35	— — ≤ 35	— — ≤ 35	— — ≥ 36	— — ≥ 36	— — ≥ 36	— — ≥ 36
CHARACTERISTICS OF FRACTION PASSING 0.425 SIEVE (No. 40): LIQUID LIMIT PLASTICITY INDEX *	— 6 max		— NP	≤ 40 ≤ 10	≥ 41 ≤ 10	≤ 40 ≥ 11	≥ 41 ≥ 11	≤ 40 ≤ 10	≥ 41 ≤ 10	≤ 40 ≥ 11	≥ 41 ≥ 11
USUAL TYPES OF CONSTITUENT MATERIALS	STONE FRAGM'TS, GRAVEL, SAND		FINE SAND	SILTY OR CLAYEY GRAVEL AND SAND				SILTY SOILS		CLAYEY SOILS	
GENERAL RATING AS A SUBGRADE	EXCELLENT TO GOOD							FAIR TO POOR			

\*Plasticity index of A-7-5 subgroup is equal to or less than  $LL - 30$ . Plasticity index of A-7-6 subgroup is greater than  $LL - 30$ .  
 NP = Non-plastic (use "0"). Symbol "—" means that the particular sieve analysis is not considered for that classification.

If the soil classification is A4-A7, then calculate the group index (GI) as shown below and report with classification. The higher the GI, the less suitable the soil. Example: A-6 with GI = 15 is less suitable than A-6 with GI = 10.

$$GI = (F - 35) [ 0.2 + 0.005 (LL - 40) ] + 0.01 (F - 15) (PI - 10)$$

- where: F = Percent passing No. 200 sieve, expressed as a whole number. This percentage is based only on the material passing the No. 200 sieve.  
 LL = Liquid limit  
 PI = Plasticity index

If the computed value of  $GI < 0$ , then use  $GI = 0$ .

# ASTM D2487-11 Standard Practice for Classification of Soils for Engineering Purposes (Unified Soil Classification System)

Criteria for Assigning Group Symbols and Group Names Using Laboratory Tests <sup>A</sup>				Soil Classification		
				Group Symbol	Group Name <sup>B</sup>	
COARSE-GRAINED SOILS	Gravels (more than 50% of coarse fraction retained on No. 4 sieve)	Clean Gravels (Less than 5% fines <sup>C</sup> )	$Cu \geq 4$ and $1 \leq Cc \leq 3^D$	GW	Well-graded gravel <sup>E</sup>	
			$Cu < 4$ and/or $[Cc < 1 \text{ or } Cc > 3]^D$	GP	Poorly graded gravel <sup>E</sup>	
		Gravels with Fines (More than 12% fines <sup>C</sup> )	Fines classify as ML or MH	GM	Silty gravel <sup>E, F, G</sup>	
			Fines classify as CL or CH	GC	Clayey gravel <sup>E, F, G</sup>	
	More than 50% retained on No. 200 sieve	Sands (50% or more of coarse fraction passes No. 4 sieve)	Clean Sands (Less than 5% fines <sup>H</sup> )	$Cu \geq 6$ and $1 \leq Cc \leq 3^D$	SW	Well-graded sand <sup>I</sup>
				$Cu < 6$ and/or $[Cc < 1 \text{ or } Cc > 3]^D$	SP	Poorly graded sand <sup>I</sup>
		Sands with Fines (More than 12% fines <sup>H</sup> )	Fines classify as ML or MH	SM	Silty sand <sup>F, G, I</sup>	
			Fines classify as CL or CH	SC	Clayey sand <sup>F, G, I</sup>	
FINE-GRAINED SOILS	Silts and Clays	inorganic	$PI > 7$ and plots on or above "A" line <sup>J</sup>	CL	Lean clay <sup>K, L, M</sup>	
			$PI < 4$ or plots below "A" line <sup>J</sup>	ML	Silt <sup>K, L, M</sup>	
	Liquid limit less than 50	organic	Liquid limit – oven dried/Liquid&#10	OL	Organic clay <sup>K, L, M, N</sup>	
			$< 0.75$		Organic silt <sup>K, L, M, O</sup>	
	50% or more passes the No. 200 sieve	Silts and Clays	inorganic	$PI$ plots on or above "A" line	CH	Fat clay <sup>K, L, M</sup>
				$PI$ plots below "A" line	MH	Elastic silt <sup>K, L, M</sup>
		Liquid limit 50 or more	organic	Liquid limit – oven dried/Liquid&#10	OH	Organic clay <sup>K, L, M, P</sup>
				$< 0.75$		Organic silt <sup>K, L, M, Q</sup>
HIGHLY ORGANIC SOILS		Primarily organic matter, dark in color, and organic odor		PT	Peat	

<sup>A</sup>Based on the material passing the 3-in. (75-mm) sieve.

<sup>B</sup>If field sample contained cobbles or boulders, or both, add "with cobbles or boulders, or both" to group name.

<sup>C</sup>Gravels with 5 to 12% fines require dual symbols:

GW-GM well-graded gravel with silt  
GW-GC well-graded gravel with clay  
GP-GM poorly graded gravel with silt  
GP-GC poorly graded gravel with clay

$$^D Cu = D_{60}/D_{10} \quad Cc = \frac{(D_{30})^2}{D_{10} \times D_{60}}$$

<sup>E</sup>If soil contains  $\geq 15\%$  sand, add "with sand" to group name.

<sup>F</sup>If fines classify as CL-ML, use dual symbol GC-GM, or SC-SM.

<sup>G</sup>If fines are organic, add "with organic fines" to group name.

<sup>H</sup>Sands with 5 to 12% fines require dual symbols:

SW-SM well-graded sand with silt  
SW-SC well-graded sand with clay  
SP-SM poorly graded sand with silt  
SP-SC poorly graded sand with clay

<sup>I</sup>If soil contains  $\geq 15\%$  gravel, add "with gravel" to group name.

<sup>J</sup>If Atterberg limits plot in hatched area, soil is a CL-ML, silty clay.

<sup>K</sup>If soil contains  $15 < 30\%$  plus No. 200, add "with sand" or "with gravel", whichever is predominant.

<sup>L</sup>If soil contains  $\geq 30\%$  plus No. 200, predominantly sand, add "sand" to group name.

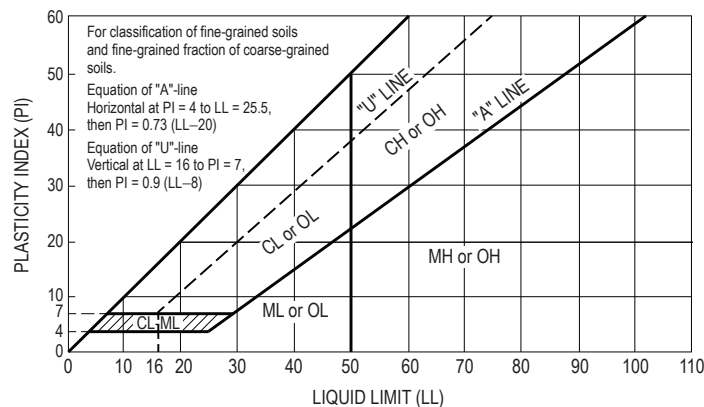
<sup>M</sup>If soil contains  $\geq 30\%$  plus No. 200, predominantly gravel, add "gravelly" to group name.

<sup>N</sup> $PI \geq 4$  and plots on or above "A" line.

<sup>O</sup> $PI < 4$  or plots below "A" line.

<sup>P</sup>PI plots on or above "A" line.

<sup>Q</sup>PI plots below "A" line.



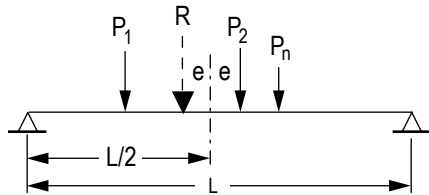


## STRUCTURAL ANALYSIS

### Influence Lines for Beams and Trusses

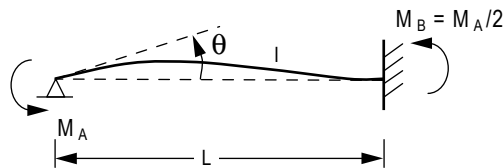
An influence line shows the variation of an effect (reaction, shear and moment in beams, bar force in a truss) caused by moving a unit load across the structure. An influence line is used to determine the position of a moveable set of loads that causes the maximum value of the effect.

### Moving Concentrated Load Sets



The **absolute maximum moment** produced in a beam by a set of "n" moving loads occurs when the resultant "R" of the load set and an adjacent load are equal distance from the centerline of the beam. In general, two possible load set positions must be considered, one for each adjacent load.

### Beam Stiffness and Moment Carryover



$$\theta = \frac{M L}{4 E I} \Rightarrow M = \left( \frac{4 E I}{L} \right) \theta = k_{AB} \theta$$

$k_{AB}$  = stiffness

$M_B = M_A/2$  = carryover

### Truss Deflection by Unit Load Method

The displacement of a truss joint caused by external effects (truss loads, member temperature change, member misfit) is found by applying a unit load at the point that corresponds to the desired displacement.

$$\Delta_{\text{joint}} = \sum_{i=1}^{\text{members}} f_i (\Delta L)_i$$

where:  $\Delta_{\text{joint}}$  = joint displacement at point of application of unit load (+ in direction of unit load)

$f_i$  = force in member "i" caused by unit load (+ tension)

$(\Delta L)_i$  = change in length caused by external effect (+ for increase in member length):

$$= \left( \frac{F L}{A E} \right)_i \text{ for bar force } F \text{ caused by external load}$$

$$= \alpha L_i (\Delta T)_i \text{ for temperature change in member } (\alpha = \text{coefficient of thermal expansion})$$

= member misfit

$L, A$  = member length and cross-sectional area

$E$  = member elastic modulus

### Frame Deflection by Unit Load Method

The displacement of any point on a frame caused by external loads is found by applying a unit load at that point that corresponds to the desired displacement:

$$\Delta = \sum_{i=1}^{\text{members}} \int_{x=0}^{x=L_i} \frac{m_i M_i}{E I_i} dx$$

where:  $\Delta$  = displacement at point of application of unit load (+ in direction of unit load)

$m_i$  = moment equation in member "i" caused by the unit load

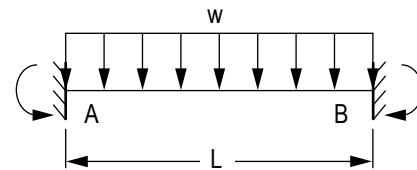
$M_i$  = moment equation in member "i" caused by loads applied to frame

$L_i$  = length of member "i"

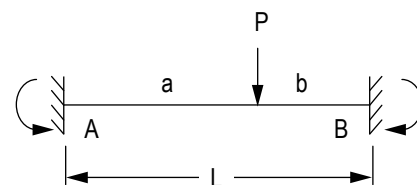
$I_i$  = moment of inertia of member "i"

If either the real loads or the unit load cause no moment in a member, that member can be omitted from the summation.

### Member Fixed-End Moments (Magnitudes)



$$FEM_{AB} = FEM_{BA} = \frac{w L^2}{12}$$



$$FEM_{AB} = \frac{P a b^2}{L^2} \quad FEM_{BA} = \frac{P a^2 b}{L^2}$$

## STABILITY, DETERMINANCY, AND CLASSIFICATION OF STRUCTURES

$m$  = number of members

$r$  = number of independent reaction components

$j$  = number of joints

$c$  = number of condition equations based on known internal moments or forces, such as internal moment of zero at a hinge

### Plane Truss

<u>Static Analysis</u>	<u>Classification</u>
------------------------	-----------------------

$m + r < 2j$	Unstable
--------------	----------

$m + r = 2j$	Stable and statically determinate
--------------	-----------------------------------

$m + r > 2j$	Stable and statically indeterminate
--------------	-------------------------------------

### Plane Frame

<u>Static Analysis</u>	<u>Classification</u>
------------------------	-----------------------

$3m + r < 3j + c$	Unstable
-------------------	----------

$3m + r = 3j + c$	Stable and statically determinate
-------------------	-----------------------------------

$3m + r > 3j + c$	Stable and statically indeterminate
-------------------	-------------------------------------

Stability also requires an appropriate arrangement of members and reaction components.

## STRUCTURAL DESIGN

### Live Load Reduction

The effect on a building member of nominal occupancy live loads may often be reduced based on the loaded floor area supported by the member. A typical model used for computing reduced live load (as found in ASCE 7 and many building codes) is:

$$L_{\text{reduced}} = L_{\text{nominal}} \left( 0.25 + \frac{15}{\sqrt{K_{LL} A_T}} \right) \geq 0.4 L_{\text{nominal}}$$

where:  $L_{\text{nominal}}$  = nominal live load given in ASCE 7 or a building code

$A_T$  = the cumulative floor tributary area supported by the member

$K_{LL} A_T$  = area of influence supported by the member

$K_{LL}$  = ratio of area of influence to the tributary area supported by the member:

$K_{LL} = 4$  (typical columns)

$K_{LL} = 2$  (typical beams and girders)

## Load Combinations using Strength Design (LRFD, USD)

Nominal loads used in following combinations:

$D$  = dead loads

$E$  = earthquake loads

$L$  = live loads (floor)

$L_r$  = live loads (roof)

$R$  = rain load

$S$  = snow load

$W$  = wind load

Load factors  $\lambda$ :  $\lambda_D$  (dead load),  $\lambda_L$  (live load), etc.

Basic combinations  $L_r/S/R$  = largest of  $L_r$ ,  $S$ ,  $R$

$L$  or  $0.8W$  = larger of  $L$ ,  $0.8W$

$1.4D$

$1.2D + 1.6L + 0.5(L_r/S/R)$

$1.2D + 1.6(L_r/S/R) + (L \text{ or } 0.8W)$

$1.2D + 1.6W + L + 0.5(L_r/S/R)$

$1.2D + 1.0E + L + 0.2S$

$0.9D + 1.6W$

$0.9D + 1.0E$

## DESIGN OF REINFORCED CONCRETE COMPONENTS (ACI 318-11)

U.S. Customary units

### Definitions

$a$  = depth of equivalent rectangular stress block, in.

$A_g$  = gross area of column, in<sup>2</sup>

$A_s$  = area of tension reinforcement, in<sup>2</sup>

$A_{st}$  = total area of longitudinal reinforcement, in<sup>2</sup>

$A_v$  = area of shear reinforcement within a distance  $s$ , in.

$b$  = width of compression face of member, in.

$\beta_1$  = ratio of depth of rectangular stress block,  $a$ , to depth to neutral axis,  $c$

$$= 0.85 \geq 0.85 - 0.05 \left( \frac{f'_c - 4,000}{1,000} \right) \geq 0.65$$

$c$  = distance from extreme compression fiber to neutral axis, in.

$d$  = distance from extreme compression fiber to centroid of nonprestressed tension reinforcement, in.

$d_t$  = distance from extreme compression fiber to extreme tension steel, in.

$E_c$  = modulus of elasticity =  $33w_c^{1.5} \sqrt{f'_c}$ , psi

$\epsilon_t$  = net tensile strain in extreme tension steel at nominal strength

$f'_c$  = compressive strength of concrete, psi

$f_y$  = yield strength of steel reinforcement, psi

$M_n$  = nominal moment strength at section, in.-lb

$\phi M_n$  = design moment strength at section, in.-lb

$M_u$  = factored moment at section, in.-lb

$P_n$  = nominal axial load strength at given eccentricity, lb

$\phi P_n$  = design axial load strength at given eccentricity, lb

$P_u$  = factored axial force at section, lb

$\rho_g$  = ratio of total reinforcement area to cross-sectional area of column =  $A_{st}/A_g$

$s$  = spacing of shear ties measured along longitudinal axis of member, in.

$V_c$  = nominal shear strength provided by concrete, lb

$V_n$  = nominal shear strength at section, lb

$\phi V_n$  = design shear strength at section, lb

$V_s$  = nominal shear strength provided by reinforcement, lb

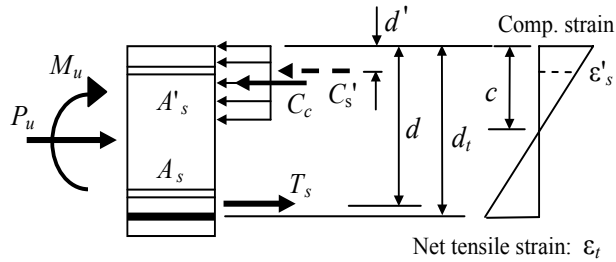
$V_u$  = factored shear force at section, lb

### ASTM STANDARD REINFORCING BARS

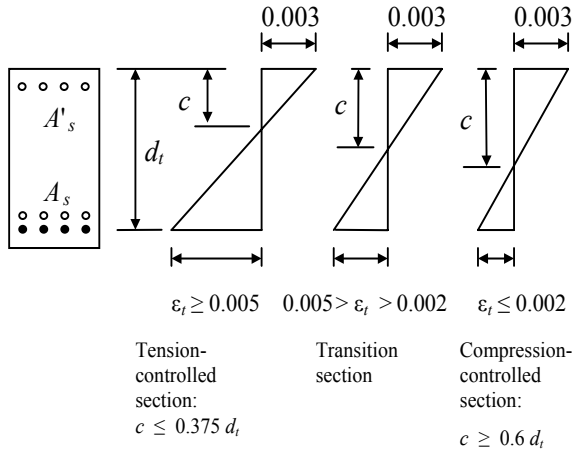
BAR SIZE	DIAMETER, IN.	AREA, IN <sup>2</sup>	WEIGHT, LB/FT
#3	0.375	0.11	0.376
#4	0.500	0.20	0.668
#5	0.625	0.31	1.043
#6	0.750	0.44	1.502
#7	0.875	0.60	2.044
#8	1.000	0.79	2.670
#9	1.128	1.00	3.400
#10	1.270	1.27	4.303
#11	1.410	1.56	5.313
#14	1.693	2.25	7.650
#18	2.257	4.00	13.60

## UNIFIED DESIGN PROVISIONS

### Internal Forces and Strains



### Strain Conditions



## RESISTANCE FACTORS, $\phi$

Tension-controlled sections ( $\varepsilon_t \geq 0.005$ ):  $\phi = 0.9$

Compression-controlled sections ( $\varepsilon_t \leq 0.002$ ):

Members with tied reinforcement  $\phi = 0.65$

Transition sections ( $0.002 < \varepsilon_t < 0.005$ ):

Members with tied reinforcement  $\phi = 0.48 + 83\varepsilon_t$

Shear and torsion  $\phi = 0.75$

Bearing on concrete  $\phi = 0.65$

## BEAMS—FLEXURE

$$\phi M_n \geq M_u$$

### For All Beams

Net tensile strain:  $a = \beta_1 c$

$$\varepsilon_t = \frac{0.003(d_t - c)}{c} = \frac{0.003(\beta_1 d_t - a)}{a}$$

Design moment strength:  $\phi M_n$

where:  $\phi = 0.9$  [ $\varepsilon_t \geq 0.005$ ]

$\phi = 0.48 + 83\varepsilon_t$  [ $0.004 \leq \varepsilon_t < 0.005$ ]

### Singly-Reinforced Beams

$$a = \frac{A_s f_y}{0.85 f'_c b}$$

$$M_n = 0.85 f'_c a b \left( d - \frac{a}{2} \right) = A_s f_y \left( d - \frac{a}{2} \right)$$

## BEAMS—SHEAR

$$\phi V_n \geq V_u$$

Nominal shear strength:

$$V_n = V_c + V_s$$

$$V_c = 2 b_w d \sqrt{f'_c}$$

$$V_s = \frac{A_v f_y d}{s} \text{ (may not exceed } 8 b_w d \sqrt{f'_c} \text{)}$$

Required and maximum-permitted stirrup spacing,  $s$

$$V_u \leq \frac{\phi V_c}{2}: \text{ No stirrups required}$$

$$V_u > \frac{\phi V_c}{2}: \text{ Use the following table (} A_v \text{ given)}$$

	$\frac{\phi V_c}{2} < V_u \leq \phi V_c$	$V_u > \phi V_c$
Required spacing	<p>Smaller of:</p> $s = \frac{A_v f_y}{50 b_w}$ $s = \frac{A_v f_y}{0.75 b_w \sqrt{f'_c}}$	$V_s = \frac{V_u}{\phi} - V_c$ $s = \frac{A_v f_y d}{V_s}$
Maximum permitted spacing	<p>Smaller of:</p> $s = \frac{d}{2}$ <p>OR</p> $s = 24"$	$V_s \leq 4 b_w d \sqrt{f'_c}$ <p>Smaller of:</p> $s = \frac{d}{2} \quad \text{OR}$ $s = 24"$
		$V_s > 4 b_w d \sqrt{f'_c}$ <p>Smaller of:</p> $s = \frac{d}{4}$ $s = 12"$

## SHORT COLUMNS

### Limits for Main Reinforcements

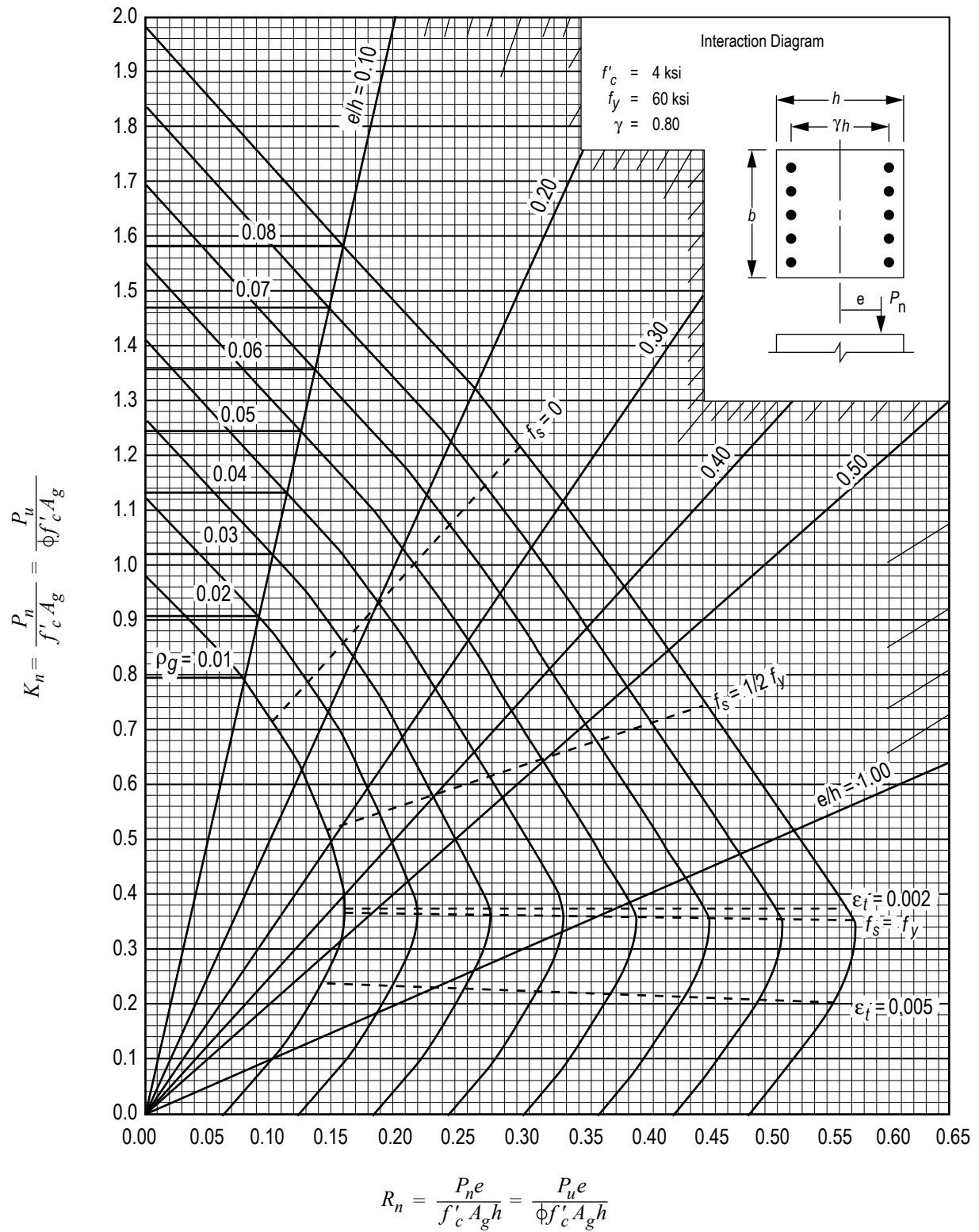
$$\rho_g = \frac{A_{st}}{A_g}$$

$$0.01 \leq \rho_g \leq 0.08$$

### Design Column Strength, Tied Columns

$$\phi = 0.65$$

$$\phi P_n = 0.80 \phi [0.85 f'_c (A_g - A_{st}) + A_{st} f_y]$$



### GRAPH A.11

Column strength interaction diagram for rectangular section with bars on end faces and  $\gamma = 0.80$  (for instructional use only).

Nilson, Arthur H., David Darwin, and Charles W. Dolan, *Design of Concrete Structures*, 13th ed., McGraw-Hill, New York, 2004.

## DESIGN OF STEEL COMPONENTS

(ANSI/AISC 360-10)

LRFD,  $E = 29,000$  ksi

### BEAMS

For doubly symmetric compact I-shaped members bent about their major axis, the *design flexural strength*  $\phi_b M_n$  is determined with  $\phi_b = 0.90$  as follows:

#### Yielding

$$M_n = M_p = F_y Z_x$$

where

$F_y$  = specified minimum yield stress

$Z_x$  = plastic section modulus about the x-axis

#### Lateral-Torsional Buckling

Based on bracing where  $L_b$  is the length between points that are either braced against lateral displacement of the compression flange or braced against twist of the cross section with respect to the length limits  $L_p$  and  $L_r$ :

When  $L_b \leq L_p$ , the limit state of lateral-torsional buckling does not apply.

When  $L_p < L_b \leq L_r$

$$M_n = C_b \left[ M_p - (M_p - 0.7 F_y S_x) \left( \frac{L_b - L_p}{L_r - L_p} \right) \right] \leq M_p$$

where

$$C_b = \frac{12.5 M_{\max}}{2.5 M_{\max} + 3 M_A + 4 M_B + 3 M_C}$$

$M_{\max}$  = absolute value of maximum moment in the unbraced segment

$M_A$  = absolute value of maximum moment at quarter point of the unbraced segment

$M_B$  = absolute value of maximum moment at centerline of the unbraced segment

$M_C$  = absolute value of maximum moment at three-quarter of the unbraced segment

#### Shear

The *design shear strength*  $\phi_v V_n$  is determined with  $\phi_v = 1.00$  for webs of rolled I-shaped members and is determined as follows:

$$V_n = 0.6 F_y (d t_w)$$

## COLUMNS

The *design compressive strength*  $\phi_c P_n$  is determined with  $\phi_c = 0.90$  for flexural buckling of members without slender elements and is determined as follows:

$$P_n = F_{cr} A_g$$

where the critical stress  $F_{cr}$  is determined as follows:

$$(a) \text{ When } \frac{KL}{r} \leq 4.71 \sqrt{\frac{E}{F_y}}, F_{cr} = [0.658 \frac{F_y}{F_c}] F_y$$

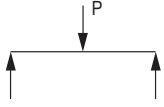
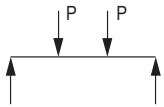
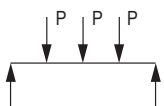
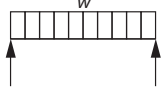
$$(b) \text{ When } \frac{KL}{r} > 4.71 \sqrt{\frac{E}{F_y}}, F_{cr} = 0.877 F_y$$

where

$KL/r$  is the effective slenderness ratio based on the column effective length ( $KL$ ) and radius of gyration ( $r$ )

$KL$  is determined from AISC Table C-A-7.1 or AISC Figures C-A-7.1 and C-A-7.2 on p. 158.

$F_e$  is the elastic buckling stress =  $\pi^2 E / (KL/r)^2$

VALUES OF $C_b$ FOR SIMPLY SUPPORTED BEAMS		
LOAD	LATERAL BRACING ALONG SPAN	$C_b$
	NONE LOAD AT MIDPOINT	1.32
	AT LOAD POINT	1.67
	NONE LOADS AT THIRD POINTS	1.14
	AT LOAD POINTS LOADS SYMMETRICALLY PLACED	1.67
	NONE LOADS AT QUARTER POINTS	1.14
	AT LOAD POINTS LOADS AT QUARTER POINTS	1.67
	NONE	1.14
	AT MIDPOINT	1.30
	AT THIRD POINTS	1.45
	AT QUARTER POINTS	1.52
	AT FIFTH POINTS	1.56

NOTE: LATERAL BRACING MUST ALWAYS BE PROVIDED AT POINTS OF SUPPORT PER AISC SPECIFICATION CHAPTER F.

Adapted from *Steel Construction Manual*, 14th ed., AISC, 2011.

## TENSION MEMBERS

### Flat bars or angles, bolted or welded

#### Definitions

Bolt diameter:  $d_b$

Nominal hole diameter:  $d_h = d_b + 1/16"$

Gross width of member:  $b_g$

Member thickness:  $t$

Connection eccentricity:  $\bar{x}$

Gross area:  $A_g = b_g t$  (use tabulated areas for angles)

Net area (parallel holes):  $A_n = \left[ b_g - \sum \left( d_h + \frac{1}{16} \right) \right] t$

Net area (staggered holes):

$$A_n = \left[ b_g - \sum \left( d_h + \frac{1}{16} \right) + \sum \frac{s^2}{4g} \right] t$$

$s$  = longitudinal spacing of consecutive holes

$g$  = transverse spacing between lines of holes

Effective area (bolted members):

$$A_e = UA_n \quad \begin{cases} \text{Flat bars: } U = 1.0 \\ \text{Angles: } U = 1 - \bar{x}/L \end{cases}$$

Effective area (welded members):

$$A_e = UA_n \quad \begin{cases} \text{Flat bars or angles with transverse} \\ \text{welds: } U = 1.0 \\ \text{Flat bars of width "w", longitudinal} \\ \text{welds of length "L" only:} \\ \quad U = 1.0 \quad (L \geq 2w) \\ \quad U = 0.87 \quad (2w > L \geq 1.5w) \\ \quad U = 0.75 \quad (1.5w > L > w) \\ \text{Angles with longitudinal welds} \\ \text{only} \\ \quad U = 1 - \bar{x}/L \end{cases}$$

### Limit States and Available Strengths

Yielding:  $\phi_y = 0.90$   
 $\phi T_n = \phi_y F_y A_g$

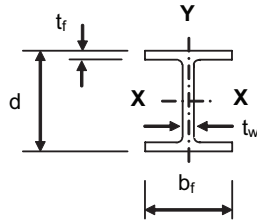
Fracture:  $\phi_f = 0.75$   
 $\phi T_n = \phi_f F_u A_e$

Block shear:  $\phi = 0.75$   
 $U_{bs} = 1.0$  (flat bars and angles)  
 $A_{gv}$  = gross area for shear  
 $A_{nv}$  = net area for shear  
 $A_{nt}$  = net area for tension

$$\phi T_n = \begin{cases} 0.75 F_u [0.6 A_{nv} + U_{bs} A_{nt}] \\ 0.75 [0.6 F_y A_{gv} + U_{bs} F_u A_{nt}] \end{cases}$$

smaller





**Table 1-1: W Shapes Dimensions and Properties**

Shape	Area	Depth	Web	Flange		Axis X-X				Axis Y-Y	
	A	d	t <sub>w</sub>	b <sub>f</sub>	t <sub>f</sub>	I	S	r	Z	I	r
	In. <sup>2</sup>	In.	In.	In.	In.	In. <sup>4</sup>	In. <sup>3</sup>	In.	In. <sup>3</sup>	In. <sup>4</sup>	In.
W24X68	20.1	23.7	0.415	8.97	0.585	1830	154	9.55	177	70.4	1.87
W24X62	18.2	23.7	0.430	7.04	0.590	1550	131	9.23	153	34.5	1.38
W24X55	16.3	23.6	0.395	7.01	0.505	1350	114	9.11	134	29.1	1.34
W21X73	21.5	21.2	0.455	8.30	0.740	1600	151	8.64	172	70.6	1.81
W21X68	20.0	21.1	0.430	8.27	0.685	1480	140	8.60	160	64.7	1.80
W21X62	18.3	21.0	0.400	8.24	0.615	1330	127	8.54	144	57.5	1.77
W21X55	16.2	20.8	0.375	8.22	0.522	1140	110	8.40	126	48.4	1.73
W21X57	16.7	21.1	0.405	6.56	0.650	1170	111	8.36	129	30.6	1.35
W21X50	14.7	20.8	0.380	6.53	0.535	984	94.5	8.18	110	24.9	1.30
W21X48	14.1	20.6	0.350	8.14	0.430	959	93.0	8.24	107	38.7	1.66
W21X44	13.0	20.7	0.350	6.50	0.450	843	81.6	8.06	95.4	20.7	1.26
W18X71	20.8	18.5	0.495	7.64	0.810	1170	127	7.50	146	60.3	1.70
W18X65	19.1	18.4	0.450	7.59	0.750	1070	117	7.49	133	54.8	1.69
W18X60	17.6	18.2	0.415	7.56	0.695	984	108	7.47	123	50.1	1.68
W18X55	16.2	18.1	0.390	7.53	0.630	890	98.3	7.41	112	44.9	1.67
W18X50	14.7	18.0	0.355	7.50	0.570	800	88.9	7.38	101	40.1	1.65
W18X46	13.5	18.1	0.360	6.06	0.605	712	78.8	7.25	90.7	22.5	1.29
W18X40	11.8	17.9	0.315	6.02	0.525	612	68.4	7.21	78.4	19.1	1.27
W16X67	19.7	16.3	0.395	10.2	0.67	954	117	6.96	130	119	2.46
W16X57	16.8	16.4	0.430	7.12	0.715	758	92.2	6.72	105	43.1	1.60
W16X50	14.7	16.3	0.380	7.07	0.630	659	81.0	6.68	92.0	37.2	1.59
W16X45	13.3	16.1	0.345	7.04	0.565	586	72.7	6.65	82.3	32.8	1.57
W16X40	11.8	16.0	0.305	7.00	0.505	518	64.7	6.63	73.0	28.9	1.57
W16X36	10.6	15.9	0.295	6.99	0.430	448	56.5	6.51	64.0	24.5	1.52
W14X74	21.8	14.2	0.450	10.1	0.785	795	112	6.04	126	134	2.48
W14X68	20.0	14.0	0.415	10.0	0.720	722	103	6.01	115	121	2.46
W14X61	17.9	13.9	0.375	9.99	0.645	640	92.1	5.98	102	107	2.45
W14X53	15.6	13.9	0.370	8.06	0.660	541	77.8	5.89	87.1	57.7	1.92
W14X48	14.1	13.8	0.340	8.03	0.595	484	70.2	5.85	78.4	51.4	1.91
W12X79	23.2	12.4	0.470	12.1	0.735	662	107	5.34	119	216	3.05
W12X72	21.1	12.3	0.430	12.0	0.670	597	97.4	5.31	108	195	3.04
W12X65	19.1	12.1	0.390	12.0	0.605	533	87.9	5.28	96.8	174	3.02
W12X58	17.0	12.2	0.360	10.0	0.640	475	78.0	5.28	86.4	107	2.51
W12X53	15.6	12.1	0.345	9.99	0.575	425	70.6	5.23	77.9	95.8	2.48
W12X50	14.6	12.2	0.370	8.08	0.640	391	64.2	5.18	71.9	56.3	1.96
W12X45	13.1	12.1	0.335	8.05	0.575	348	57.7	5.15	64.2	50.0	1.95
W12X40	11.7	11.9	0.295	8.01	0.515	307	51.5	5.13	57.0	44.1	1.94
W10x60	17.6	10.2	0.420	10.1	0.680	341	66.7	4.39	74.6	116	2.57
W10x54	15.8	10.1	0.370	10.0	0.615	303	60.0	4.37	66.6	103	2.56
W10x49	14.4	10.0	0.340	10.0	0.560	272	54.6	4.35	60.4	93.4	2.54
W10x45	13.3	10.1	0.350	8.02	0.620	248	49.1	4.32	54.9	53.4	2.01
W10x39	11.5	9.92	0.315	7.99	0.530	209	42.1	4.27	46.8	45.0	1.98

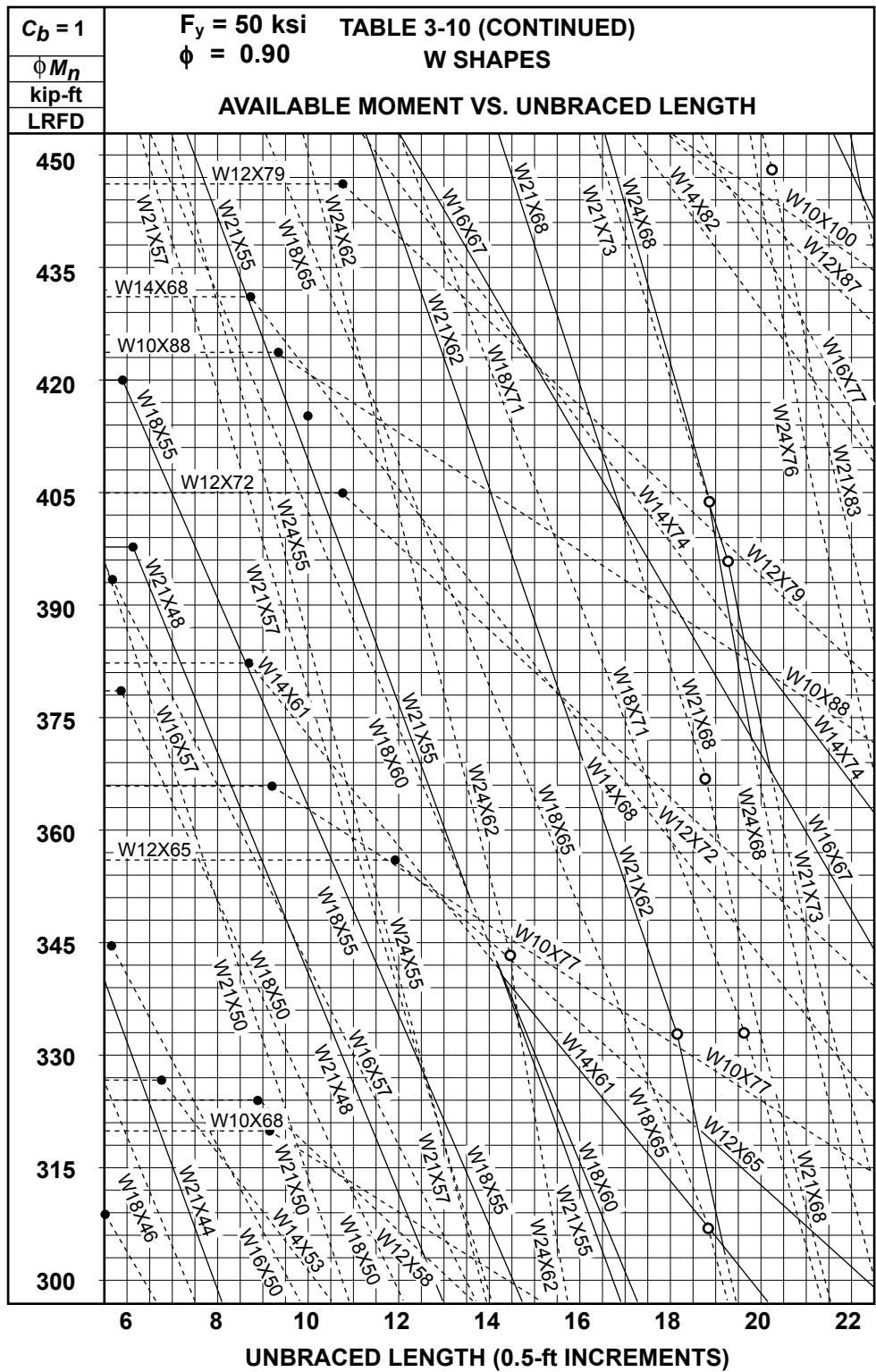
Adapted from *Steel Construction Manual*, 14th ed., AISC, 2011.

<div> <div><b>Z<sub>x</sub></b></div> <div>AISC Table 3-2</div> <div><b>W Shapes – Selection by Z<sub>x</sub></b></div> <div> F<sub>y</sub> = 50 ksi  φ<sub>b</sub> = 0.90  φ<sub>v</sub> = 1.00 </div> </div>								
Shape	Z <sub>x</sub> in. <sup>3</sup>	φ <sub>b</sub> M <sub>px</sub> kip-ft	φ <sub>b</sub> M <sub>rx</sub> kip-ft	φ <sub>b</sub> BF kips	L <sub>p</sub> ft.	L <sub>r</sub> ft.	I <sub>x</sub> in. <sup>4</sup>	φ <sub>v</sub> V <sub>nx</sub> kips
<b>W24 x 55</b>	<b>134</b>	<b>503</b>	<b>299</b>	<b>22.2</b>	<b>4.73</b>	<b>13.9</b>	<b>1350</b>	<b>251</b>
W18 x 65	133	499	307	14.9	5.97	18.8	1070	248
W12 x 87	132	495	310	5.76	10.8	43.0	740	194
W16 x 67	130	488	307	10.4	8.69	26.1	954	194
W10 x 100	130	488	294	4.01	9.36	57.7	623	226
W21 x 57	129	484	291	20.1	4.77	14.3	1170	256
<b>W21 x 55</b>	<b>126</b>	<b>473</b>	<b>289</b>	<b>16.3</b>	<b>6.11</b>	<b>17.4</b>	<b>1140</b>	<b>234</b>
W14 x 74	126	473	294	8.03	8.76	31.0	795	191
W18 x 60	123	461	284	14.5	5.93	18.2	984	227
W12 x 79	119	446	281	5.67	10.8	39.9	662	175
W14 x 68	115	431	270	7.81	8.69	29.3	722	175
W10 x 88	113	424	259	3.95	9.29	51.1	534	197
<b>W18 x 55</b>	<b>112</b>	<b>420</b>	<b>258</b>	<b>13.9</b>	<b>5.90</b>	<b>17.5</b>	<b>890</b>	<b>212</b>
<b>W21 x 50</b>	<b>110</b>	<b>413</b>	<b>248</b>	<b>18.3</b>	<b>4.59</b>	<b>13.6</b>	<b>984</b>	<b>237</b>
W12 x 72	108	405	256	5.59	10.7	37.4	597	158
<b>W21 x 48</b>	<b>107</b>	<b>398</b>	<b>244</b>	<b>14.7</b>	<b>6.09</b>	<b>16.6</b>	<b>959</b>	<b>217</b>
W16 x 57	105	394	242	12.0	5.56	18.3	758	212
W14 x 61	102	383	242	7.46	8.65	27.5	640	156
W18 x 50	101	379	233	13.1	5.83	17.0	800	192
W10 x 77	97.6	366	225	3.90	9.18	45.2	455	169
W12 x 65	96.8	356	231	5.41	11.9	35.1	533	142
<b>W21 x 44</b>	<b>95.4</b>	<b>358</b>	<b>214</b>	<b>16.8</b>	<b>4.45</b>	<b>13.0</b>	<b>843</b>	<b>217</b>
W16 x 50	92.0	345	213	11.4	5.62	17.2	659	185
W18 x 46	90.7	340	207	14.6	4.56	13.7	712	195
W14 x 53	87.1	327	204	7.93	6.78	22.2	541	155
W12 x 58	86.4	324	205	5.66	8.87	29.9	475	132
W10 x 68	85.3	320	199	3.86	9.15	40.6	394	147
W16 x 45	82.3	309	191	10.8	5.55	16.5	586	167
<b>W18 x 40</b>	<b>78.4</b>	<b>294</b>	<b>180</b>	<b>13.3</b>	<b>4.49</b>	<b>13.1</b>	<b>612</b>	<b>169</b>
W14 x 48	78.4	294	184	7.66	6.75	21.1	484	141
W12 x 53	77.9	292	185	5.48	8.76	28.2	425	125
W10 x 60	74.6	280	175	3.80	9.08	36.6	341	129
<b>W16 x 40</b>	<b>73.0</b>	<b>274</b>	<b>170</b>	<b>10.1</b>	<b>5.55</b>	<b>15.9</b>	<b>518</b>	<b>146</b>
W12 x 50	71.9	270	169	5.97	6.92	23.9	391	135
W8 x 67	70.1	263	159	2.60	7.49	47.7	272	154
W14 x 43	69.6	261	164	7.24	6.68	20.0	428	125
W10 x 54	66.6	250	158	3.74	9.04	33.7	303	112
<b>W18 x 35</b>	<b>66.5</b>	<b>249</b>	<b>151</b>	<b>12.3</b>	<b>4.31</b>	<b>12.4</b>	<b>510</b>	<b>159</b>
W12 x 45	64.2	241	151	5.75	6.89	22.4	348	121
W16 x 36	64.0	240	148	9.31	5.37	15.2	448	140
W14 x 38	61.5	231	143	8.10	5.47	16.2	385	131
W10 x 49	60.4	227	143	3.67	8.97	31.6	272	102
W8 x 58	59.8	224	137	2.56	7.42	41.7	228	134
W12 x 40	57.0	214	135	5.50	6.85	21.1	307	106
W10 x 45	54.9	206	129	3.89	7.10	26.9	248	106





$$M_{rx} = (0.7F_y)S_x$$

$$BF = \frac{M_{px} - M_{rx}}{L_r - L_p}$$

Adapted from *Steel Construction Manual*, 14th ed., AISC, 2011.



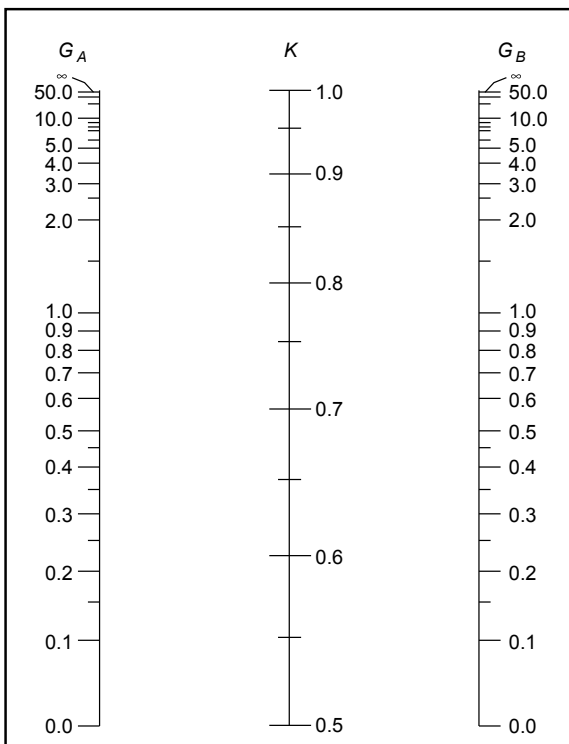
*Steel Construction Manual*, 14th ed., AISC, 2011.

TABLE C-A-7.1 APPROXIMATE VALUES OF EFFECTIVE LENGTH FACTOR, $K$						
BUCKLED SHAPE OF COLUMN IS SHOWN BY DASHED LINE.	(a)	(b)	(c)	(d)	(e)	(f)
THEORETICAL $K$ VALUE	0.5	0.7	1.0	1.0	2.0	2.0
RECOMMENDED DESIGN VALUE WHEN IDEAL CONDITIONS ARE APPROXIMATED	0.65	0.80	1.2	1.0	2.10	2.0
END CONDITION CODE	 ROTATION FIXED AND TRANSLATION FIXED  ROTATION FREE AND TRANSLATION FIXED  ROTATION FIXED AND TRANSLATION FREE  ROTATION FREE AND TRANSLATION FREE					

FOR COLUMN ENDS SUPPORTED BY, BUT NOT RIGIDLY CONNECTED TO, A FOOTING OR FOUNDATION,  $G$  IS THEORETICALLY INFINITY BUT UNLESS DESIGNED AS A TRUE FRICTION-FREE PIN, MAY BE TAKEN AS 10 FOR PRACTICAL DESIGNS. IF THE COLUMN END IS RIGIDLY ATTACHED TO A PROPERLY DESIGNED FOOTING,  $G$  MAY BE TAKEN AS 1.0. SMALLER VALUES MAY BE USED IF JUSTIFIED BY ANALYSIS.

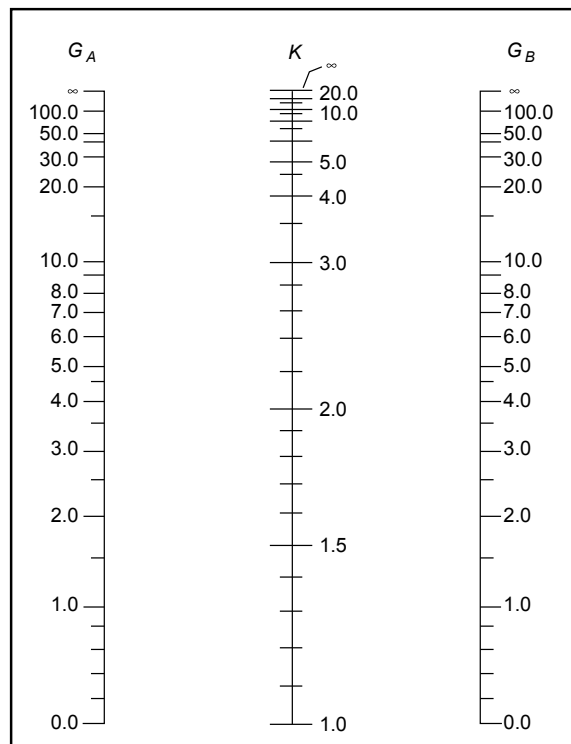
**AISC Figure C-A-7.1**

Alignment chart, sidesway inhibited (braced frame)



**AISC Figure C-A-7.2**

Alignment chart, sidesway uninhibited (moment frame)



AISC Table 4-22  
Available Critical Stress  $\phi_c F_{cr}$  for Compression Members  
 $F_y = 50$  ksi  $\phi_c = 0.90$

$\frac{KL}{r}$	$\phi F_{cr}$ ksi	$\frac{KL}{r}$	$\phi F_{cr}$ ksi	$\frac{KL}{r}$	$\phi F_{cr}$ ksi	$\frac{KL}{r}$	$\phi F_{cr}$ ksi	$\frac{KL}{r}$	$\phi F_{cr}$ ksi
1	45.0	41	39.8	81	27.9	121	15.4	161	8.72
2	45.0	42	39.5	82	27.5	122	15.2	162	8.61
3	45.0	43	39.3	83	27.2	123	14.9	163	8.50
4	44.9	44	39.1	84	26.9	124	14.7	164	8.40
5	44.9	45	38.8	85	26.5	125	14.5	165	8.30
6	44.9	46	38.5	86	26.2	126	14.2	166	8.20
7	44.8	47	38.3	87	25.9	127	14.0	167	8.10
8	44.8	48	38.0	88	25.5	128	13.8	168	8.00
9	44.7	49	37.7	89	25.2	129	13.6	169	7.89
10	44.7	50	37.5	90	24.9	130	13.4	170	7.82
11	44.6	51	37.2	91	24.6	131	13.2	171	7.73
12	44.5	52	36.9	92	24.2	132	13.0	172	7.64
13	44.4	53	36.7	93	23.9	133	12.8	173	7.55
14	44.4	54	36.4	94	23.6	134	12.6	174	7.46
15	44.3	55	36.1	95	23.3	135	12.4	175	7.38
16	44.2	56	35.8	96	22.9	136	12.2	176	7.29
17	44.1	57	35.5	97	22.6	137	12.0	177	7.21
18	43.9	58	35.2	98	22.3	138	11.9	178	7.13
19	43.8	59	34.9	99	22.0	139	11.7	179	7.05
20	43.7	60	34.6	100	21.7	140	11.5	180	6.97
21	43.6	61	34.3	101	21.3	141	11.4	181	6.90
22	43.4	62	34.0	102	21.0	142	11.2	182	6.82
23	43.3	63	33.7	103	20.7	143	11.0	183	6.75
24	43.1	64	33.4	104	20.4	144	10.9	184	6.67
25	43.0	65	33.0	105	20.1	145	10.7	185	6.60
26	42.8	66	32.7	106	19.8	146	10.6	186	6.53
27	42.7	67	32.4	107	19.5	147	10.5	187	6.46
28	42.5	68	32.1	108	19.2	148	10.3	188	6.39
29	42.3	69	31.8	109	18.9	149	10.2	189	6.32
30	42.1	70	31.4	110	18.6	150	10.0	190	6.26
31	41.9	71	31.1	111	18.3	151	9.91	191	6.19
32	41.8	72	30.8	112	18.0	152	9.78	192	6.13
33	41.6	73	30.5	113	17.7	153	9.65	193	6.06
34	41.4	74	30.2	114	17.4	154	9.53	194	6.00
35	41.2	75	29.8	115	17.1	155	9.40	195	5.94
36	40.9	76	29.5	116	16.8	156	9.28	196	5.88
37	40.7	77	29.2	117	16.5	157	9.17	197	5.82
38	40.5	78	28.8	118	16.2	158	9.05	198	5.76
39	40.3	79	28.5	119	16.0	159	8.94	199	5.70
40	40.0	80	28.2	120	15.7	160	8.82	200	5.65

Adapted from *Steel Construction Manual*, 14th ed., AISC, 2011.

I Selected W14, W12, W10				AISC Table 4-1 Available Strength in Axial Compression, kips—W shapes LRFD: $\phi P_n$										F <sub>y</sub> = 50 ksi $\phi_c$ = 0.90			
Shape wt/ft	W14				W12							W10					
	74	68	61	53	48	58	53	50	45	40	60	54	49	45	39		
0	980	899	806	702	636	767	701	657	590	526	794	712	649	597	516		
6	922	844	757	633	573	722	659	595	534	475	750	672	612	543	469		
7	901	826	740	610	552	707	644	574	516	458	734	658	599	525	452		
8	878	804	721	585	529	689	628	551	495	439	717	643	585	505	435		
9	853	781	700	557	504	670	610	526	472	419	698	625	569	483	415		
10	826	755	677	528	477	649	590	499	448	397	677	607	551	460	395		
11	797	728	652	497	449	627	569	471	422	375	655	586	533	435	373		
12	766	700	626	465	420	603	547	443	396	351	631	565	513	410	351		
13	734	670	599	433	391	578	525	413	370	328	606	543	493	384	328		
14	701	639	572	401	361	553	501	384	343	304	581	520	471	358	305		
15	667	608	543	369	332	527	477	354	317	280	555	496	450	332	282		
16	632	576	515	338	304	500	452	326	291	257	528	472	428	306	260		
17	598	544	486	308	276	473	427	297	265	234	501	448	405	281	238		
18	563	512	457	278	250	446	402	270	241	212	474	423	383	256	216		
19	528	480	428	250	224	420	378	244	217	191	447	399	360	233	195		
20	494	448	400	226	202	393	353	220	196	172	420	375	338	210	176		
22	428	387	345	186	167	342	306	182	162	142	367	327	295	174	146		
24	365	329	293	157	140	293	261	153	136	120	317	282	254	146	122		
26	311	281	250	133	120	249	222	130	116	102	270	241	216	124	104		
28	268	242	215	115	103	215	192	112	99.8	88.0	233	208	186	107	90.0		
30	234	211	187	100	89.9	187	167	97.7	87.0	76.6	203	181	162	93.4	78.4		
32	205	185	165	88.1		165	147	82.9	76.4	67.3	179	159	143	82.1	68.9		
34	182	164	146			146	130				158	141	126				
36	162	146	130			130	116				141	126	113				
38	146	131	117			117	104				127	113	101				
40	131	119	105			105	93.9				114	102	91.3				

Heavy line indicates KL/r equal to or greater than 200

Adapted from *Steel Construction Manual*, 14th ed., AISC, 2011

## HYDROLOGY/WATER RESOURCES

### NRCS (SCS) Rainfall-Runoff

$$Q = \frac{(P - 0.2S)^2}{P + 0.8S}$$

$$S = \frac{1,000}{CN} - 10$$

$$CN = \frac{1,000}{S + 10}$$

$P$  = precipitation (inches)

$S$  = maximum basin retention (inches)

$Q$  = runoff (inches)

$CN$  = curve number

### Rational Formula

$$Q = CIA, \text{ where}$$

$A$  = watershed area (acres)

$C$  = runoff coefficient

$I$  = rainfall intensity (in./hr)

$Q$  = peak discharge (cfs)

### Darcy's Law

$$Q = -KA(dh/dx), \text{ where}$$

$Q$  = discharge rate (ft<sup>3</sup>/sec or m<sup>3</sup>/s)

$K$  = hydraulic conductivity (ft/sec or m/s)

$h$  = hydraulic head (ft or m)

$A$  = cross-sectional area of flow (ft<sup>2</sup> or m<sup>2</sup>)

$$q = -K(dh/dx)$$

$q$  = specific discharge (also called Darcy velocity or superficial velocity)

$$v = q/n = -K/n(dh/dx)$$

$v$  = average seepage velocity

$n$  = effective porosity

**Unit hydrograph:** The direct runoff hydrograph that would result from one unit of runoff occurring uniformly in space and time over a specified period of time.

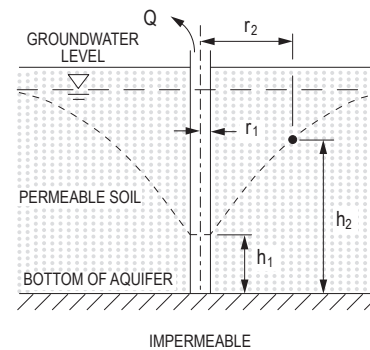
**Transmissivity,  $T$ :** The product of hydraulic conductivity and thickness,  $b$ , of the aquifer ( $L^2T^{-1}$ ).

**Storativity or storage coefficient of an aquifer,  $S$ :**

The volume of water taken into or released from storage per unit surface area per unit change in potentiometric (piezometric) head.

## Well Drawdown

### Unconfined aquifer



### Dupuit's Formula

$$Q = \frac{\pi k (h_2^2 - h_1^2)}{\ln \left( \frac{r_2}{r_1} \right)}$$

where

$Q$  = flow rate of water drawn from well (cfs)

$k$  = coefficient of permeability of soil (fps)

$h_1$  = height of water surface above bottom of aquifer at perimeter of well (ft)

$h_2$  = height of water surface above bottom of aquifer at distance  $r_2$  from well centerline (ft)

$r_1$  = radius to water surface at perimeter of well, i.e., radius of well (ft)

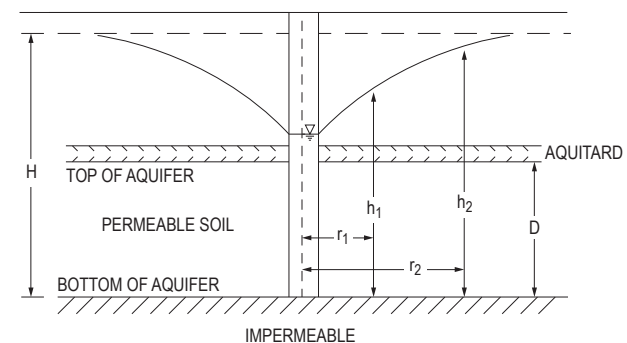
$r_2$  = radius to water surface whose height is  $h_2$  above bottom of aquifer (ft)

$\ln$  = natural logarithm

$Q/D_w$  = specific capacity

$D_w$  = well drawdown (ft)

Confined aquifer:



$$Q = \frac{2\pi T (h_2 - h_1)}{\ln \left( \frac{r_2}{r_1} \right)}$$

where

$T = KD$  = transmissivity (ft<sup>2</sup>/sec)

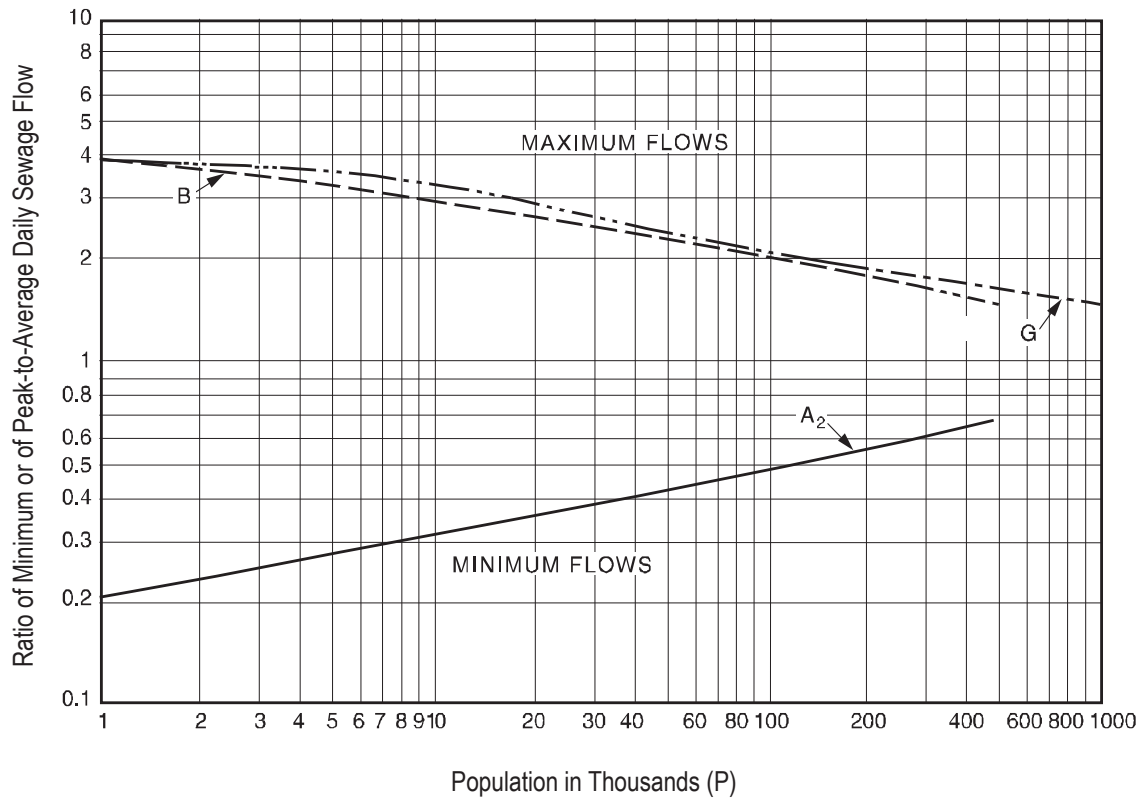
$D$  = thickness of confined aquifer (ft)

$h_1, h_2$  = heights of piezometric surface above bottom of aquifer (ft)

$r_1, r_2$  = radii from pumping well (ft)

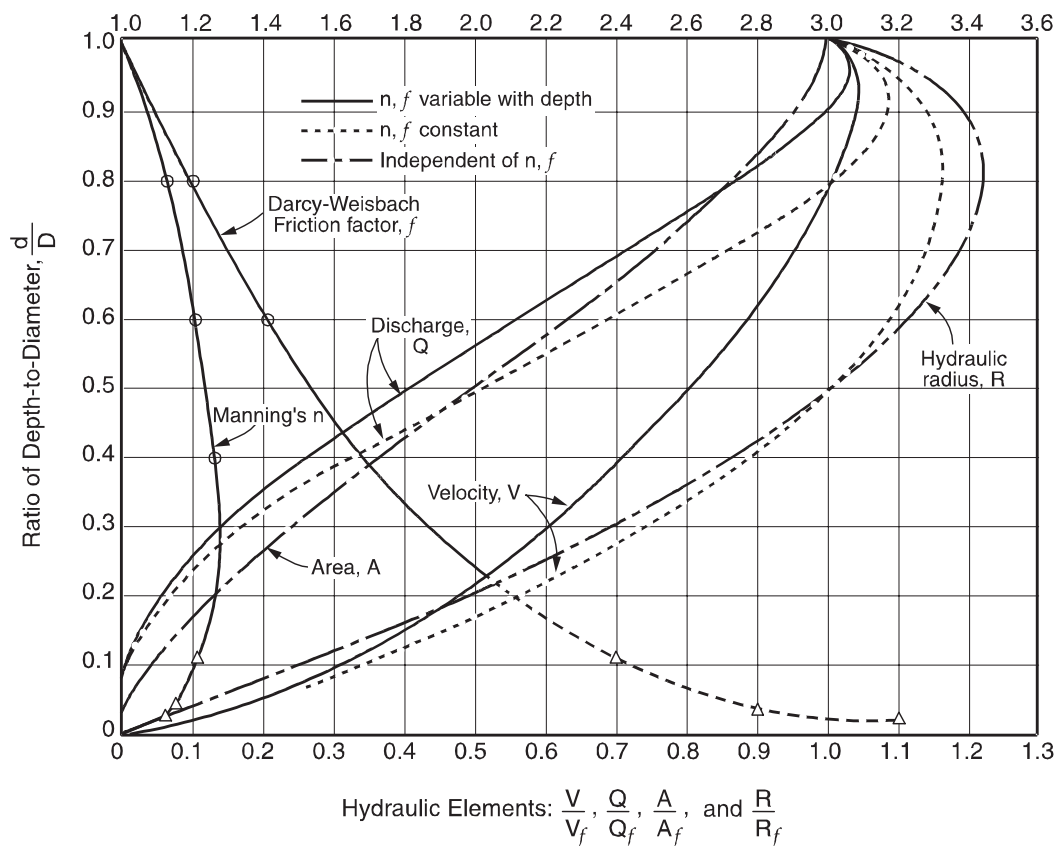
$\ln$  = natural logarithm

## Sewage Flow Ratio Curves



## Hydraulic-Elements Graph for Circular Sewers

Values of:  $\frac{f}{f_f}$  and  $\frac{n}{n_f}$





## Open-Channel Flow

### Specific Energy

$$E = \alpha \frac{V^2}{2g} + y = \frac{\alpha Q^2}{2gA^2} + y, \text{ where}$$

$E$  = specific energy

$Q$  = discharge

$V$  = velocity

$y$  = depth of flow

$A$  = cross-sectional area of flow

$\alpha$  = kinetic energy correction factor, usually 1.0

Critical Depth = that depth in a channel at minimum specific energy

$$\frac{Q^2}{g} = \frac{A^3}{T}$$

where  $Q$  and  $A$  are as defined above,

$g$  = acceleration due to gravity

$T$  = width of the water surface

For rectangular channels

$$y_c = \left( \frac{q^2}{g} \right)^{1/3}, \text{ where}$$

$y_c$  = critical depth

$q$  = unit discharge =  $Q/B$

$B$  = channel width

$g$  = acceleration due to gravity

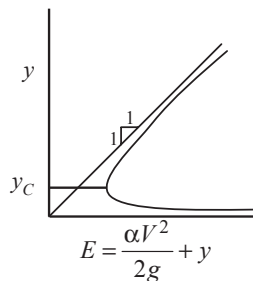
Froude Number = ratio of inertial forces to gravity forces

$$Fr = \frac{V}{\sqrt{gy_h}}, \text{ where}$$

$V$  = velocity

$y_h$  = hydraulic depth =  $A/T$

### Specific Energy Diagram



*Alternate depths:* depths with the same specific energy.

*Uniform flow:* a flow condition where depth and velocity do not change along a channel.

## Manning's Equation

$$Q = \frac{K}{n} A R_H^{2/3} S^{1/2}$$

$Q$  = discharge (ft<sup>3</sup>/sec or m<sup>3</sup>/s)

$K$  = 1.0 for SI units, 1.486 for USCS units

$A$  = cross-sectional area of flow (ft<sup>2</sup> or m<sup>2</sup>)

$R_H$  = hydraulic radius =  $A/P$  (ft or m)

$P$  = wetted perimeter (ft or m)

$S$  = slope of hydraulic surface (ft/ft or m/m)

$n$  = Manning's roughness coefficient

Normal depth (uniform flow depth)

$$A R_H^{2/3} = \frac{Qn}{K S^{1/2}}$$

### Weir Formulas

#### Rectangular

Free discharge suppressed

$$Q = CLH^{3/2}$$

Free discharge contracted

$$Q = C(L - 0.2H)H^{3/2}$$

#### V-Notch

$$Q = CH^{5/2}, \text{ where}$$

$Q$  = discharge (cfs or m<sup>3</sup>/s)

$C$  = 3.33 for rectangular weir (USCS units)

$C$  = 1.84 for rectangular weir (SI units)

$C$  = 2.54 for 90° V-notch weir (USCS units)

$C$  = 1.40 for 90° V-notch weir (SI units)

$L$  = weir length (ft or m)

$H$  = head (depth of discharge over weir) ft or m

### Hazen-Williams Equation

$$V = k_1 C R_H^{0.63} S^{0.54}, \text{ where}$$

$C$  = roughness coefficient

$k_1$  = 0.849 for SI units

$k_1$  = 1.318 for USCS units

$R_H$  = hydraulic radius (ft or m)

$S$  = slope of energy grade line

$$= h_f/L \text{ (ft/ft or m/m)}$$

$V$  = velocity (ft/sec or m/s)

### Circular Pipe Head Loss Equation (Head Loss Expressed in Feet)

$$h_f = \frac{4.73 L}{C^{1.852} D^{4.87}} Q^{1.852}, \text{ where}$$

- $h_f$  = head loss (ft)  
 $L$  = pipe length (ft)  
 $D$  = pipe diameter (ft)  
 $Q$  = flow (cfs)  
 $C$  = Hazen-Williams coefficient

### Circular Pipe Head Loss Equation (Head Loss Expressed as Pressure)

#### U.S. Customary Units

$$P = \frac{4.52 Q^{1.85}}{C^{1.85} D^{4.87}}, \text{ where}$$

- $P$  = pressure loss (psi per foot of pipe)  
 $Q$  = flow (gpm)  
 $D$  = pipe diameter (inches)  
 $C$  = Hazen-Williams coefficient

#### SI Units

$$P = \frac{6.05 Q^{1.85}}{C^{1.85} D^{4.87}} \times 10^5, \text{ where}$$

- $P$  = pressure loss (bars per meter of pipe)  
 $Q$  = flow (liters/minute)  
 $D$  = pipe diameter (mm)

#### Values of Hazen-Williams Coefficient $C$

Pipe Material	$C$
Ductile iron	140
Concrete (regardless of age)	130
Cast iron:	
New	130
5 yr old	120
20 yr old	100
Welded steel, new	120
Wood stave (regardless of age)	120
Vitrified clay	110
Riveted steel, new	110
Brick sewers	100
Asbestos-cement	140
Plastic	150

### TRANSPORTATION

#### U.S. Customary Units

- $a$  = deceleration rate (ft/sec<sup>2</sup>)  
 $A$  = absolute value of algebraic difference in grades (%)  
 $e$  = superelevation (%)  
 $f$  = side friction factor  
 $\pm G$  = percent grade divided by 100 (uphill grade "+")  
 $h_1$  = height of driver's eyes above the roadway surface (ft)  
 $h_2$  = height of object above the roadway surface (ft)  
 $L$  = length of curve (ft)  
 $L_s$  = spiral transition length (ft)  
 $R$  = radius of curve (ft)  
 $SSD$  = stopping sight distance (ft)  
 $t$  = driver reaction time (sec)  
 $V$  = design speed (mph)  
 $v$  = vehicle approach speed (fps)  
 $W$  = width of intersection, curb-to-curb (ft)  
 $l$  = length of vehicle (ft)  
 $y$  = length of yellow interval to nearest 0.1 sec (sec)  
 $r$  = length of red clearance interval to nearest 0.1 sec (sec)

#### Vehicle Signal Change Interval

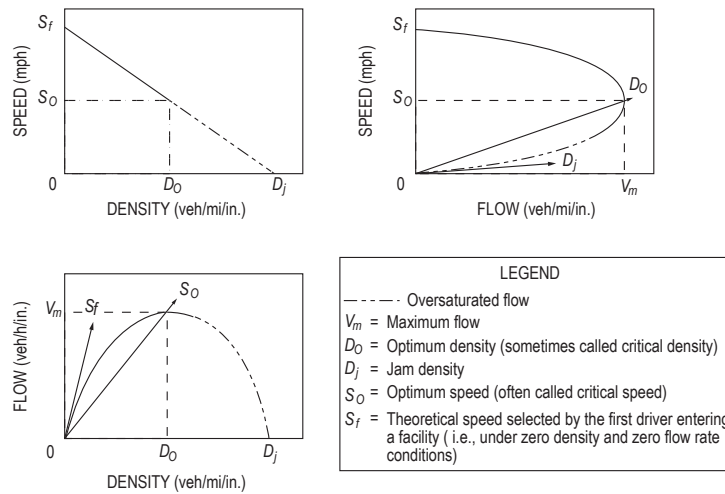
$$y = t + \frac{v}{2a \pm 64.4 G}$$

$$r = \frac{W + l}{v}$$

#### Stopping Sight Distance

$$SSD = 1.47 Vt + \frac{V^2}{30 \left( \left( \frac{a}{32.2} \right) \pm G \right)}$$

## Traffic Flow Relationships ( $q = kv$ )



Vertical Curves: Sight Distance Related to Curve Length		
	$S \leq L$	$S > L$
Crest Vertical Curve General equation:	$L = \frac{AS^2}{100(\sqrt{2h_1} + \sqrt{2h_2})^2}$	$L = 2S - \frac{200(\sqrt{h_1} + \sqrt{h_2})^2}{A}$
Standard Criteria: $h_1 = 3.50$ ft and $h_2 = 2.0$ ft:	$L = \frac{AS^2}{2,158}$	$L = 2S - \frac{2,158}{A}$
Sag Vertical Curve (based on standard headlight criteria)	$L = \frac{AS^2}{400 + 3.5S}$	$L = 2S - \left( \frac{400 + 3.5S}{A} \right)$
Sag Vertical Curve (based on riding comfort)	$L = \frac{AV^2}{46.5}$	
Sag Vertical Curve (based on adequate sight distance under an overhead structure to see an object beyond a sag vertical curve)	$L = \frac{AS^2}{800 \left( C - \frac{h_1 + h_2}{2} \right)}$	$L = 2S - \frac{800}{A} \left( C - \frac{h_1 + h_2}{2} \right)$
$C$ = vertical clearance for overhead structure (overpass) located within 200 feet of the midpoint of the curve		

Horizontal Curves	
Side friction factor (based on superelevation)	$0.01e + f = \frac{V^2}{15R}$
Spiral Transition Length	$L_s = \frac{3.15V^3}{RC}$ <p><math>C</math> = rate of increase of lateral acceleration [use 1 ft/sec<sup>3</sup> unless otherwise stated]</p>
Sight Distance (to see around obstruction)	$HSO = R \left[ 1 - \cos \left( \frac{28.65S}{R} \right) \right]$ <p>HSO = Horizontal sight line offset</p>

- ◆ AASHTO, *A Policy on Geometric Design of Highways and Streets*, 6th ed., 2011. Used by permission.
- Compiled from AASHTO, *A Policy on Geometric Design of Highways and Streets*, 6th ed., 2011.

### Horizontal Curve Formulas

$D$  = Degree of Curve, Arc Definition  
 $PC$  = Point of Curve (also called  $BC$ )  
 $PT$  = Point of Tangent (also called  $EC$ )  
 $PI$  = Point of Intersection  
 $I$  = Intersection Angle (also called  $\Delta$ )  
     Angle Between Two Tangents  
 $L$  = Length of Curve, from  $PC$  to  $PT$   
 $T$  = Tangent Distance  
 $E$  = External Distance  
 $R$  = Radius  
 $LC$  = Length of Long Chord  
 $M$  = Length of Middle Ordinate  
 $c$  = Length of Sub-Chord  
 $d$  = Angle of Sub-Chord  
 $l$  = Curve Length for Sub-Chord

$$R = \frac{5729.58}{D}$$

$$R = \frac{LC}{2 \sin(I/2)}$$

$$T = R \tan(I/2) = \frac{LC}{2 \cos(I/2)}$$

$$L = RI \frac{\pi}{180} = \frac{I}{D} 100$$

$$M = R[1 - \cos(I/2)]$$

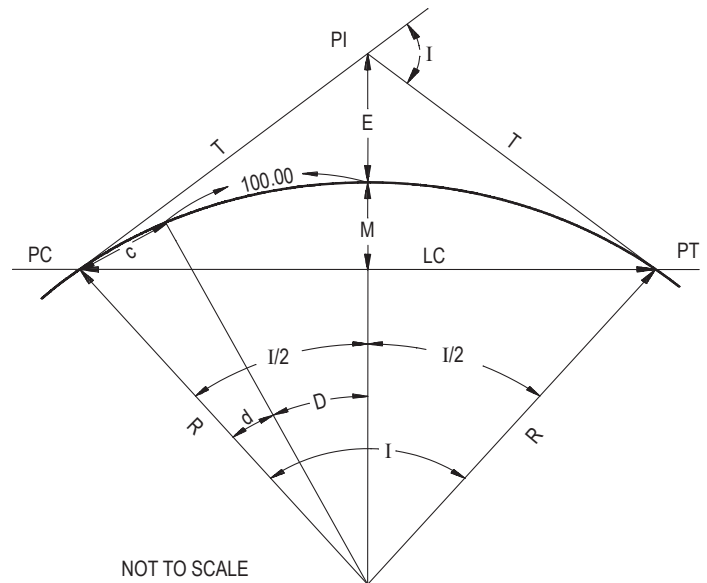
$$\frac{R}{E + R} = \cos(I/2)$$

$$\frac{R - M}{R} = \cos(I/2)$$

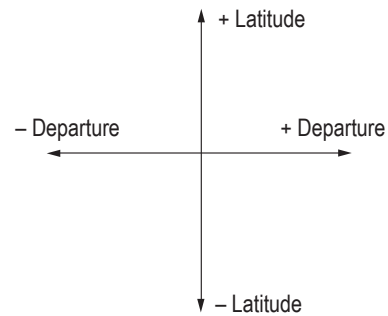
$$c = 2R \sin(d/2)$$

$$l = Rd \left( \frac{\pi}{180} \right)$$

$$E = R \left[ \frac{1}{\cos(I/2)} - 1 \right]$$

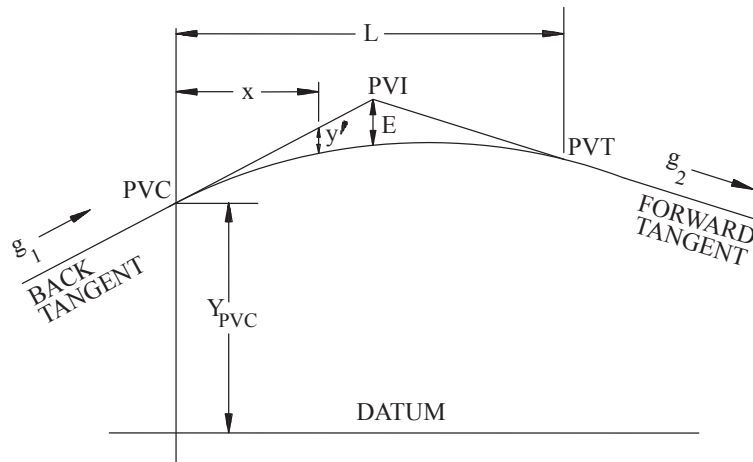


### LATITUDES AND DEPARTURES



Deflection angle per 100 feet of arc length equals  $D/2$

## Vertical Curve Formulas



VERTICAL CURVE FORMULAS  
NOT TO SCALE

$L$  = Length of curve

$PVC$  = Point of vertical curvature

$PVI$  = Point of vertical intersection

$PVT$  = Point of vertical tangency

$g_1$  = Grade of back tangent

$x$  = Horizontal distance from  $PVC$   
to point on curve

$g_2$  = Grade of forward tangent

$a$  = Parabola constant

$y$  = Tangent offset

$E$  = Tangent offset at  $PVI = AL/800$

$r$  = Rate of change of grade

$K$  = Rate of vertical curvature

$$x_m = \text{Horizontal distance to min/max elevation on curve} = -\frac{g_1}{2a} = \frac{g_1 L}{g_1 - g_2}$$

$$\text{Tangent elevation} = Y_{PVC} + g_1 x \text{ and } = Y_{PVI} + g_2 (x - L/2)$$

$$\text{Curve elevation} = Y_{PVC} + g_1 x + ax^2 = Y_{PVC} + g_1 x + [(g_2 - g_1)/(2L)]x^2$$

$$y = ax^2 \quad a = \frac{g_2 - g_1}{2L} \quad E = a\left(\frac{L}{2}\right)^2 \quad r = \frac{g_2 - g_1}{L} \quad K = \frac{L}{A}$$

## EARTHWORK FORMULAS

Average End Area Formula,  $V = L(A_1 + A_2)/2$

Prismoidal Formula,  $V = L(A_1 + 4A_m + A_2)/6$

where  $A_m$  = area of mid-section

$L$  = distance between  $A_1$  and  $A_2$

Pyramid or Cone,  $V = h(\text{Area of Base})/3$

## AREA FORMULAS

Area by Coordinates:  $\text{Area} = [X_A(Y_B - Y_N) + X_B(Y_C - Y_A) + X_C(Y_D - Y_B) + \dots + X_N(Y_A - Y_{N-1})]/2$

$$\text{Trapezoidal Rule: Area} = w\left(\frac{h_1 + h_n}{2} + h_2 + h_3 + h_4 + \dots + h_{n-1}\right)$$

$w$  = common interval

$$\text{Simpson's 1/3 Rule: Area} = w\left[h_1 + 2\left(\sum_{k=3,5,\dots}^{n-2} h_k\right) + 4\left(\sum_{k=2,4,\dots}^{n-1} h_k\right) + h_n\right]/3$$

$n$  must be odd number of measurements  
(only for Simpson's 1/3 Rule)

## Highway Pavement Design

AASHTO Structural Number Equation	
$SN = a_1D_1 + a_2D_2 + \dots + a_nD_n$ , where $SN$ = structural number for the pavement $a_i$ = layer coefficient and $D_i$ = thickness of layer (inches).	

Gross Axle Load		Load Equivalency Factors		Gross Axle Load		Load Equivalency Factors	
kN	lb	Single Axles	Tandem Axles	kN	lb	Single Axles	Tandem Axles
4.45	1,000	0.00002		187.0	42,000	25.64	2.51
8.9	2,000	0.00018		195.7	44,000	31.00	3.00
17.8	4,000	0.00209		200.0	45,000	34.00	3.27
22.25	5,000	0.00500		204.5	46,000	37.24	3.55
<b>26.7</b>	<b>6,000</b>	<b>0.01043</b>		<b>213.5</b>	<b>48,000</b>	<b>44.50</b>	<b>4.17</b>
35.6	8,000	0.0343		222.4	50,000	52.88	4.86
44.5	10,000	0.0877	0.00688	231.3	52,000		5.63
53.4	12,000	0.189	0.0144	240.2	54,000		6.47
62.3	14,000	0.360	0.0270	244.6	55,000		6.93
<b>66.7</b>	<b>15,000</b>	<b>0.478</b>	<b>0.0360</b>	<b>249.0</b>	<b>56,000</b>		<b>7.41</b>
71.2	16,000	0.623	0.0472	258.0	58,000		8.45
80.0	18,000	1.000	0.0773	267.0	60,000		9.59
89.0	20,000	1.51	0.1206	275.8	62,000		10.84
97.8	22,000	2.18	0.180	284.5	64,000		12.22
<b>106.8</b>	<b>24,000</b>	<b>3.03</b>	<b>0.260</b>	<b>289.0</b>	<b>65,000</b>		<b>12.96</b>
111.2	25,000	3.53	0.308	293.5	66,000		13.73
115.6	26,000	4.09	0.364	302.5	68,000		15.38
124.5	28,000	5.39	0.495	311.5	70,000		17.19
133.5	30,000	6.97	0.658	320.0	72,000		19.16
<b>142.3</b>	<b>32,000</b>	<b>8.88</b>	<b>0.857</b>	<b>329.0</b>	<b>74,000</b>		<b>21.32</b>
151.2	34,000	11.18	1.095	333.5	75,000		22.47
155.7	35,000	12.50	1.23	338.0	76,000		23.66
160.0	36,000	13.93	1.38	347.0	78,000		26.22
169.0	38,000	17.20	1.70	356.0	80,000		28.99
<b>178.0</b>	<b>40,000</b>	<b>21.08</b>	<b>2.08</b>				
Note: kN converted to lb are within 0.1 percent of lb shown							

## Superpave

### PERFORMANCE-GRADED (PG) BINDER GRADING SYSTEM

PERFORMANCE GRADE	PG 52							PG 58					PG 64				
	-10	-16	-22	-28	-34	-40	-46	-16	-22	-28	-34	-40	-16	-22	-28	-34	-40
AVERAGE 7-DAY MAXIMUM PAVEMENT DESIGN TEMPERATURE, °C <sup>a</sup>	<52							<58					<64				
MINIMUM PAVEMENT DESIGN TEMPERATURE, °C <sup>a</sup>	>-10	>-16	>-22	>-28	>-34	>-40	>-46	>-16	>-22	>-28	>-34	>-40	>-16	>-22	>-28	>-34	>-40
ORIGINAL BINDER																	
FLASH POINT TEMP, T48: MINIMUM °C	230																
VISCOSITY, ASTM D 4402: <sup>b</sup> MAXIMUM, 3 Pa-s (3,000 cP), TEST TEMP, °C	135																
DYNAMIC SHEAR, TP5: <sup>c</sup> G*/sin δ , MINIMUM, 1.00 kPa TEST TEMPERATURE @ 10 rad/sec., °C	52							58					64				
ROLLING THIN FILM OVEN (T240) OR THIN FILM OVEN (T179) RESIDUE																	
MASS LOSS, MAXIMUM, %	1.00																
DYNAMIC SHEAR, TP5: G*/sin δ , MINIMUM, 2.20 kPa TEST TEMP @ 10 rad/sec. °C	52							58					64				
PRESSURE AGING VESSEL RESIDUE (PP1)																	
PAV AGING TEMPERATURE, °C <sup>d</sup>	90							100					100				
DYNAMIC SHEAR, TP5: G*/sin δ , MAXIMUM, 5,000 kPa TEST TEMP @ 10 rad/sec. °C	25	22	19	16	13	10	7	25	22	19	16	13	28	25	22	19	16
PHYSICAL HARDENING <sup>e</sup>	REPORT																
CREEP STIFFNESS, TP1: <sup>f</sup> S, MAXIMUM, 300 MPa M-VALUE, MINIMUM, 0.300 TEST TEMP, @ 60 sec., °C	0	-6	-12	-18	-24	-30	-36	-6	-12	-18	-24	-30	-6	-12	-18	-24	-30
DIRECT TENSION, TP3: <sup>f</sup> FAILURE STRAIN, MINIMUM, 1.0% TEST TEMP @ 1.0 mm/min, °C	0	-6	-12	-18	-24	-30	-36	-6	-12	-18	-24	-30	-6	-12	-18	-24	-30

Federal Highway Administration Report FHWA-SA-95-03, "Background of Superpave Asphalt Mixture Design and Analysis," Nov. 1994.

## Superpave Mixture Design: Compaction Requirements

SUPERPAVE GYRATORY COMPACTION EFFORT												
TRAFFIC, MILLION ESALS	AVERAGE DESIGN HIGH AIR TEMPERATURE											
	< 39°C			39° – 40°C			41° – 42°C			42° – 43°C		
	N <sub>int</sub>	N <sub>des</sub>	N <sub>max</sub>	N <sub>int</sub>	N <sub>des</sub>	N <sub>max</sub>	N <sub>int</sub>	N <sub>des</sub>	N <sub>max</sub>	N <sub>int</sub>	N <sub>des</sub>	N <sub>max</sub>
< 0.3	7	68	104	7	74	114	7	78	121	7	82	127
< 1	7	76	117	7	83	129	7	88	138	8	93	146
< 3	7	86	134	8	95	150	8	100	158	8	105	167
< 10	8	96	152	8	106	169	8	113	181	9	119	192
< 30	8	109	174	9	121	195	9	128	208	9	135	220
< 100	9	126	204	9	139	228	9	146	240	10	153	253
≥ 100	9	142	233	10	158	262	10	165	275	10	177	288

VFA REQUIREMENTS @ 4% AIR VOIDS	
TRAFFIC, MILLION ESALS	DESIGN VFA (%)
< 0.3	70 – 80
< 1	65 – 78
< 3	65 – 78
< 10	65 – 75
< 30	65 – 75
< 100	65 – 75
≥ 100	65 – 75

VMA REQUIREMENTS @ 4% AIR VOIDS					
NOMINAL MAXIMUM AGGREGATE SIZE (mm)	9.5	12.5	19.0	25.0	37.5
MINIMUM VMA (%)	15	14	13	12	11

COMPACTION KEY			
SUPERPAVE GYRATORY COMPACTION	N <sub>int</sub>	N <sub>des</sub>	N <sub>max</sub>
PERCENT OF Gmm	≤ 89%	96%	≤ 98%

Federal Highway Administration Report FHWA-SA-95-03, "Background of Superpave Asphalt Mixture Design and Analysis," Nov. 1994.

### Gravity Model

$$T_{ij} = P_i \left[ \frac{A_j F_{ij} K_{ij}}{\sum_j A_j F_{ij} K_{ij}} \right]$$

where

$T_{ij}$  = number of trips that are produced in zone  $i$  and attracted to zone  $j$

$P_i$  = total number of trips produced in zone  $i$

$A_j$  = number of trips attracted to zone  $j$

$F_{ij}$  = a value that is an inverse function of travel time between zones  $i$  and  $j$

$K_{ij}$  = socioeconomic adjustment factor for interchange  $ij$

### Logit Models

$$U_x = \sum_{i=1}^n a_i X_i$$

where

$U_x$  = utility of mode  $x$

$n$  = number of attributes

$X_i$  = attribute value (time, cost, and so forth)

$a_i$  = coefficient value for attributes  $i$  (negative, since the values are disutilities)

If two modes, auto ( $A$ ) and transit ( $T$ ), are being considered, the probability of selecting the auto mode  $A$  can be written as

$$P(A) = \frac{e^{U_A}}{e^{U_A} + e^{U_T}} \quad P(x) = \frac{e^{U_x}}{\sum_{e=1}^n e^{U_{xi}}}$$



## Traffic Safety Equations

$$RMEV = \frac{A \times 1,000,000}{V}$$

where

$RMEV$  = crash rate per million entering vehicles

$A$  = number of crashes, total or by type occurring in a single year at the location

$V$  =  $ADT \times 365$

$ADT$  = average daily traffic entering intersection

$$RMVM = \frac{A \times 100,000,000}{VMT}$$

where

$RMVM$  = crash rate per million vehicle miles

$A$  = number of crashes, total or by type at the study location, during a given period

$VMT$  = vehicle miles of travel during the given period

=  $ADT \times (\text{number of days in study period}) \times (\text{length of road})$

$$\text{Crashes prevented} = N \times CR \frac{(ADT \text{ after improvement})}{(ADT \text{ before improvement})}$$

where

$N$  = expected number of crashes if countermeasure is not implemented and if the traffic volume remains the same

$CR = CR_1 + (1 - CR_1)CR_2 + (1 - CR_1)(1 - CR_2)CR_3 + \dots + (1 - CR_1) \dots (1 - CR_{m-1}) CR_m$   
= overall crash reduction factor for multiple mutually exclusive improvements at a single site

$CR_i$  = crash reduction factor for a specific countermeasure  $i$

$m$  = number of countermeasures at the site

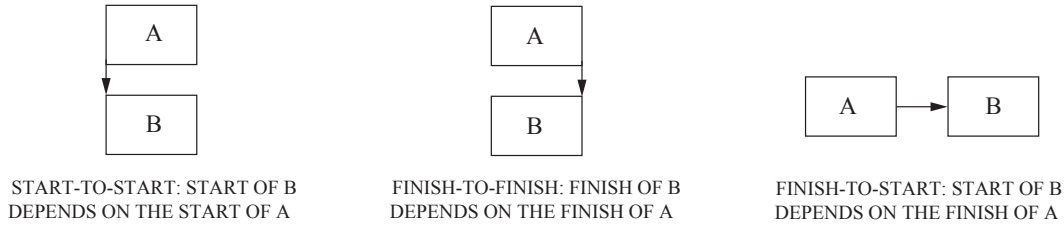
Garber, Nicholas J., and Lester A. Hoel, *Traffic and Highway Engineering*, 4th ed., Cengage Learning, 2009.

## CONSTRUCTION

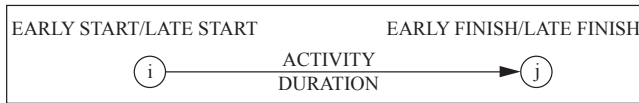
Construction project scheduling and analysis questions may be based on either the activity-on-node method or the activity-on-arrow method.

## CPM PRECEDENCE RELATIONSHIPS

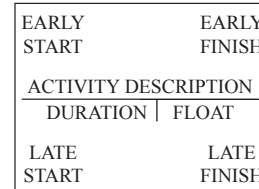
### ACTIVITY-ON-NODE



### ACTIVITY-ON-ARROW ANNOTATION



### ACTIVITY-ON-NODE ANNOTATION



## Nomenclature

ES = Early start = Latest EF of predecessors

EF = Early finish = ES + duration

LS = Late start = LF – duration

LF = Late finish = Earliest LS of successors

D = Duration

Float = LS – ES or LF – EF

## Determining the Project Size Modifier

Square Foot Base Size							
Building Type	Median Cost per S.F.	Typical Size Gross S.F.	Typical Range Gross S.F.	Building Type	Median Cost per S.F.	Typical Size Gross S.F.	Typical Range Gross S.F.
Apartments, Low Rise	\$ 54.05	21,000	9,700–37,200	Jails	\$ 165.00	13,700	7,500–28,000
Apartments, Mid Rise	68.25	50,000	32,000–100,000	Libraries	97.30	12,000	7,000–31,000
Apartments, High Rise	78.30	310,000	100,000–650,000	Medical Clinics	93.15	7,200	4,200–15,700
Auditoriums	90.35	25,000	7,600–39,000	Medical Offices	87.50	6,000	4,000–15,000
Auto Sales	55.90	20,000	10,800–28,600	Motels	67.00	27,000	15,800–51,000
Banks	121.00	4,200	2,500–7,500	Nursing Homes	89.95	23,000	15,000–37,000
Churches	81.60	9,000	5,300–13,200	Offices, Low Rise	73.00	8,600	4,700–19,000
Clubs, Country	81.40	6,500	4,500–15,000	Offices, Mid Rise	76.65	52,000	31,300–83,100
Clubs, Social	79.15	10,000	6,000–13,500	Offices, High Rise	98.05	260,000	151,000–468,000
Clubs, YMCA	81.60	28,300	12,800–39,400	Police Stations	122.00	10,500	4,000–19,000
Colleges (Class)	107.00	50,000	23,500–98,500	Post Offices	90.40	12,400	6,800–30,000
Colleges (Science Lab)	156.00	45,600	16,600–80,000	Power Plants	678.00	7,500	1,000–20,000
College (Student Union)	119.00	33,400	16,000–85,000	Religious Education	74.85	9,000	6,000–12,000
Community Center	85.05	9,400	5,300–16,700	Research	127.00	19,000	6,300–45,000
Court Houses	116.00	32,400	17,800–106,000	Restaurants	110.00	4,400	2,800–6,000
Dept. Stores	50.50	90,000	44,000–122,000	Retail Stores	53.70	7,200	4,000–17,600
Dormitories, Low Rise	87.20	24,500	13,400–40,000	Schools, Elementary	78.20	41,000	24,500–55,000
Dormitories, Mid Rise	113.00	55,600	36,100–90,000	Schools, Jr. High	79.65	92,000	52,000–119,000
Factories	48.95	26,400	12,900–50,000	Schools, Sr. High	79.65	101,000	50,500–175,000
Fire Stations	85.45	5,800	4,000–8,700	Schools, Vocational	79.35	37,000	20,500–82,000
Fraternity Houses	84.10	12,500	8,200–14,800	Sports Arenas	66.45	15,000	5,000–40,000
Funeral Homes	94.00	7,800	4,500–11,000	Supermarkets	53.85	20,000	12,000–30,000
Garages, Commercial	59.70	9,300	5,000–13,600	Swimming Pools	125.00	13,000	7,800–22,000
Garages, Municipal	76.40	8,300	4,500–12,600	Telephone Exchange	145.00	4,500	1,200–10,600
Garages, Parking	31.30	163,000	76,400–225,300	Theaters	79.70	10,500	8,800–17,500
Gymnasiums	78.95	19,200	11,600–41,000	Town Halls	87.65	10,800	4,800–23,400
Hospitals	149.00	55,000	27,200–125,000	Warehouses	36.15	25,000	8,000–72,000
House (Elderly)	73.90	37,000	21,000–66,000	Warehouse and Office	41.75	25,000	8,000–72,000
Housing (Public)	68.45	36,000	14,400–74,400				
Ice Rinks	76.00	29,000	27,200–33,600				

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**Earned-Value Analysis**

BCWS = Budgeted cost of work scheduled (Planned)

ACWP = Actual cost of work performed (Actual)

BCWP = Budgeted cost of work performed (Earned)

**Variances**

CV = BCWP – ACWP (Cost variance = Earned – Actual)

SV = BCWP – BCWS (Schedule variance = Earned – Planned)

**Indices**

$CPI = \frac{BCWP}{ACWP}$  (Cost Performance Index =  $\frac{\text{Earned}}{\text{Actual}}$ )

$SPI = \frac{BCWP}{BCWS}$  (Schedule Performance Index =  $\frac{\text{Earned}}{\text{Planned}}$ )

**Forecasting**

BAC = Original project estimate (Budget at completion)

$ETC = \frac{BAC - BCWP}{CPI}$  (Estimate to complete)

$EAC = (ACWP + ETC)$  (Estimate to complete)

# ENVIRONMENTAL ENGINEERING

## AIR POLLUTION

### Nomenclature

$$\frac{\mu\text{g}}{\text{m}^3} = \text{ppb} \times \frac{P(\text{MW})}{RT}$$

ppb = parts per billion

$P$  = pressure (atm)

$R$  = ideal gas law constant  
= 0.0821 L·atm/(mole·K)

$T$  = absolute temperature, K = 273.15 + °C

MW = molecular weight (g/mole)

### Atmospheric Dispersion Modeling (Gaussian)

$\sigma_y$  and  $\sigma_z$  as a function of downwind distance and stability class, see following figures.

$$C = \frac{Q}{2\pi u \sigma_y \sigma_z} \exp\left(-\frac{1}{2} \frac{y^2}{\sigma_y^2}\right) \left[ \exp\left(-\frac{1}{2} \frac{(z-H)^2}{\sigma_z^2}\right) + \exp\left(-\frac{1}{2} \frac{(z+H)^2}{\sigma_z^2}\right) \right]$$

where

$C$  = steady-state concentration at a point ( $x, y, z$ ) ( $\mu\text{g}/\text{m}^3$ )

$Q$  = emissions rate ( $\mu\text{g}/\text{s}$ )

$\sigma_y$  = horizontal dispersion parameter (m)

$\sigma_z$  = vertical dispersion parameter (m)

$u$  = average wind speed at stack height (m/s)

$y$  = horizontal distance from plume centerline (m)

$z$  = vertical distance from ground level (m)

$H$  = effective stack height (m) =  $h + \Delta h$

where  $h$  = physical stack height

$\Delta h$  = plume rise

$x$  = downwind distance along plume centerline (m)

Maximum concentration at ground level and directly downwind from an elevated source.

$$C_{\max} = \frac{Q}{\pi u \sigma_y \sigma_z} \exp\left(-\frac{1}{2} \frac{(H^2)}{\sigma_z^2}\right)$$

where variables are as above except

$C_{\max}$  = maximum ground-level concentration

$\sigma_z = \frac{H}{\sqrt{2}}$  for neutral atmospheric conditions

### Selected Properties of Air

Nitrogen ( $\text{N}_2$ ) by volume 78.09%

Oxygen ( $\text{O}_2$ ) by volume 20.94%

Argon (Ar) by volume 0.93%

Absolute viscosity,  $\mu$

at 80°F 0.045 lbm/(hr-ft)

at 100°F 0.047 lbm/(hr-ft)

Density

at 80°F 0.0734 lbm/ft

at 100°F 0.0708 lbm/ft

The dry adiabatic lapse rate  $\Gamma_{\text{AD}}$  is 0.98°C per 100 m (5.4°F per 1,000 ft). This is the rate at which dry air cools adiabatically with altitude. The actual (environmental) lapse rate  $\Gamma$  is compared to  $\Gamma_{\text{AD}}$  to determine stability as follows:

Lapse Rate	Stability Condition
$\Gamma > \Gamma_{\text{AD}}$	Unstable
$\Gamma = \Gamma_{\text{AD}}$	Neutral
$\Gamma < \Gamma_{\text{AD}}$	Stable

### Atmospheric Stability Under Various Conditions

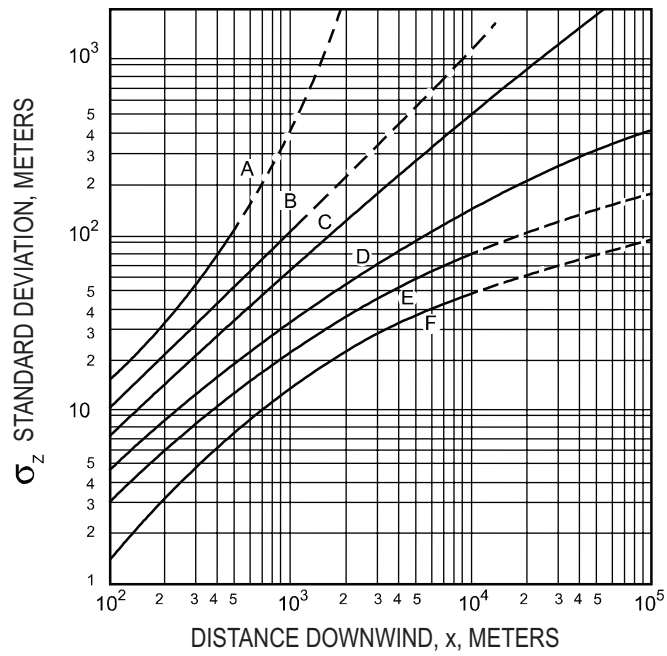
Surface Wind Speed <sup>a</sup> (m/s)	Day Solar Insolation			Night Cloudiness <sup>e</sup>	
	Strong <sup>b</sup>	Moderate <sup>c</sup>	Slight <sup>d</sup>	Cloudy (≥4/8)	Clear (≤3/8)
<2	A	A-B <sup>f</sup>	B	E	F
2-3	A-B	B	C	E	F
3-5	B	B-C	C	D	E
5-6	C	C-D	D	D	D
>6	C	D	D	D	D

**Notes:**

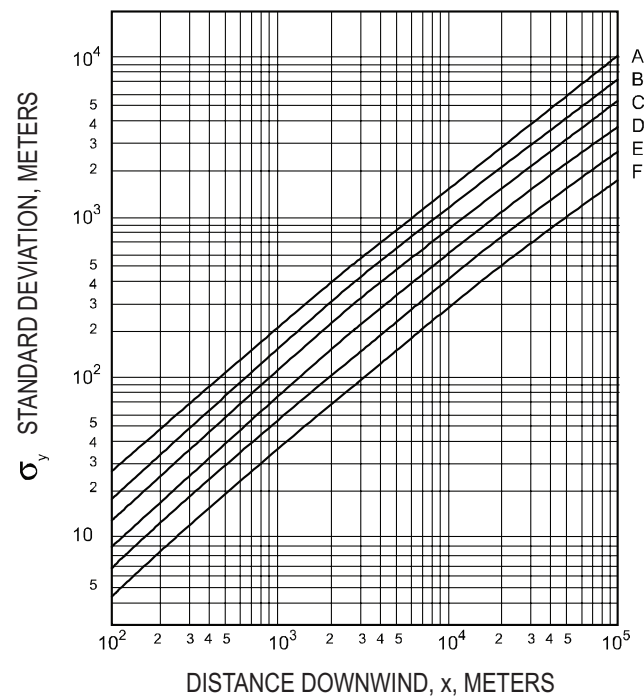
- a. Surface wind speed is measured at 10 m above the ground.
  - b. Corresponds to clear summer day with sun higher than 60° above the horizon.
  - c. Corresponds to a summer day with a few broken clouds, or a clear day with sun 35-60° above the horizon.
  - d. Corresponds to a fall afternoon, or a cloudy summer day, or clear summer day with the sun 15-35°.
  - e. Cloudiness is defined as the fraction of sky covered by the clouds.
  - f. For A-B, B-C, or C-D conditions, average the values obtained for each.
- \* A = Very unstable                      D = Neutral  
      B = Moderately unstable            E = Slightly stable  
      C = Slightly unstable                F = Stable

Regardless of wind speed, Class D should be assumed for overcast conditions, day or night.

Turner, D.B., "Workbook of Atmospheric Dispersion Estimates: An Introduction to Dispersion Modeling," 2nd ed., Lewis Publishing/CRC Press, Florida, 1994.



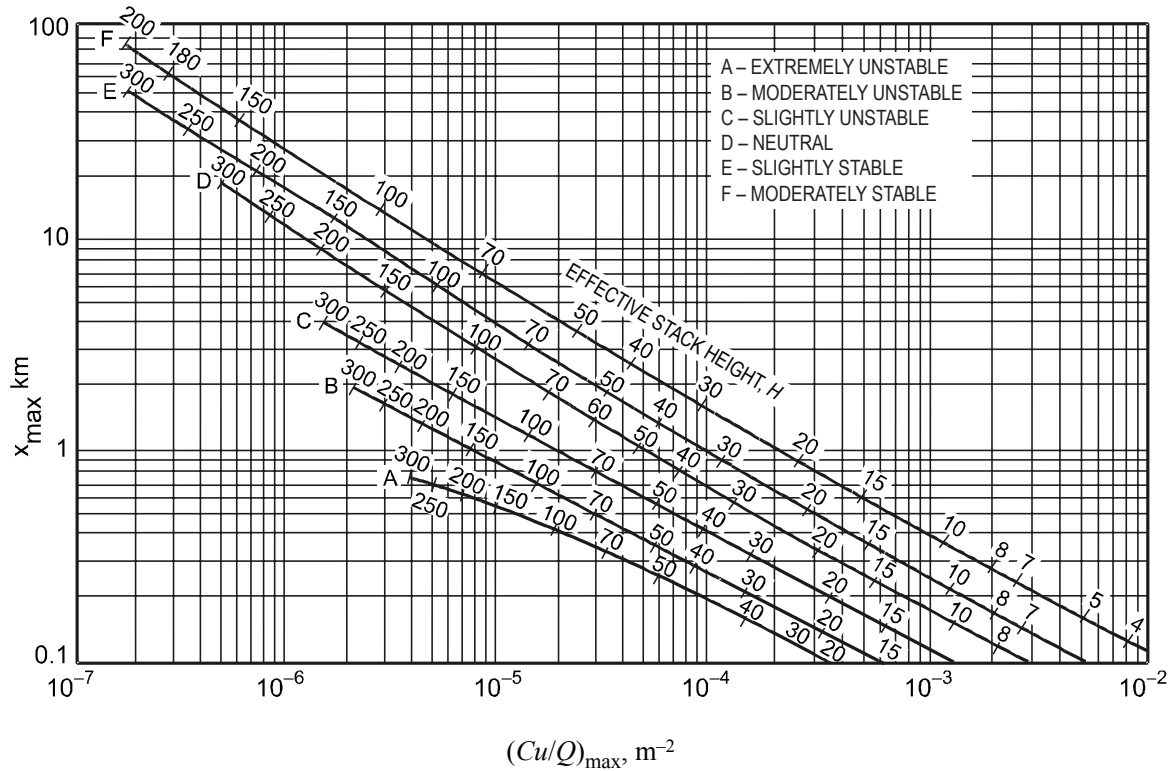
**VERTICAL STANDARD DEVIATIONS OF A PLUME**



**HORIZONTAL STANDARD DEVIATIONS OF A PLUME**

- A – EXTREMELY UNSTABLE
- B – MODERATELY UNSTABLE
- C – SLIGHTLY UNSTABLE
- D – NEUTRAL
- E – SLIGHTLY STABLE
- F – MODERATELY STABLE

Downwind distance where the maximum concentration occurs,  $x_{\max}$ , versus  $(Cu/Q)_{\max}$  as a function of stability class



**NOTES:** Effective stack height shown on curves numerically.

$x_{\max}$  = distance along plume centerline to the point of maximum concentration

$$(Cu/Q)_{\max} = e^{[a + b \ln H + c (\ln H)^2 + d (\ln H)^3]}$$

$H$  = effective stack height, stack height + plume rise, m

**Values of Curve-Fit Constants for Estimating  $(Cu/Q)_{\max}$  from  $H$  as a Function of Atmospheric Stability**

Stability	Constants			
	$a$	$b$	$c$	$d$
A	-1.0563	-2.7153	0.1261	0
B	-1.8060	-2.1912	0.0389	0
C	-1.9748	-1.9980	0	0
D	-2.5302	-1.5610	-0.0934	0
E	-1.4496	-2.5910	0.2181	-0.0343
F	-1.0488	-3.2252	0.4977	-0.0765

Adapted from Ranchoux, R.J.P., 1976.

♦ Turner, D.B., "Workbook of Atmospheric Dispersion Estimates: An Introduction to Dispersion Modeling," 2nd ed., Lewis Publishing/CRC Press, Florida, 1994.

## Cyclone

### Cyclone Collection (Particle Removal) Efficiency

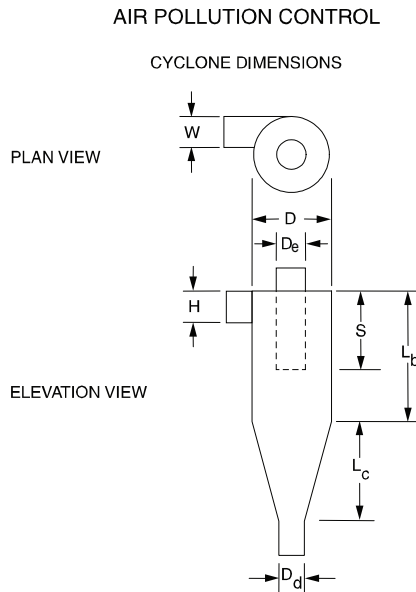
$$\eta = \frac{1}{1 + (d_{pc}/d_p)^2}, \text{ where}$$

$d_{pc}$  = diameter of particle collected with 50% efficiency

$d_p$  = diameter of particle of interest

$\eta$  = fractional particle collection efficiency

◆



### Cyclone 50% Collection Efficiency for Particle Diameter

$$d_{pc} = \left[ \frac{9\mu W}{2\pi N_e V_i (\rho_p - \rho_g)} \right]^{0.5}, \text{ where}$$

$d_{pc}$  = diameter of particle that is collected with 50% efficiency (m)

$\mu$  = dynamic viscosity of gas (kg/m•s)

$W$  = inlet width of cyclone (m)

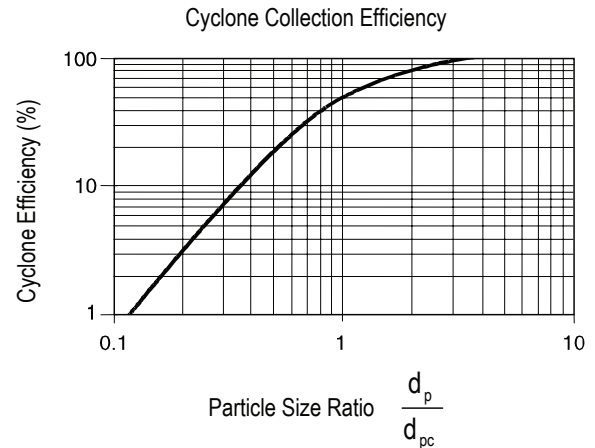
$N_e$  = number of effective turns gas makes in cyclone

$V_i$  = inlet velocity into cyclone (m/s)

$\rho_p$  = density of particle (kg/m<sup>3</sup>)

$\rho_g$  = density of gas (kg/m<sup>3</sup>)

◆



### Cyclone Effective Number of Turns Approximation

$$N_e = \frac{1}{H} \left[ L_b + \frac{L_c}{2} \right], \text{ where}$$

$N_e$  = number of effective turns gas makes in cyclone

$H$  = inlet height of cyclone (m)

$L_b$  = length of body cyclone (m)

$L_c$  = length of cone of cyclone (m)

◆

### Cyclone Ratio of Dimensions to Body Diameter

Dimension	High Efficiency	Conventional	High Throughput
Inlet height, $H$	0.44	0.50	0.80
Inlet width, $W$	0.21	0.25	0.35
Body length, $L_b$	1.40	1.75	1.70
Cone length, $L_c$	2.50	2.00	2.00
Vortex finder length, $S$	0.50	0.60	0.85
Gas exit diameter, $D_e$	0.40	0.50	0.75
Dust outlet diameter, $D_d$	0.40	0.40	0.40

◆ Adapted from Cooper, David C., and F.C. Alley, *Air Pollution Control: A Design Approach*, 2nd ed., Waveland Press, Illinois, 1986.



## Baghouse

### Air-to-Cloth Ratio for Baghouses

Dust	Shaker/Woven Reverse Air/Woven [m <sup>3</sup> /(min•m <sup>2</sup> )]	Pulse Jet/Felt [m <sup>3</sup> /(min•m <sup>2</sup> )]
alumina	0.8	2.4
asbestos	0.9	3.0
bauxite	0.8	2.4
carbon black	0.5	1.5
coal	0.8	2.4
cocoa	0.8	3.7
clay	0.8	2.7
cement	0.6	2.4
cosmetics	0.5	3.0
enamel frit	0.8	2.7
feeds, grain	1.1	4.3
feldspar	0.7	2.7
fertilizer	0.9	2.4
flour	0.9	3.7
fly ash	0.8	1.5
graphite	0.6	1.5
gypsum	0.6	3.0
iron ore	0.9	3.4
iron oxide	0.8	2.1
iron sulfate	0.6	1.8
lead oxide	0.6	1.8
leather dust	1.1	3.7
lime	0.8	3.0
limestone	0.8	2.4
mica	0.8	2.7
paint pigments	0.8	2.1
paper	1.1	3.0
plastics	0.8	2.1
quartz	0.9	2.7
rock dust	0.9	2.7
sand	0.8	3.0
sawdust (wood)	1.1	3.7
silica	0.8	2.1
slate	1.1	3.7
soap detergents	0.6	1.5
spices	0.8	3.0
starch	0.9	2.4
sugar	0.6	2.1
talc	0.8	3.0
tobacco	1.1	4.0

U.S. EPA OAQPS Control Cost Manual, 4th ed., EPA 450/3-90-006 (NTIS PB 90-169954), January 1990.

## Electrostatic Precipitator Efficiency

Deutsch-Anderson equation:

$$\eta = 1 - e^{(-WA/Q)}$$

where

$\eta$  = fractional collection efficiency

$W$  = terminal drift velocity

$A$  = total collection area

$Q$  = volumetric gas flow rate

Note that any consistent set of units can be used for  $W$ ,  $A$ , and  $Q$  (for example, ft/min, ft<sup>2</sup>, and ft<sup>3</sup>/min).

## Incineration

$$DRE = \frac{W_{in} - W_{out}}{W_{in}} \times 100\%$$

where

$DRE$  = destruction and removal efficiency (%)

$W_{in}$  = mass feed rate of a particular POHC (kg/h or lb/h)

$W_{out}$  = mass emission rate of the same POHC (kg/h or lb/h)

POHC = principal organic hazardous contaminant

$$CE = \frac{CO_2}{CO_2 + CO} \times 100\%$$

$CO_2$  = volume concentration (dry) of  $CO_2$   
(parts per million, volume, ppm<sub>v</sub>)

$CO$  = volume concentration (dry) of  $CO$  (ppm<sub>v</sub>)

$CE$  = combustion efficiency

## Kiln Formula

$$t = \frac{2.28 L/D}{SN}$$

where

$t$  = mean residence time, min

$L/D$  = internal length-to-diameter ratio

$S$  = kiln rake slope, in./ft of length

$N$  = rotational speed, rev/min

## Energy Content of Waste

Typical Waste Values	Moisture, %	Energy, Btu/lb
Food Waste	70	2,000
Paper	6	7,200
Cardboard	5	7,000
Plastics	2	14,000
Wood	20	8,000
Glass	2	60
Bi-metallic Cans	3	300

## FATE AND TRANSPORT

### Mass Calculations

Mass balance:  $\text{Mass}_{\text{in}} = \text{Mass}_{\text{out}}$

$M = CQ = CV$

Continuity equation =  $Q = vA$

$M$  = mass

$C$  = concentration

$Q$  = flow rate

$V$  = volume

$v$  = velocity

$A$  = cross-sectional area of flow

$M$  (lb/day) =  $C$  (mg/L)  $\times Q$  (MGD)  $\times 8.34$  [lb-L/(mg-MG)]

where:

MGD = million gallons per day

MG = million gallons

### Microbial Kinetics

#### BOD Exertion

$$y_t = L(1 - e^{-kt})$$

where

$k$  = BOD decay rate constant (base e, days<sup>-1</sup>)

$L$  = ultimate BOD (mg/L)

$t$  = time (days)

$y_t$  = the amount of BOD exerted at time  $t$  (mg/L)

#### Stream Modeling

Streeter Phelps

$$D = \frac{k_1 L_0}{k_2 - k_1} [\exp(-k_1 t) - \exp(-k_2 t)] + D_0 \exp(-k_2 t)$$

$$t_c = \frac{1}{k_2 - k_1} \ln \left[ \frac{k_2}{k_1} \left( 1 - D_0 \frac{(k_2 - k_1)}{k_1 L_0} \right) \right]$$

$$DO = DO_{\text{sat}} - D$$

where

$D$  = dissolved oxygen deficit (mg/L)

$DO$  = dissolved oxygen concentration (mg/L)

$D_0$  = initial dissolved oxygen deficit in mixing zone (mg/L)

$DO_{\text{sat}}$  = saturated dissolved oxygen concentration (mg/L)

$k_1$  = deoxygenation rate constant, base e (days<sup>-1</sup>)

$k_2$  = reaeration rate constant, base e (days<sup>-1</sup>)

$L_0$  = initial BOD ultimate in mixing zone (mg/L)

$t$  = time (days)

$t_c$  = time at which minimum dissolved oxygen occurs (days)

## Monod Kinetics—Substrate Limited Growth

Continuous flow systems where growth is limited by one substrate (chemostat):

$$\mu = \frac{Y k_m S}{K_s + S} - k_d = \mu_{\text{max}} \frac{S}{K_s + S} - k_d$$

### Multiple Limiting Substrates

$$\frac{\mu}{\mu_{\text{max}}} = [\mu_1(S_1)][\mu_2(S_2)][\mu_3(S_3)] \dots [\mu_n(S_n)]$$

$$\text{where } \mu_i = \frac{S_i}{K_{s_i} + S_i} \text{ for } i = 1 \text{ to } n$$

### Non-steady State Continuous Flow

$$\frac{dx}{dt} = Dx_0 + (\mu - k_d - D)x$$

### Steady State Continuous Flow

$$\mu = D \text{ with } k_d \ll \mu$$

### Product production at steady state, single substrate limiting

$$X_1 = Y_{P/S}(S_0 - S_i)$$

$k_d$  = microbial death rate or endogenous decay rate constant (time<sup>-1</sup>)

$k_m$  = maximum growth rate constant (time<sup>-1</sup>)

$K_s$  = saturation constant or half-velocity constant  
[= concentration at  $\mu_{\text{max}}/2$ ]

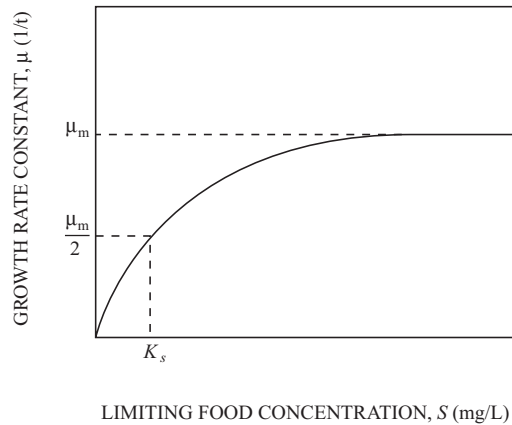
$S$  = concentration of substrate in solution (mass/unit volume)

$Y$  = yield coefficient [(mass/L product)/(mass/L food used)]

$\mu$  = specific growth rate (time<sup>-1</sup>)

$\mu_{\text{max}}$  = maximum specific growth rate (time<sup>-1</sup>) =  $Y k_m$

- ♦ Monod growth rate constant as a function of limiting food concentration.



- $X_1$  = product (mg/L)
- $V_r$  = volume (L)
- $D$  = dilution rate (flow  $f$ /reactor volume  $V_r$ ;  $\text{hr}^{-1}$ )
- $f$  = flow rate (L/hr)
- $\mu_i$  = growth rate with one or multiple limiting substrates ( $\text{hr}^{-1}$ )
- $S_i$  = substrate  $i$  concentration (mass/unit volume)
- $S_0$  = initial substrate concentration (mass/unit volume)
- $Y_{P/S}$  = product yield per unit of substrate (mass/mass)
- $p$  = product concentration (mass/unit volume)
- $x$  = cell concentration (mass/unit volume)
- $x_0$  = initial cell concentration (mass/unit volume)
- $t$  = time (time)

#### Kinetic Temperature Corrections

$$k_T = k_{20} (\theta)^{T-20}$$

Activated sludge:  $\theta = 1.136$  ( $T > 20^\circ\text{C}$ )  
 $\theta = 1.056$  ( $T < 20^\circ\text{C}$ )

Reaeration  $\theta = 1.024$

Biotowers  $\theta = 1.035$

Trickling Filters  $\theta = 1.072$

- ♦ Davis, M.L., and D. Cornwell, *Introduction to Environmental Engineering*, 3rd ed., McGraw-Hill, 1998.

#### **Partition Coefficients**

##### Bioconcentration Factor $BCF$

The amount of a chemical to accumulate in aquatic organisms.

$$BCF = C_{\text{org}}/C$$

where

$C_{\text{org}}$  = equilibrium concentration in organism (mg/kg or ppm)

$C$  = concentration in water (ppm)

##### Octanol-Water Partition Coefficient

The ratio of a chemical's concentration in the octanol phase to its concentration in the aqueous phase of a two-phase octanol-water system.

$$K_{ow} = C_o/C_w$$

where

$C_o$  = concentration of chemical in octanol phase (mg/L or  $\mu\text{g/L}$ )

$C_w$  = concentration of chemical in aqueous phase (mg/L or  $\mu\text{g/L}$ )

##### Organic Carbon Partition Coefficient $K_{oc}$

$$K_{oc} = C_{\text{soil}}/C_{\text{water}}$$

where

$C_{\text{soil}}$  = concentration of chemical in organic carbon component of soil ( $\mu\text{g}$  adsorbed/kg organic C, or ppb)

$C_{\text{water}}$  = concentration of chemical in water (ppb or  $\mu\text{g/kg}$ )

##### Retardation Factor $R$

$$R = 1 + (\rho/\eta)K_d$$

where

$\rho$  = bulk density

$\eta$  = porosity

$K_d$  = distribution coefficient

##### Soil - Water Partition Coefficient $K_{sw} = K_p$

$$K_{sw} = X/C$$

where

$X$  = concentration of chemical in soil (ppb or  $\mu\text{g/kg}$ )

$C$  = concentration of chemical in water (ppb or  $\mu\text{g/kg}$ )

$$K_{sw} = K_{oc}f_{oc}$$

$f_{oc}$  = fraction of organic carbon in the soil (dimensionless)

♦ **Steady-State Reactor Parameters (Constant Density Systems)**

**Comparison of Steady-State Retention Times ( $\theta$ ) for Decay Reactions of Different Order<sup>a</sup>**

Reaction Order	r	Equations for Mean Retention Times ( $\theta$ )		
		Ideal Batch	Ideal Plug Flow	Ideal CMFR
Zero <sup>b</sup>	$-k$	$\frac{(C_o - C_t)}{k}$	$\frac{(C_o - C_t)}{k}$	$\frac{(C_o - C_t)}{k}$
First	$-kC$	$\frac{\ln(C_o/C_t)}{k}$	$\frac{\ln(C_o/C_t)}{k}$	$\frac{(C_o/C_t) - 1}{k}$
Second	$-kC^2$	$\frac{1}{k} \left[ \frac{1}{C_t} - \frac{1}{C_o} \right]$	$\frac{(C_o/C_t) - 1}{kC_o}$	$\frac{(C_o/C_t) - 1}{kC_t}$

<sup>a</sup> $C_o$  = initial concentration or influent concentration;  $C_t$  = final condition or effluent concentration.

<sup>b</sup>Expressions are valid for  $k\theta \leq C_o$ ; otherwise  $C_t = 0$ .

**Comparison of Steady-State Performance for Decay Reactions of Different Order<sup>a</sup>**

Reaction Order	r	Equations for $C_t$		
		Ideal Batch	Ideal Plug Flow	Ideal CMFR
Zero <sup>b</sup> $t \leq C_o/k$	$-k$	$C_o - kt$	$C_o - k\theta$	$C_o - k\theta$
$t > C_o/k$		0		
First	$-kC$	$C_o[\exp(-kt)]$	$C_o[\exp(-k\theta)]$	$\frac{C_o}{1 + k\theta}$
Second	$-kC^2$	$\frac{C_o}{1 + ktC_o}$	$\frac{C_o}{1 + k\theta C_o}$	$\frac{(4k\theta C_o + 1)^{1/2} - 1}{2k\theta}$

<sup>a</sup> $C_o$  = initial concentration or influent concentration;  $C_t$  = final condition or effluent concentration.

<sup>b</sup>Time conditions are for ideal batch reactor only.

♦ Davis, M.L., and S.J. Masten, *Principles of Environmental Engineering and Science*, McGraw-Hill, 2004.

## LANDFILL

### Break-Through Time for Leachate to Penetrate a Clay Liner

$$t = \frac{d^2 \eta}{K(d + h)}$$

where

$t$  = breakthrough time (yr)

$d$  = thickness of clay liner (ft)

$\eta$  = porosity

$K$  = hydraulic conductivity (ft/yr)

$h$  = hydraulic head (ft)

Typical porosity values for clays with a coefficient of permeability in the range of  $10^{-6}$  to  $10^{-8}$  cm/s vary from 0.1 to 0.3.

### Effect of Overburden Pressure

$$SW_p = SW_i + \frac{p}{a + bp}$$

where

$SW_p$  = specific weight of the waste material at pressure  $p$  (lb/yd<sup>3</sup>) (typical 1,750 to 2,150)

$SW_i$  = initial compacted specific weight of waste (lb/yd<sup>3</sup>) (typical 1,000)

$p$  = overburden pressure (lb/in<sup>2</sup>)

$a$  = empirical constant (yd<sup>3</sup>/in<sup>2</sup>)

$b$  = empirical constant (yd<sup>3</sup>/lb)

### Gas Flux

$$N_A = \frac{D\eta^{4/3}(C_{A_{\text{atm}}} - C_{A_{\text{fill}}})}{L}$$

where

$N_A$  = gas flux of compound  $A$ , [g/(cm<sup>2</sup> • s)][lb • mol/(ft<sup>2</sup> • d)]

$C_{A_{\text{atm}}}$  = concentration of compound  $A$  at the surface of the landfill cover, g/cm<sup>3</sup> (lb • mol/ft<sup>3</sup>)

$C_{A_{\text{fill}}}$  = concentration of compound  $A$  at the bottom of the landfill cover, g/cm<sup>3</sup> (lb • mol/ft<sup>3</sup>)

$L$  = depth of the landfill cover, cm (ft)

Typical values for the coefficient of diffusion for methane and carbon dioxide are 0.20 cm<sup>2</sup>/s (18.6 ft<sup>2</sup>/d) and 0.13 cm<sup>2</sup>/s (12.1 ft<sup>2</sup>/d), respectively.

$D$  = diffusion coefficient, cm<sup>2</sup>/s (ft<sup>2</sup>/d)

$\eta_{\text{gas}}$  = gas-filled porosity, cm<sup>3</sup>/cm<sup>3</sup> (ft<sup>3</sup>/ft<sup>3</sup>)

$\eta$  = porosity, cm<sup>3</sup>/cm<sup>3</sup> (ft<sup>3</sup>/ft<sup>3</sup>)

## Soil Landfill Cover Water Balance

$$\Delta S_{\text{LC}} = P - R - \text{ET} - \text{PER}_{\text{sw}}$$

where

$\Delta S_{\text{LC}}$  = change in the amount of water held in storage in a unit volume of landfill cover (in.)

$P$  = amount of precipitation per unit area (in.)

$R$  = amount of runoff per unit area (in.)

$\text{ET}$  = amount of water lost through evapotranspiration per unit area (in.)

$\text{PER}_{\text{sw}}$  = amount of water percolating through the unit area of landfill cover into compacted solid waste (in.)

## POPULATION MODELING

### Population Projection Equations

#### Linear Projection = Algebraic Projection

$$P_t = P_0 + k\Delta t$$

where

$P_t$  = population at time  $t$

$P_0$  = population at time zero

$k$  = growth rate

$\Delta t$  = elapsed time in years relative to time zero

#### Log Growth = Exponential Growth = Geometric Growth

$$P_t = P_0 e^{k\Delta t}$$

$$\ln P_t = \ln P_0 + k\Delta t$$

where

$P_t$  = population at time  $t$

$P_0$  = population at time zero

$k$  = growth rate

$\Delta t$  = elapsed time in years relative to time zero

## RADIATION

### Effective Half-Life

Effective half-life,  $\tau_e$ , is the combined radioactive and biological half-life.

$$\frac{1}{\tau_e} = \frac{1}{\tau_r} + \frac{1}{\tau_b}$$

where

$\tau_r$  = radioactive half-life

$\tau_b$  = biological half-life

### Half-Life

$$N = N_0 e^{-0.693 t/\tau}$$

where

$N_0$  = original number of atoms

$N$  = final number of atoms

$t$  = time

$\tau$  = half-life

Flux at distance 2 = (Flux at distance 1)  $(r_1/r_2)^2$

where  $r_1$  and  $r_2$  are distances from source.

The half-life of a biologically degraded contaminant assuming a first-order rate constant is given by:

$$t_{1/2} = \frac{0.693}{k}$$

$k$  = rate constant ( $\text{time}^{-1}$ )

$t_{1/2}$  = half-life (time)

### Daughter Product Activity

$$N_2 = \frac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

where  $\lambda_{1,2}$  = decay constants ( $\text{time}^{-1}$ )

$N_{10}$  = initial activity (curies) of parent nuclei

$t$  = time

### Daughter Product Maximum Activity Time

$$t' = \frac{\ln \lambda_2 - \ln \lambda_1}{\lambda_2 - \lambda_1}$$

### Inverse Square Law

$$\frac{I_1}{I_2} = \frac{(R_2)^2}{(R_1)^2}$$

where  $I_{1,2}$  = Radiation intensity at locations 1 and 2

$R_{1,2}$  = Distance from the source at locations 1 and 2

## SAMPLING AND MONITORING

### Data Quality Objectives (DQO) for Sampling Soils and Solids

Investigation Type	Confidence Level (1- $\alpha$ ) (%)	Power (1- $\beta$ ) (%)	Minimum Detectable Relative Difference (%)
Preliminary site investigation	70–80	90–95	10–30
Emergency clean-up	80–90	90–95	10–20
Planned removal and remedial response operations	90–95	90–95	10–20

Confidence level:  $1 - (\text{Probability of a Type I error}) = 1 - \alpha$  = size probability of not making a Type I error.

Power =  $1 - (\text{Probability of a Type II error}) = 1 - \beta$  = probability of not making a Type II error.

EPA Document "EPA/600/8-89/046" *Soil Sampling Quality Assurance User's Guide*, Chapter 7.

$$CV = (100 * s) / \bar{x}$$

CV = coefficient of variation

$s$  = standard deviation of sample

$\bar{x}$  = sample average

Minimum Detectable Relative Difference = Relative increase over background  $[100 (\mu_s - \mu_B) / \mu_B]$  to be detectable with a probability  $(1 - \beta)$

**Number of Samples Required in a One-Sided One-Sample t-Test to Achieve a Minimum Detectable Relative Difference at Confidence Level (1- $\alpha$ ) and Power (1- $\beta$ )**

Coefficient of Variation (%)	Power (%)	Confidence Level (%)	Minimum Detectable Relative Difference (%)				
			5	10	20	30	40
15	95	99	145	39	12	7	5
		95	99	26	8	5	3
		90	78	21	6	3	3
		80	57	15	4	2	2
	90	99	120	32	11	6	5
		95	79	21	7	4	3
		90	60	16	5	3	2
		80	41	11	3	2	1
	80	99	94	26	9	6	5
		95	58	16	5	3	3
		90	42	11	4	2	2
		80	26	7	2	2	1
25	95	99	397	102	28	14	9
		95	272	69	19	9	6
		90	216	55	15	7	5
		80	155	40	11	5	3
	90	99	329	85	24	12	8
		95	272	70	19	9	6
		90	166	42	12	6	4
		80	114	29	8	4	3
	80	99	254	66	19	10	7
		95	156	41	12	6	4
		90	114	30	8	4	3
		80	72	19	5	3	2
35	95	99	775	196	42	25	15
		95	532	134	35	17	10
		90	421	106	28	13	8
		80	304	77	20	9	6
	90	99	641	163	43	21	13
		95	421	107	28	14	8
		90	323	82	21	10	6
		80	222	56	15	7	4
	80	99	495	126	34	17	11
		95	305	78	21	10	7
		90	222	57	15	7	5
		80	140	36	10	5	3

## WASTEWATER TREATMENT AND TECHNOLOGIES

### Activated Sludge

$$X_A = \frac{\theta_c Y (S_0 - S_e)}{\theta (1 + k_d \theta_c)}$$

Steady State Mass Balance around Secondary Clarifier:

$$(Q_0 + Q_R)X_A = Q_e X_e + Q_R X_r + Q_w X_w$$

$$\theta_c = \text{Solids residence time} = \frac{V(X_A)}{Q_w X_w + Q_e X_e}$$

$$\text{Sludge volume/day: } Q_s = \frac{M(100)}{\rho_s (\% \text{ solids})}$$

$$\text{SVI} = \frac{\text{Sludge volume after settling (mL/L)} * 1,000}{\text{MLSS (mg/L)}}$$

$k_d$  = microbial death ratio; kinetic constant; day<sup>-1</sup>; typical range 0.1–0.01, typical domestic wastewater value = 0.05 day<sup>-1</sup>

$S_e$  = effluent BOD or COD concentration (kg/m<sup>3</sup>)

$S_0$  = influent BOD or COD concentration (kg/m<sup>3</sup>)

$X_A$  = biomass concentration in aeration tank (MLSS or MLVSS kg/m<sup>3</sup>)

$Y$  = yield coefficient (kg biomass/kg BOD or COD consumed); range 0.4–1.2

$\theta$  = hydraulic residence time =  $V/Q$

◆

### Design and Operational Parameters for Activated-Sludge Treatment of Municipal Wastewater

Type of Process	Mean cell residence time ( $\theta_c$ , d)	Food-to-mass ratio [(kg BOD <sub>5</sub> / (day·kg MLSS)]	Volumetric loading (kg BOD <sub>5</sub> /m <sup>3</sup> )	Hydraulic residence time in aeration basin ( $\theta$ , h)	Mixed liquor suspended solids (MLSS, mg/L)	Recycle ratio ( $Q_r/Q$ )	Flow regime*	BOD <sub>5</sub> removal efficiency (%)	Air supplied (m <sup>3</sup> /kg BOD <sub>5</sub> )
Tapered aeration	5–15	0.2–0.4	0.3–0.6	4–8	1,500–3,000	0.25–0.5	PF	85–95	45–90
Conventional	4–15	0.2–0.4	0.3–0.6	4–8	1,500–3,000	0.25–0.5	PF	85–95	45–90
Step aeration	4–15	0.2–0.4	0.6–1.0	3–5	2,000–3,500	0.25–0.75	PF	85–95	45–90
Completely mixed	4–15	0.2–0.4	0.8–2.0	3–5	3,000–6,000	0.25–1.0	CM	85–95	45–90
Contact stabilization	4–15	0.2–0.6	1.0–1.2			0.25–1.0			45–90
Contact basin				0.5–1.0	1,000–3,000		PF	80–90	
Stabilization basin				4–6	4,000–10,000		PF		
High-rate aeration	4–15	0.4–1.5	1.6–16	0.5–2.0	4,000–10,000	1.0–5.0	CM	75–90	25–45
Pure oxygen	8–20	0.2–1.0	1.6–4	1–3	6,000–8,000	0.25–0.5	CM	85–95	
Extended aeration	20–30	0.05–0.15	0.16–0.40	18–24	3,000–6,000	0.75–1.50	CM	75–90	90–125

\*PF = plug flow, CM = completely mixed.

### ◆ Blowers

$$P_w = \frac{W R T_1}{C n e} \left[ \left( \frac{P_2}{P_1} \right)^{0.283} - 1 \right]$$

$C$  = 29.7 (constant for SI unit conversion)  
= 550 ft-lb/(sec-hp) (U.S. Customary Units)

$P_w$  = power requirement (hp)

$W$  = weight of flow of air (lb/sec)

$R$  = engineering gas constant for air = 53.3 ft-lb/(lb air·°R)

Solids loading rate =  $Q X_A$

For activated sludge secondary clarifier  $Q = Q_0 + Q_R$

Organic loading rate (volumetric) =  $Q_0 S_0 / Vol$

Organic loading rate (F:M) =  $Q_0 S_0 / (Vol X_A)$

Organic loading rate (surface area) =  $Q_0 S_0 / A_M$

$\rho_s$  = density of solids

$A$  = surface area of unit

$A_M$  = surface area of media in fixed-film reactor

$A_x$  = cross-sectional area of channel

$M$  = sludge production rate (dry weight basis)

$Q_0$  = influent flow rate

$Q_e$  = effluent flow rate

$Q_w$  = waste sludge flow rate

$\rho_s$  = wet sludge density

$R$  = recycle ratio =  $Q_R/Q_0$

$Q_R$  = recycle flow rate =  $Q_0 R$

$X_e$  = effluent suspended solids concentration

$X_w$  = waste sludge suspended solids concentration

$V$  = aeration basin volume

$Q$  = flow rate

$X_r$  = recycled sludge suspended solids concentration

$T_1$  = absolute inlet temperature (°R)

$P_1$  = absolute inlet pressure (lbf/in<sup>2</sup>)

$P_2$  = absolute outlet pressure (lbf/in<sup>2</sup>)

$n$  =  $(k - 1)/k = 0.283$  for air

$e$  = efficiency (usually 0.70 <  $e$  < 0.90)

◆ Metcalf and Eddy, *Wastewater Engineering: Treatment, Disposal, and Reuse*, 3rd ed., McGraw-Hill, 1991.



## Facultative Pond

BOD Loading Total System  $\leq 35$  pounds BOD<sub>5</sub>/(acre-day)

Minimum = 3 ponds

Depth = 3–8 ft

Minimum  $t = 90$ –120 days

## Biotower

### Fixed-Film Equation without Recycle

$$\frac{S_e}{S_0} = e^{-kD/q^n}$$

### Fixed-Film Equation with Recycle

$$\frac{S_e}{S_a} = \frac{e^{-kD/q^n}}{(1 + R) - R(e^{-kD/q^n})}$$

where

$S_e$  = effluent BOD<sub>5</sub> (mg/L)

$S_0$  = influent BOD<sub>5</sub> (mg/L)

$R$  = recycle ratio =  $Q_0/Q_R$

$Q_R$  = recycle flow rate

$$S_a = \frac{S_0 + RS_e}{1 + R}$$

$D$  = depth of biotower media (m)

$q$  = hydraulic loading [ $\text{m}^3/(\text{m}^2 \cdot \text{min})$ ]

$$= (Q_0 + RQ_0)/A_{\text{plan}} \text{ (with recycle)}$$

$k$  = treatability constant; functions of wastewater and medium ( $\text{min}^{-1}$ ); range 0.01–0.1; for municipal wastewater and modular plastic media  $0.06 \text{ min}^{-1}$  @  $20^\circ\text{C}$

$$k_T = k_{20}(1.035)^{T-20}$$

$n$  = coefficient relating to media characteristics; modular plastic,  $n = 0.5$

## ♦ Aerobic Digestion

### Design criteria for aerobic digesters<sup>a</sup>

Parameter	Value
Sludge retention time, d	
At $20^\circ\text{C}$	40
At $15^\circ\text{C}$	60
Solids loading, lb volatile solids/ $\text{ft}^3 \cdot \text{d}$	0.1–0.3
Oxygen requirements, lb O <sub>2</sub> /lb solids destroyed	
Cell tissue	~2.3
BOD <sub>5</sub> in primary sludge	1.6–1.9
Energy requirements for mixing	
Mechanical aerators, hp/ $10^3 \text{ ft}^3$	0.7–1.50
Diffused-air mixing, $\text{ft}^3/10^3 \text{ ft}^3 \cdot \text{min}$	20–40
Dissolved-oxygen residual in liquid, mg/L	1–2
Reduction in volatile suspended solids, %	40–50

### Tank Volume

$$V = \frac{Q_i(X_i + FS_i)}{X_d(k_d P_v + 1/\theta_c)}$$

where

$V$  = volume of aerobic digester ( $\text{ft}^3$ )

$Q_i$  = influent average flowrate to digester ( $\text{ft}^3/\text{d}$ )

$X_i$  = influent suspended solids (mg/L)

$F$  = fraction of the influent BOD<sub>5</sub> consisting of raw primary sludge (expressed as a decimal)

$S_i$  = influent BOD<sub>5</sub> (mg/L)

$X_d$  = digester suspended solids (mg/L); typically  $X_d = (0.7)X_i$

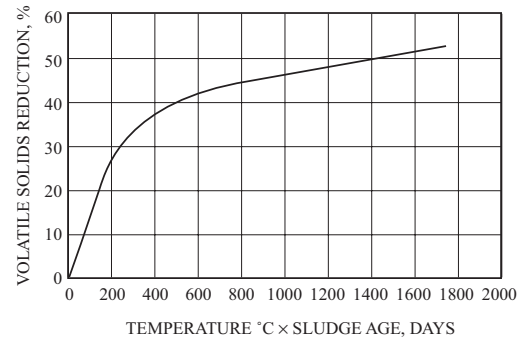
$k_d$  = reaction-rate constant ( $\text{d}^{-1}$ )

$P_v$  = volatile fraction of digester suspended solids (expressed as a decimal)

$\theta_c$  = solids residence time (sludge age) (d)

$FS_i$  can be neglected if primary sludge is not included on the sludge flow to the digester.

♦



VOLATILE SOLIDS REDUCTION IN AN AEROBIC DIGESTER AS A FUNCTION OF DIGESTER LIQUID TEMPERATURE AND DIGESTER SLUDGE AGE

## • Anaerobic Digestion

### Design parameters for anaerobic digesters

Parameter	Standard-rate	High-rate
Solids residence time, d	30–90	10–20
Volatile solids loading, $\text{kg}/\text{m}^3/\text{d}$	0.5–1.6	1.6–6.4
Digested solids concentration, %	4–6	4–6
Volatile solids reduction, %	35–50	45–55
Gas production ( $\text{m}^3/\text{kg}$ VSS added)	0.5–0.55	0.6–0.65
Methane content, %	65	65

### Standard Rate

$$\text{Reactor Volume} = \frac{V_1 + V_2}{2} t_r + V_2 t_s$$

### High Rate

#### First stage

$$\text{Reactor Volume} = V_1 t_r$$

#### Second Stage

$$\text{Reactor Volume} = \frac{V_1 + V_2}{2} t_t + V_2 t_s$$

where

$V_1$  = raw sludge input (volume/day)

$V_2$  = digested sludge accumulation (volume/day)

$t_r$  = time to react in a high-rate digester = time to react and thicken in a standard-rate digester

$t_t$  = time to thicken in a high-rate digester

$t_s$  = storage time

♦ Tchobanoglous, G., and Metcalf and Eddy, *Wastewater Engineering: Treatment, Disposal, and Reuse*, 4th ed., McGraw-Hill, 2003.

• Peavy, HS, D.R. Rowe, and G. Tchobanoglous, *Environmental Engineering*, McGraw-Hill, 1985.

## WATER TREATMENT TECHNOLOGIES

### Activated Carbon Adsorption

#### Freundlich Isotherm

$$\frac{x}{m} = X = KC_e^{1/n}$$

where

$x$  = mass of solute adsorbed

$m$  = mass of adsorbent

$X$  = mass ratio of the solid phase—that is, the mass of adsorbed solute per mass of adsorbent

$C_e$  = equilibrium concentration of solute, mass/volume

$K, n$  = experimental constants

#### Linearized Form

$$\ln \frac{x}{m} = \frac{1}{n} \ln C_e + \ln K$$

For linear isotherm,  $n = 1$

#### Langmuir Isotherm

$$\frac{x}{m} = X = \frac{aKC_e}{1 + KC_e}$$

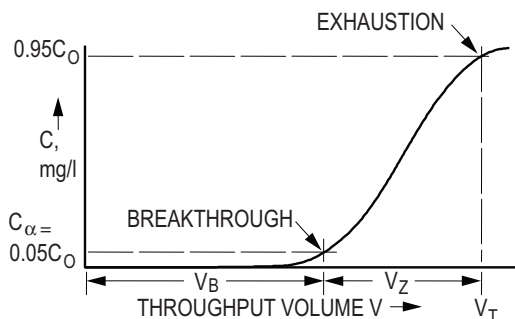
where

$a$  = mass of adsorbed solute required to saturate completely a unit mass of adsorbent

$K$  = experimental constant

#### Linearized Form

$$\frac{m}{x} = \frac{1}{a} + \frac{1}{aK} \frac{1}{C_e}$$



#### Depth of Sorption Zone

$$Z_s = Z \left[ \frac{V_Z}{V_T - 0.5V_Z} \right]$$

where

$$V_Z = V_T - V_B$$

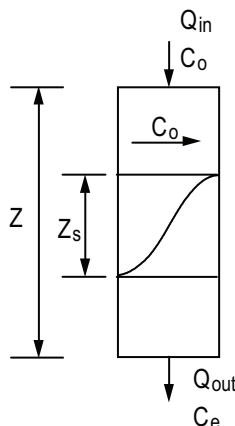
$Z_s$  = depth of sorption zone

$Z$  = total carbon depth

$V_T$  = total volume treated at exhaustion ( $C = 0.95 C_0$ )

$V_B$  = total volume at breakthrough ( $C = C_\alpha = 0.05 C_0$ )

$C_0$  = concentration of contaminant in influent



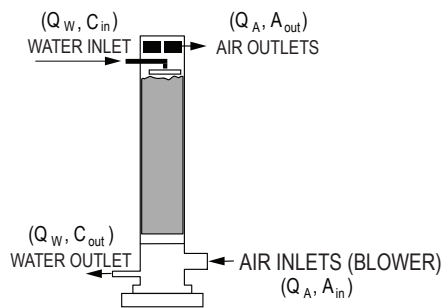
### Air Stripping

$P_i = HC_i$  = Henry's Law

$P_i$  = partial pressure of component  $i$ , atm

$H$  = Henry's Law constant, atm·m<sup>3</sup>/kmol

$C_i$  = concentration of component  $i$  in solvent, kmol/m<sup>3</sup>



$$A_{out} = H C_{in}$$

$$Q_W \cdot C_{in} = Q_A H C_{in}$$

$$Q_W = Q_A H$$

$$H' (Q_A / Q_W) = 1$$

where

$A_{out}$  = concentration in the effluent air (kmol/m<sup>3</sup>); in this formulation of the equation  $A_{in}$  and  $C_{out}$  are assumed to be negligible for simplicity.

$Q_W$  = water flow rate (m<sup>3</sup>/s)

$Q_A$  = air flow rate (m<sup>3</sup>/s)

$A_{in}$  = concentration of contaminant in air (kmol/m<sup>3</sup>)

$C_{out}$  = concentration of contaminants in effluent water (kmol/m<sup>3</sup>)

$C_{in}$  = concentration of contaminants in influent water (kmol/m<sup>3</sup>)

#### Stripper Packing Height = $Z$

$$Z = HTU \times NTU$$

Assuming rapid equilibrium:

$$NTU = \left( \frac{R_S}{R_S - 1} \right) \ln \left( \frac{(C_{in}/C_{out})(R_S - 1) + 1}{R_S} \right)$$

where

$NTU$  = number of transfer units

$H$  = Henry's Law constant

$H' = H/RT$  = dimensionless Henry's Law constant

$T$  = temperature in units consistent with  $R$

$R$  = universal gas constant, atm·m<sup>3</sup>/(kmol·K)

$R_S$  = stripping factor  $H'(Q_A/Q_W)$

$C_{in}$  = concentration in the influent water (kmol/m<sup>3</sup>)

$C_{out}$  = concentration in the effluent water (kmol/m<sup>3</sup>)

$HTU$  = Height of Transfer Units =  $\frac{L}{M_W K_L a}$

where

$L$  = liquid molar loading rate [kmol/(s·m<sup>2</sup>)]

$M_W$  = molar density of water

(55.6 kmol/m<sup>3</sup>) = 3.47 lbmol/ft<sup>3</sup>

$K_L a$  = overall transfer rate constant (s<sup>-1</sup>)

## Clarifier

Overflow rate = Hydraulic loading rate =  $v_o = Q/A_{\text{surface}}$

$v_o$  = critical settling velocity

= terminal settling velocity of smallest particle that is 100% removed

Weir loading = weir overflow rate, WOR =  $Q/\text{Weir Length}$

Horizontal velocity = approach velocity =  $v_h$

$$= Q/A_{\text{cross-section}} = Q/A_x$$

Hydraulic residence time =  $V/Q = \theta$

where

$Q$  = flow rate

$A_x$  = cross-sectional area

$A$  = surface area, plan view

$V$  = tank volume

### Typical Primary Clarifier Efficiency Percent Removal

	Overflow rates			
	1,200 (gpd/ft <sup>2</sup> )	1,000 (gpd/ft <sup>2</sup> )	800 (gpd/ft <sup>2</sup> )	600 (gpd/ft <sup>2</sup> )
	48.9 (m/d)	40.7 (m/d)	32.6 (m/d)	24.4 (m/d)
	54%	58%	64%	68%
Suspended Solids				
BOD <sub>5</sub>	30%	32%	34%	36%

## Weir Loadings

1. Water Treatment—weir overflow rates should not exceed 20,000 gpd/ft
2. Wastewater Treatment
  - a. Flow  $\leq 1$  MGD: weir overflow rates should not exceed 10,000 gpd/ft
  - b. Flow  $> 1$  MGD: weir overflow rates should not exceed 15,000 gpd/ft

## Horizontal Velocities

1. Water Treatment—horizontal velocities should not exceed 0.5 fpm
2. Wastewater Treatment—no specific requirements (use the same criteria as for water)

## Dimensions

1. Rectangular Tanks
  - a. Length:Width ratio = 3:1 to 5:1
  - b. Basin width is determined by the scraper width (or multiples of the scraper width)
  - c. Bottom slope is set at 1%
2. Circular Tanks
  - a. Diameters up to 200 ft
  - b. Diameters must match the dimensions of the sludge scraping mechanism
  - c. Bottom slope is less than 8%

## Design Criteria for Sedimentation Basins

Type of Basin	Overflow Rate				Solids Loading Rate				Hydraulic Residence Time (hr)	Depth (ft)
	Average (gpd/ft <sup>2</sup> )	Peak (m <sup>3</sup> /m <sup>2</sup> ·d)	Average (gpd/ft <sup>2</sup> )	Peak (m <sup>3</sup> /m <sup>2</sup> ·d)	Average (lb/ft <sup>2</sup> ·d)	Peak (kg/m <sup>2</sup> ·h)	Average (lb/ft <sup>2</sup> ·h)	Peak (kg/m <sup>2</sup> ·h)		
<b>Water Treatment</b>										
Clarification following coagulation and flocculation:										
Alum coagulation	350–550	14–22							4–8	12–16
Ferric coagulation	550–700	22–28							4–8	12–16
Upflow clarifiers										
Groundwater	1,500–2,200	61–90							1	
Surface water	1,000–1,500	41–61							4	
Clarification following lime-soda softening										
Conventional	550–1,000	22–41							2–4	
Upflow clarifiers										
Groundwater	1,000–2,500	41–102							1	
Surface water	1,000–1,800	41–73							4	
<b>Wastewater Treatment</b>										
Primary clarifiers	800–1,200	32–49	1,200–2,000	50–80					2	10–12
Settling basins following fixed film reactors	400–800	16–33							2	
Settling basins following air-activated sludge reactors										
All configurations EXCEPT extended aeration	400–700	16–28							2	12–15
Extended aeration	200–400	8–16	1,000–1,200	40–64	19–29	4–6	38	8	2	12–15
Settling basins following chemical flocculation reactors	800–1,200		600–800	24–32	5–24	1–5	34	7	2	

## Settling Equations

### General Spherical

$$v_t = \sqrt{\frac{4g(\rho_p - \rho_f)d}{3C_D\rho_f}}$$

where

$C_D$  = drag coefficient  
 =  $24/\text{Re}$  (Laminar;  $\text{Re} \leq 1.0$ )  
 =  $24/\text{Re} + 3/(\text{Re}^{1/2}) + 0.34$  (Transitional)  
 =  $0.4$  (Turbulent;  $\text{Re} \geq 10^4$ )

$\text{Re}$  = Reynolds number  $\frac{v_t \rho d}{\mu}$

$g$  = gravitational constant

$\rho_p$  and  $\rho_f$  = density of particle and fluid respectively

$d$  = diameter of sphere

$\mu$  = bulk viscosity of liquid = absolute viscosity

$v_t$  = terminal settling velocity

### Stokes' Law

$$v_t = \frac{g(\rho_p - \rho_f)d^2}{18\mu} = \frac{g\rho_f(S.G. - 1)d^2}{18\mu}$$

Approach velocity = horizontal velocity =  $Q/A_x$

Hydraulic loading rate =  $Q/A$

Hydraulic residence time =  $V/Q = \theta$

where

$Q$  = flow rate

$A_x$  = cross-sectional area

$A$  = surface area, plan view

$V$  = tank volume

$\rho_f$  = fluid mass density

$S.G.$  = specific gravity

## Filtration Equations

Filter bay length:width ratio = 1.2:1 to 1.5:1

Effective size =  $d_{10}$

Uniformity coefficient =  $d_{60}/d_{10}$

$d_x$  = diameter of particle class for which  $x\%$  of sample is less than (units meters or feet)

Filter equations can be used with any consistent set of units.

## Head Loss Through Clean Bed

### Rose Equation

Monosized Media

$$h_f = \frac{1067(v_s)^2 LC_D}{g\eta^4 d}$$

Multisized Media

$$h_f = \frac{1067(v_s)^2 L}{g\eta^4} \sum \frac{C_{Dij} x_{ij}}{d_{ij}}$$

### Carmen-Kozeny Equation

Monosized Media

$$h_f = \frac{f' L (1 - \eta) v_s^2}{\eta^3 g d}$$

Multisized Media

$$h_f = \frac{L(1 - \eta) v_s^2}{\eta^3 g} \sum \frac{f'_{ij} x_{ij}}{d_{ij}}$$

$$f' = \text{friction factor} = 150 \left( \frac{1 - \eta}{\text{Re}} \right) + 1.75$$

where

$h_f$  = head loss through the clean bed (m of  $\text{H}_2\text{O}$ )

$L$  = depth of filter media (m)

$\eta$  = porosity of bed = void volume/total volume

$v_s$  = filtration rate = empty bed approach velocity

=  $Q/A_{\text{plan}}$  (m/s)

$g$  = gravitational acceleration ( $\text{m/s}^2$ )

$\text{Re}$  = Reynolds number =  $\frac{v_s \rho d}{\mu}$

$d_{ij}$ ,  $d$  = diameter of filter media particles; arithmetic average of adjacent screen openings (m)

$i$  = filter media (sand, anthracite, garnet)

$j$  = filter media particle size

$x_{ij}$  = mass fraction of media retained between adjacent sieves

$f'_{ij}$  = friction factors for each media fraction

$C_D$  = drag coefficient as defined in settling velocity equations

## Bed Expansion

Monosized

$$L_f = \frac{L_o(1 - \eta_o)}{1 - \left( \frac{v_B}{v_t} \right)^{0.22}}$$

Multisized

$$L_f = L_o(1 - \eta_o) \sum \frac{x_{ij}}{1 - \left( \frac{v_B}{v_{t,ij}} \right)^{0.22}}$$

$$\eta_f = \left( \frac{v_B}{v_t} \right)^{0.22}$$

where

$L_f$  = depth of fluidized filter media (m)

$v_B$  = backwash velocity (m/s),  $Q_B/A_{\text{plan}}$

$Q_B$  = backwash flowrate

$v_t$  = terminal setting velocity

$\eta_f$  = porosity of fluidized bed

$L_o$  = initial bed depth

$\eta_o$  = initial bed porosity

## Lime-Soda Softening Equations

- Carbon dioxide removal  

$$\text{CO}_2 + \text{Ca(OH)}_2 \rightarrow \text{CaCO}_3(\text{s}) + \text{H}_2\text{O}$$
- Calcium carbonate hardness removal  

$$\text{Ca(HCO}_3)_2 + \text{Ca(OH)}_2 \rightarrow 2\text{CaCO}_3(\text{s}) + 2\text{H}_2\text{O}$$
- Calcium non-carbonate hardness removal  

$$\text{CaSO}_4 + \text{Na}_2\text{CO}_3 \rightarrow \text{CaCO}_3(\text{s}) + 2\text{Na}^+ + \text{SO}_4^{2-}$$
- Magnesium carbonate hardness removal  

$$\text{Mg(HCO}_3)_2 + 2\text{Ca(OH)}_2 \rightarrow 2\text{CaCO}_3(\text{s}) + \text{Mg(OH)}_2(\text{s}) + 2\text{H}_2\text{O}$$
- Magnesium non-carbonate hardness removal  

$$\text{MgSO}_4 + \text{Ca(OH)}_2 + \text{Na}_2\text{CO}_3 \rightarrow \text{CaCO}_3(\text{s}) + \text{Mg(OH)}_2(\text{s}) + 2\text{Na}^+ + \text{SO}_4^{2-}$$
- Destruction of excess alkalinity  

$$2\text{HCO}_3^- + \text{Ca(OH)}_2 \rightarrow \text{CaCO}_3(\text{s}) + \text{CO}_3^{2-} + 2\text{H}_2\text{O}$$
- Recarbonation  

$$\text{Ca}^{2+} + 2\text{OH}^- + \text{CO}_2 \rightarrow \text{CaCO}_3(\text{s}) + \text{H}_2\text{O}$$

Molecular Formulas	Molecular Weight	<i>n</i>	Equivalent Weight
		# Equiv per mole	
CO <sub>3</sub> <sup>2-</sup>	60.0	2	30.0
CO <sub>2</sub>	44.0	2	22.0
Ca(OH) <sub>2</sub>	74.1	2	37.1
CaCO <sub>3</sub>	100.1	2	50.0
Ca(HCO <sub>3</sub> ) <sub>2</sub>	162.1	2	81.1
CaSO <sub>4</sub>	136.1	2	68.1
Ca <sup>2+</sup>	40.1	2	20.0
H <sup>+</sup>	1.0	1	1.0
HCO <sub>3</sub> <sup>-</sup>	61.0	1	61.0
Mg(HCO <sub>3</sub> ) <sub>2</sub>	146.3	2	73.2
Mg(OH) <sub>2</sub>	58.3	2	29.2
MgSO <sub>4</sub>	120.4	2	60.2
Mg <sup>2+</sup>	24.3	2	12.2
Na <sup>+</sup>	23.0	1	23.0
Na <sub>2</sub> CO <sub>3</sub>	106.0	2	53.0
OH <sup>-</sup>	17.0	1	17.0
SO <sub>4</sub> <sup>2-</sup>	96.1	2	48.0

## Rapid Mix and Flocculator Design

$$G = \sqrt{\frac{P}{\mu V}} = \sqrt{\frac{\gamma H_L}{t \mu}}$$

$$Gt = 10^4 \text{ to } 10^5$$

where

$G$  = root mean square velocity gradient (mixing intensity)  
 [ft/(sec-ft) or m/(s•m)]

$P$  = power to the fluid (ft-lb/sec or N•m/s)

$V$  = volume (ft<sup>3</sup> or m<sup>3</sup>)

$\mu$  = dynamic viscosity [lb/(ft-sec) or Pa•s]

$\gamma$  = specific weight of water (lb/ft<sup>3</sup> or N/m<sup>3</sup>)

$H_L$  = head loss (ft or m)

$t$  = time (sec or s)

## Reel and Paddle

$$P = \frac{C_D A_p \rho_f v_r^3}{2}$$

where

$C_D$  = drag coefficient = 1.8 for flat blade with a L:W > 20:1

$A_p$  = area of blade (m<sup>2</sup>) perpendicular to the direction of travel through the water

$\rho_f$  = density of H<sub>2</sub>O (kg/m<sup>3</sup>)

$v_p$  = velocity of paddle (m/s)

$v_r$  = relative or effective paddle velocity

=  $v_p$  • slip coefficient

slip coefficient = 0.5 to 0.75

## Turbulent Flow Impeller Mixer

$$P = K_T (n)^3 (D_i)^5 \rho_f$$

where

$K_T$  = impeller constant (see table)

$n$  = rotational speed (rev/sec)

$D_i$  = impeller diameter (m)

## Values of the Impeller Constant $K_T$ (Assume Turbulent Flow)

Type of Impeller	$K_T$
Propeller, pitch of 1, 3 blades	0.32
Propeller, pitch of 2, 3 blades	1.00
Turbine, 6 flat blades, vaned disc	6.30
Turbine, 6 curved blades	4.80
Fan turbine, 6 blades at 45°	1.65
Shrouded turbine, 6 curved blades	1.08
Shrouded turbine, with stator, no baffles	1.12

*Note:* Constant assumes baffled tanks having four baffles at the tank wall with a width equal to 10% of the tank diameter.

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## Reverse Osmosis

### Osmotic Pressure of Solutions of Electrolytes

$$\Pi = \phi v \frac{n}{V} RT$$

where

$\Pi$  = osmotic pressure, Pa

$\phi$  = osmotic coefficient

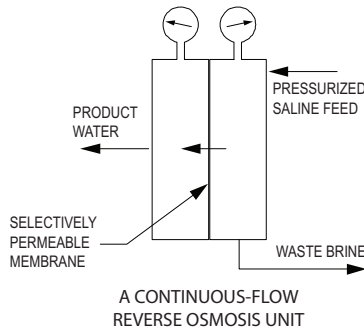
$v$  = number of ions formed from one molecule of electrolyte

$n$  = number of moles of electrolyte

$V$  = specific volume of solvent, m<sup>3</sup>/kmol

$R$  = universal gas constant, Pa • m<sup>3</sup>/(kmol • K)

$T$  = absolute temperature, K



### Salt Flux through the Membrane

$$J_s = (D_s K_s / \Delta Z)(C_{in} - C_{out})$$

where

$J_s$  = salt flux through the membrane [kmol/(m<sup>2</sup> • s)]

$D_s$  = diffusivity of the solute in the membrane (m<sup>2</sup>/s)

$K_s$  = solute distribution coefficient (dimensionless)

$C$  = concentration (kmol/m<sup>3</sup>)

$\Delta Z$  = membrane thickness (m)

$$J_s = K_p (C_{in} - C_{out})$$

$K_p$  = membrane solute mass transfer coefficient

$$= \frac{D_s K_s}{\Delta Z} (L/t, m/s)$$

### Water Flux

$$J_w = W_p (\Delta P - \Delta \pi)$$

where

$J_w$  = water flux through the membrane [kmol/(m<sup>2</sup> • s)]

$W_p$  = coefficient of water permeation, a characteristic of the particular membrane [kmol/(m<sup>2</sup> • s • Pa)]

$\Delta P$  = pressure differential across membrane =  $P_{in} - P_{out}$  (Pa)

$\Delta \pi$  = osmotic pressure differential across membrane

$$\pi_{in} - \pi_{out} (Pa)$$

## Ultrafiltration

$$J_w = \frac{\epsilon r^2 \int \Delta P}{8 \mu \delta}$$

where

$\epsilon$  = membrane porosity

$r$  = membrane pore size

$\Delta P$  = net transmembrane pressure

$\mu$  = viscosity

$\delta$  = membrane thickness

$J_w$  = volumetric flux (m/s)

### Disinfection

Chlorine contact chamber length:width ratio = 20:1 to 50:1

$$T = TDT \times BF$$

$T$  = time that the water is in contact with the disinfectant (min)

$TDT$  = theoretical detention time (min)

$$TDT (\text{min-mg/L}) = C \times T$$

$BF$  = baffling factor

$C$  = residual disinfectant concentration measured during peak hourly flow (mg/L)

◆

### Baffling Factors

Baffling Condition	Baffling Factor	Baffling Description
Unbaffled (mixed flow)	0.1	None, agitated basin, very low length to width ratio, high inlet and outlet flow velocities.
Poor	0.3	Single or multiple unbaffled inlets and outlets, no intra-basin baffles.
Average	0.5	Baffled inlet or outlet with some intra-basin baffles.
Superior	0.7	Perforated inlet baffle, serpentine or perforated intra-basin baffles, outlet weir or perforated launders.
Perfect (plug flow)	1.0	Very high length to width ratio (pipeline flow), perforated inlet, outlet, and intra-basin baffles.

◆ Guidance Manual LT1ESWTR Disinfection Profiling and Benchmarking, U.S. Environmental Protection Agency, 2003.

◆ **Removal and Inactivation Requirements**

<b>Microorganism</b>	<b>Required Log Reduction</b>	<b>Treatment</b>
<i>Giardia</i>	3-log (99.9%)	Removal and/or inactivation
Viruses	4-log (99.99%)	Removal and/or inactivation
<i>Cryptosporidium</i>	2-log (99%)	Removal

◆ **Typical Removal Credits and Inactivation Requirements for Various Treatment Technologies**

<b>Process</b>	<b>Typical Log Removal Credits</b>		<b>Resulting Disinfection Log Inactivation Requirements</b>	
	<i>Giardia</i>	Viruses	<i>Giardia</i>	Viruses
Conventional Treatment	2.5	2.0	0.5	2.0
Direct Filtration	2.0	1.0	1.0	3.0
Slow Sand Filtration	2.0	2.0	1.0	2.0
Diatomaceous Earth Filtration	2.0	1.0	1.0	3.0
Unfiltered	0	0	3.0	4.0

◆ *Guidance Manual LT1ESWTR Disinfection Profiling and Benchmarking*, U.S. Environmental Protection Agency, 2003.



### CT Values\* For 3-LOG Inactivation Of *Giardia* Cysts By Free Chlorine

Chlorine Concentration (mg/L)	Temperature <= 0.5°C							Temperature = 5°C							Temperature = 10°C						
	pH							pH							pH						
	<=6.0	6.5	7.0	7.5	8.0	8.5	9.0	<=6.0	6.5	7.0	7.5	8.0	8.5	9.0	<=6.0	6.5	7.0	7.5	8.0	8.5	9.0
<=0.4	137	163	195	237	277	329	390	97	117	139	166	198	236	279	73	88	104	125	149	177	209
0.6	141	168	200	239	286	342	407	100	120	143	171	204	244	291	75	90	107	128	153	183	218
0.8	145	172	205	246	295	354	422	103	122	146	175	210	252	301	78	92	110	131	158	189	226
1.0	148	176	210	253	304	365	437	105	125	149	179	216	260	312	79	94	112	134	162	195	234
1.2	152	180	215	259	313	376	451	107	127	152	183	221	267	320	80	95	114	137	166	200	240
1.4	155	184	221	266	321	387	464	109	130	155	187	227	274	329	82	98	116	140	170	206	247
1.6	157	189	226	273	329	397	477	111	132	158	192	232	281	337	83	99	119	144	174	211	253
1.8	162	193	231	279	338	407	489	114	135	162	196	238	287	345	86	101	122	147	179	215	259
2.0	165	197	236	286	346	417	500	116	138	165	200	243	294	353	87	104	124	150	182	221	265
2.2	169	201	242	297	353	426	511	118	140	169	204	248	300	361	89	105	127	153	186	225	271
2.4	172	205	247	298	361	435	522	120	143	172	209	253	306	368	90	107	129	157	190	230	276
2.6	175	209	252	304	368	444	533	122	146	175	213	258	312	375	92	110	131	160	194	234	281
2.8	178	213	257	310	375	452	543	124	148	178	217	263	318	382	93	111	134	163	197	239	287
3.0	181	217	261	316	382	460	552	126	151	182	221	268	324	389	95	113	137	166	201	243	292

Chlorine Concentration (mg/L)	Temperature = 15°C							Temperature = 20°C							Temperature = 25°C						
	pH							pH							pH						
	<=6.0	6.5	7.0	7.5	8.0	8.5	9.0	<=6.0	6.5	7.0	7.5	8.0	8.5	9.0	<=6.0	6.5	7.0	7.5	8.0	8.5	9.0
<=0.4	49	59	70	83	99	118	140	36	44	52	62	74	89	105	24	29	35	42	50	59	70
0.6	50	60	72	86	102	122	146	38	45	54	64	77	92	109	25	30	36	43	51	61	73
0.8	52	61	73	88	105	126	151	39	46	55	66	79	95	113	26	31	37	44	53	63	75
1.0	53	63	75	90	108	130	156	39	47	56	67	81	98	117	26	31	37	45	54	65	78
1.2	54	64	76	92	111	134	160	40	48	57	69	83	100	120	27	32	38	46	55	67	80
1.4	55	65	78	94	114	137	165	41	49	58	70	85	103	123	27	33	39	47	57	69	82
1.6	56	66	79	96	116	141	169	42	50	59	72	87	105	126	28	33	40	48	58	70	84
1.8	57	68	81	98	119	144	173	43	51	61	74	89	106	129	29	34	41	49	60	72	86
2.0	58	69	83	100	122	147	177	44	52	62	75	91	110	132	29	35	41	50	61	74	88
2.2	59	70	85	102	124	150	181	44	53	63	77	93	113	135	30	35	42	51	62	75	90
2.4	60	72	86	105	127	153	184	45	54	65	78	95	115	138	30	36	43	52	63	77	92
2.6	61	73	88	107	129	156	188	46	55	66	80	97	117	141	31	37	44	53	65	78	94
2.8	62	74	89	109	132	159	191	47	56	67	81	99	119	143	31	37	45	54	66	80	96
3.0	63	76	91	111	134	162	195	47	57	68	83	101	122	146	32	38	46	55	67	81	97

\*Although units did not appear in the original tables, units are min-mg/L

### CT VALUES\* FOR 4-LOG INACTIVATION OF VIRUSES BY FREE CHLORINE

Temperature (°C)	pH	
	6–9	10
0.5	12	90
5	8	60
10	6	45
15	4	30
20	3	22
25	2	15

\*Although units did not appear in the original tables, units are min-mg/L

◆ Guidance Manual LT1ESWTR Disinfection Profiling and Benchmarking, U.S. Environmental Protection Agency, 2003.



# ELECTRICAL AND COMPUTER ENGINEERING

## UNITS

The basic electrical units are coulombs for charge, volts for voltage, amperes for current, and ohms for resistance and impedance.

## ELECTROSTATICS

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon r^2} \mathbf{a}_{r12}, \text{ where}$$

$\mathbf{F}_2$  = the force on charge 2 due to charge 1

$Q_i$  = the  $i$ th point charge

$r$  = the distance between charges 1 and 2

$\mathbf{a}_{r12}$  = a unit vector directed from 1 to 2

$\epsilon$  = the permittivity of the medium

For free space or air:

$$\epsilon = \epsilon_0 = 8.85 \times 10^{-12} \text{ farads/meter}$$

### Electrostatic Fields

Electric field intensity  $\mathbf{E}$  (volts/meter) at point 2 due to a point charge  $Q_1$  at point 1 is

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon r^2} \mathbf{a}_{r12}$$

For a line charge of density  $\rho_L$  coulombs/meter on the  $z$ -axis, the radial electric field is

$$\mathbf{E}_L = \frac{\rho_L}{2\pi\epsilon r} \mathbf{a}_r$$

For a sheet charge of density  $\rho_s$  coulombs/meter<sup>2</sup> in the  $x$ - $y$  plane:

$$\mathbf{E}_s = \frac{\rho_s}{2\epsilon} \mathbf{a}_z, z > 0$$

Gauss' law states that the integral of the electric flux density  $\mathbf{D} = \epsilon\mathbf{E}$  over a closed surface is equal to the charge enclosed or

$$Q_{\text{encl}} = \oint_S \epsilon \mathbf{E} \cdot d\mathbf{S}$$

The force on a point charge  $Q$  in an electric field with intensity  $\mathbf{E}$  is  $\mathbf{F} = Q\mathbf{E}$ .

The work done by an external agent in moving a charge  $Q$  in an electric field from point  $p_1$  to point  $p_2$  is

$$W = -Q \int_{p_1}^{p_2} \mathbf{E} \cdot d\mathbf{l}$$

The energy  $W_E$  stored in an electric field  $\mathbf{E}$  is

$$W_E = (1/2) \iiint_V \epsilon |\mathbf{E}|^2 dV$$

## Voltage

The potential difference  $V$  between two points is the work per unit charge required to move the charge between the points.

For two parallel plates with potential difference  $V$ , separated by distance  $d$ , the strength of the  $E$  field between the plates is

$$E = \frac{V}{d}$$

directed from the + plate to the - plate.

## Current

Electric current  $i(t)$  through a surface is defined as the rate of charge transport through that surface or

$$i(t) = dq(t)/dt, \text{ which is a function of time } t$$

since  $q(t)$  denotes instantaneous charge.

A constant current  $i(t)$  is written as  $I$ , and the vector current density in amperes/m<sup>2</sup> is defined as  $\mathbf{J}$ .

## Magnetic Fields

For a current-carrying wire on the  $z$ -axis

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{I \mathbf{a}_\phi}{2\pi r}, \text{ where}$$

$\mathbf{H}$  = the magnetic field strength (amperes/meter)

$\mathbf{B}$  = the magnetic flux density (tesla)

$\mathbf{a}_\phi$  = the unit vector in positive  $\phi$  direction in cylindrical coordinates

$I$  = the current

$\mu$  = the permeability of the medium

For air:  $\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$

Force on a current-carrying conductor in a uniform magnetic field is

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}, \text{ where}$$

$\mathbf{L}$  = the length vector of a conductor

The energy stored  $W_H$  in a magnetic field  $\mathbf{H}$  is

$$W_H = (1/2) \iiint_V \mu |\mathbf{H}|^2 dV$$

## Induced Voltage

Faraday's Law states for a coil of  $N$  turns enclosing flux  $\phi$ :

$$v = -N d\phi/dt, \text{ where}$$

$v$  = the induced voltage

$\phi$  = the average flux (webers) enclosed by each turn

$$\phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

## Resistivity

For a conductor of length  $L$ , electrical resistivity  $\rho$ , and cross-sectional area  $A$ , the resistance is

$$R = \frac{\rho L}{A}$$

For metallic conductors, the resistivity and resistance vary linearly with changes in temperature according to the following relationships:

$$\rho = \rho_0 [1 + \alpha(T - T_0)], \text{ and}$$

$$R = R_0 [1 + \alpha(T - T_0)], \text{ where}$$

$\rho_0$  = resistivity at  $T_0$

$R_0$  = resistance at  $T_0$

$\alpha$  = temperature coefficient

Ohm's Law:  $V = IR$ ;  $v(t) = i(t) R$

## Resistors in Series and Parallel

For series connections, the current in all resistors is the same and the equivalent resistance for  $n$  resistors in series is

$$R_s = R_1 + R_2 + \dots + R_n$$

For parallel connections of resistors, the voltage drop across each resistor is the same and the equivalent resistance for  $n$  resistors in parallel is

$$R_p = 1 / (1/R_1 + 1/R_2 + \dots + 1/R_n)$$

For two resistors  $R_1$  and  $R_2$  in parallel

$$R_p = \frac{R_1 R_2}{R_1 + R_2}$$

## Power Absorbed by a Resistive Element

$$P = VI = \frac{V^2}{R} = I^2 R$$

## Kirchhoff's Laws

Kirchhoff's voltage law for a closed path is expressed by

$$\sum V_{\text{rises}} = \sum V_{\text{drops}}$$

Kirchhoff's current law for a closed surface is

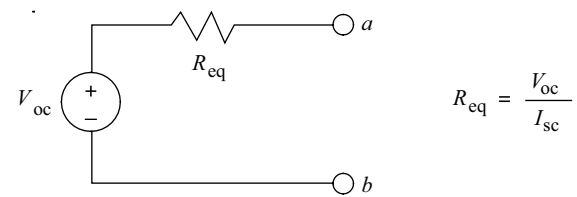
$$\sum I_{\text{in}} = \sum I_{\text{out}}$$

## SOURCE EQUIVALENTS

For an arbitrary circuit

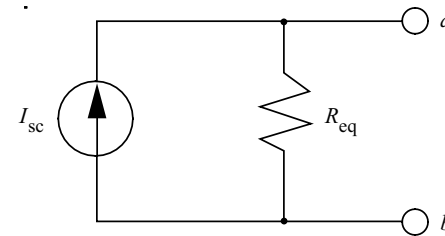


The Thévenin equivalent is



The open circuit voltage  $V_{oc}$  is  $V_a - V_b$ , and the short circuit current is  $I_{sc}$  from  $a$  to  $b$ .

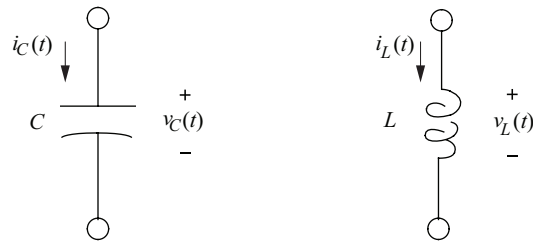
The Norton equivalent circuit is



where  $I_{sc}$  and  $R_{eq}$  are defined above.

A load resistor  $R_L$  connected across terminals  $a$  and  $b$  will draw maximum power when  $R_L = R_{eq}$ .

## CAPACITORS AND INDUCTORS



The charge  $q_C(t)$  and voltage  $v_C(t)$  relationship for a capacitor  $C$  in farads is

$$C = q_C(t) / v_C(t) \quad \text{or} \quad q_C(t) = C v_C(t)$$

A parallel plate capacitor of area  $A$  with plates separated a distance  $d$  by an insulator with a permittivity  $\epsilon$  has a capacitance

$$C = \frac{\epsilon A}{d}$$

$\epsilon$  is often given as  $\epsilon = \epsilon_r (\epsilon_0)$  where  $\epsilon_r$  is the relative permittivity or dielectric constant and  $\epsilon_0 = 8.85 \times 10^{-12}$  F/m.

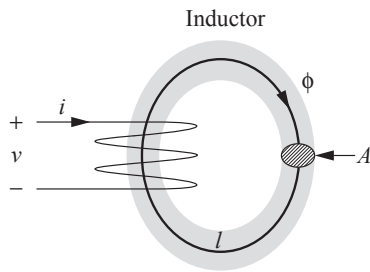
The current-voltage relationships for a capacitor are

$$v_C(t) = v_C(0) + \frac{1}{C} \int_0^t i_C(\tau) d\tau$$

and  $i_C(t) = C (dv_C/dt)$

The energy stored in a capacitor is expressed in joules and given by

$$\text{Energy} = C v_C^2 / 2 = q_C^2 / 2C = q_C v_C / 2$$



The inductance  $L$  (henrys) of a coil of  $N$  turns wound on a core with cross-sectional area  $A$  ( $\text{m}^2$ ), permeability  $\mu$  and flux  $\phi$  with a mean path of  $l$  (m) is given as:

$$L = N^2 \mu A / l$$

$$N\phi = Li$$

and using Faraday's law, the voltage-current relations for an inductor are

$$v_L(t) = L (di_L/dt)$$

$$i_L(t) = i_L(0) + \frac{1}{L} \int_0^t v_L(\tau) d\tau, \text{ where}$$

$v_L$  = inductor voltage

$L$  = inductance (henrys)

$i_L$  = inductor current (amperes)

The energy stored in an inductor is expressed in joules and given by

$$\text{Energy} = Li_L^2/2$$

### Capacitors and Inductors in Parallel and Series

Capacitors in Parallel

$$C_P = C_1 + C_2 + \dots + C_n$$

Capacitors in Series

$$C_S = \frac{1}{1/C_1 + 1/C_2 + \dots + 1/C_n}$$

Inductors in Parallel

$$L_P = \frac{1}{1/L_1 + 1/L_2 + \dots + 1/L_n}$$

Inductors in Series

$$L_S = L_1 + L_2 + \dots + L_n$$

### AC CIRCUITS

For a sinusoidal voltage or current of frequency  $f$  (Hz) and period  $T$  (seconds),

$$f = 1/T = \omega/(2\pi), \text{ where}$$

$\omega$  = the angular frequency in radians/s

#### Average Value

For a periodic waveform (either voltage or current) with period  $T$ ,

$$X_{\text{ave}} = (1/T) \int_0^T x(t) dt$$

The average value of a full-wave rectified sinusoid is

$$X_{\text{ave}} = (2X_{\text{max}})/\pi$$

and half this for half-wave rectification, where

$X_{\text{max}}$  = the peak amplitude of the sinusoid.

#### Effective or RMS Values

For a periodic waveform with period  $T$ , the rms or effective value is

$$X_{\text{eff}} = X_{\text{rms}} = \left[ (1/T) \int_0^T x^2(t) dt \right]^{1/2}$$

For a sinusoidal waveform and full-wave rectified sine wave,

$$X_{\text{eff}} = X_{\text{rms}} = X_{\text{max}}/\sqrt{2}$$

For a half-wave rectified sine wave,

$$X_{\text{eff}} = X_{\text{rms}} = X_{\text{max}}/2$$

For a periodic signal,

$$X_{\text{rms}} = \sqrt{X_{\text{dc}}^2 + \sum_{n=1}^{\infty} X_n^2} \text{ where}$$

$X_{\text{dc}}$  is the dc component of  $x(t)$

$X_n$  is the rms value of the  $n$ th harmonic

#### Sine-Cosine Relations and Trigonometric Identities

$$\cos(\omega t) = \sin(\omega t + \pi/2) = -\sin(\omega t - \pi/2)$$

$$\sin(\omega t) = \cos(\omega t - \pi/2) = -\cos(\omega t + \pi/2)$$

Other trigonometric identities for sinusoids are given in the section on Trigonometry.

#### Phasor Transforms of Sinusoids

$$P[V_{\text{max}} \cos(\omega t + \phi)] = V_{\text{rms}} \angle \phi = \mathbf{V}$$

$$P[I_{\text{max}} \cos(\omega t + \theta)] = I_{\text{rms}} \angle \theta = \mathbf{I}$$

For a circuit element, the impedance is defined as the ratio of phasor voltage to phasor current.

$$\mathbf{Z} = \mathbf{V}/\mathbf{I}$$

For a resistor,  $\mathbf{Z}_R = R$

For a capacitor,  $\mathbf{Z}_C = \frac{1}{j\omega C} = jX_C$

For an inductor,

$$\mathbf{Z}_L = j\omega L = jX_L, \text{ where}$$

$X_C$  and  $X_L$  are the capacitive and inductive reactances respectively defined as

$$X_C = -\frac{1}{\omega C} \text{ and } X_L = \omega L$$

Impedances in series combine additively while those in parallel combine according to the reciprocal rule just as in the case of resistors.

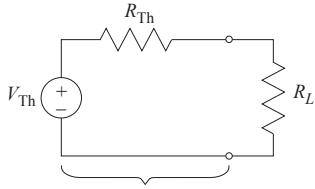
## Maximum Power-Transfer Theorem

### DC Circuits

Maximum power transfer to the load  $R_L$  occurs when  $R_L = R_{Th}$ .

$$P_{\max} = \frac{V_{Th}^2}{4 R_{Th}}$$

$$\text{Efficiency: } \eta = \frac{P_L}{P_S} = \frac{R_L}{R_L + R_{Th}}$$

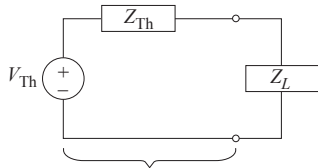


Thevenin Equivalent Circuit

### AC Circuits

In an ac circuit maximum power transfer to the load impedance  $Z_L$  occurs when the load impedance equals the complex conjugate of the Thevenin equivalent impedance:

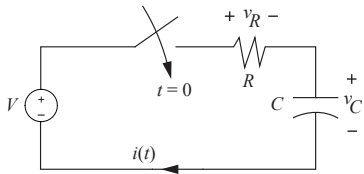
$$Z_L = Z_{Th}^*$$



Thevenin Equivalent Circuit

\*If the load is purely resistive ( $R_L$ ) then for maximum power transfer  $R_L = |Z_{Th}|$

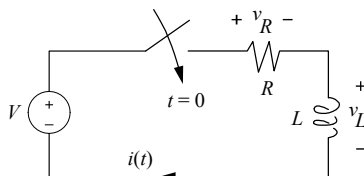
## RC AND RL TRANSIENTS



$$t \geq 0; v_C(t) = v_C(0)e^{-t/RC} + V(1 - e^{-t/RC})$$

$$i(t) = \left\{ [V - v_C(0)]/R \right\} e^{-t/RC}$$

$$v_R(t) = i(t)R = [V - v_C(0)]e^{-t/RC}$$



$$t \geq 0; i(t) = i(0)e^{-Rt/L} + \frac{V}{R}(1 - e^{-Rt/L})$$

$$v_R(t) = i(t)R = i(0)Re^{-Rt/L} + V(1 - e^{-Rt/L})$$

$$v_L(t) = L(di/dt) = -i(0)Re^{-Rt/L} + Ve^{-Rt/L}$$

where  $v(0)$  and  $i(0)$  denote the initial conditions and the parameters  $RC$  and  $L/R$  are termed the respective circuit time constants.

## RESONANCE

The radian resonant frequency for both parallel and series resonance situations is

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2\pi f_0 \text{ rad/s}$$

### Series Resonance

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$Z = R \text{ at resonance}$$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$BW = \frac{\omega_0}{Q} \text{ rad/s}$$

### Parallel Resonance

$$\omega_0 L = \frac{1}{\omega_0 C}$$

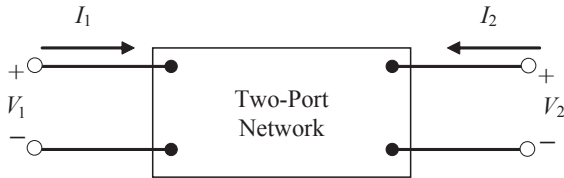
$$Z = R \text{ at resonance}$$

$$Q = \omega_0 RC = \frac{R}{\omega_0 L}$$

$$BW = \frac{\omega_0}{Q} \text{ rad/s}$$

## TWO-PORT PARAMETERS

A two-port network consists of two input and two output terminals as shown.



A two-port network may be represented by an equivalent circuit using a set of two-port parameters. Three commonly used sets of parameters are impedance, admittance, and hybrid parameters. The following table describes the equations used for each of these sets of parameters.

Parameter Type	Equations	Definitions
Impedance ( $z$ )	$V_1 = z_{11}I_1 + z_{12}I_2$ $V_2 = z_{21}I_1 + z_{22}I_2$	$z_{11} = \frac{V_1}{I_1} \Big _{I_2=0}$ $z_{12} = \frac{V_1}{I_2} \Big _{I_1=0}$ $z_{21} = \frac{V_2}{I_1} \Big _{I_2=0}$ $z_{22} = \frac{V_2}{I_2} \Big _{I_1=0}$
Admittance ( $y$ )	$I_1 = y_{11}V_1 + y_{12}V_2$ $I_2 = y_{21}V_1 + y_{22}V_2$	$y_{11} = \frac{I_1}{V_1} \Big _{V_2=0}$ $y_{12} = \frac{I_1}{V_2} \Big _{V_1=0}$ $y_{21} = \frac{I_2}{V_1} \Big _{V_2=0}$ $y_{22} = \frac{I_2}{V_2} \Big _{V_1=0}$
Hybrid ( $h$ )	$V_1 = h_{11}I_1 + h_{12}V_2$ $I_2 = h_{21}I_1 + h_{22}V_2$	$h_{11} = \frac{V_1}{I_1} \Big _{V_2=0}$ $h_{12} = \frac{V_1}{V_2} \Big _{I_1=0}$ $h_{21} = \frac{I_2}{I_1} \Big _{V_2=0}$ $h_{22} = \frac{I_2}{V_2} \Big _{I_1=0}$

## AC POWER

### Complex Power

Real power  $P$  (watts) is defined by

$$P = (\frac{1}{2})V_{\max} I_{\max} \cos \theta$$

$$= V_{\text{rms}} I_{\text{rms}} \cos \theta$$

where  $\theta$  is the angle measured from **V** to **I**. If **I** leads (lags) **V**, then the power factor ( $pf$ ),

$$pf = \cos \theta$$

is said to be a leading (lagging)  $pf$ .

Reactive power  $Q$  (vars) is defined by

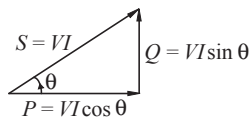
$$Q = (\frac{1}{2})V_{\max} I_{\max} \sin \theta$$

$$= V_{\text{rms}} I_{\text{rms}} \sin \theta$$

Complex power **S** (volt-amperes) is defined by

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = P + jQ,$$

where  $\mathbf{I}^*$  is the complex conjugate of the phasor current.



Complex Power Triangle (Inductive Load)

For resistors,  $\theta = 0$ , so the real power is

$$P = V_{\text{rms}} I_{\text{rms}} = V_{\text{rms}}^2 / R = I_{\text{rms}}^2 R$$

### Balanced Three-Phase (3- $\phi$ ) Systems

The 3- $\phi$  line-phase relations are

for a delta

$$V_L = V_P$$

$$I_L = \sqrt{3} I_P$$

for a wye

$$V_L = \sqrt{3} V_P = \sqrt{3} V_{LN}$$

$$I_L = I_P$$

where subscripts  $L$  and  $P$  denote line and phase respectively.

A balanced 3- $\phi$ , delta-connected load impedance can be converted to an equivalent wye-connected load impedance using the following relationship

$$\mathbf{Z}_{\Delta} = 3\mathbf{Z}_Y$$

The following formulas can be used to determine 3- $\phi$  power for balanced systems.

$$\mathbf{S} = P + jQ$$

$$|\mathbf{S}| = 3V_P I_P = \sqrt{3} V_L I_L$$

$$\mathbf{S} = 3V_P \mathbf{I}_P^* = \sqrt{3} V_L I_L (\cos \theta_P + j \sin \theta_P)$$

For balanced 3- $\phi$ , wye- and delta-connected loads

$$\mathbf{S} = \frac{V_L^2}{Z_Y^*} \quad \mathbf{S} = 3 \frac{V_L^2}{Z_{\Delta}^*}$$

where

$S$  = total 3- $\phi$  complex power (VA)

$|S|$  = total 3- $\phi$  apparent power (VA)

$P$  = total 3- $\phi$  real power (W)

$Q$  = total 3- $\phi$  reactive power (var)

$\theta_P$  = power factor angle of each phase

$V_L$  = rms value of the line-to-line voltage

$V_{LN}$  = rms value of the line-to-neutral voltage

$I_L$  = rms value of the line current

$I_P$  = rms value of the phase current

For a 3- $\phi$ , wye-connected source or load with line-to-neutral voltages and a positive phase sequence

$$V_{an} = V_P \angle 0^\circ$$

$$V_{bn} = V_P \angle -120^\circ$$

$$V_{cn} = V_P \angle 120^\circ$$

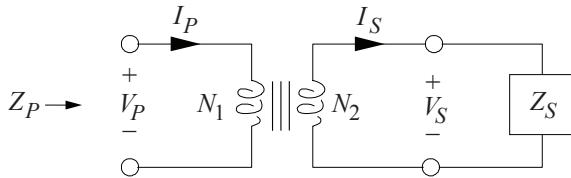
The corresponding line-to-line voltages are

$$V_{ab} = \sqrt{3} V_P \angle 30^\circ$$

$$V_{bc} = \sqrt{3} V_P \angle -90^\circ$$

$$V_{ca} = \sqrt{3} V_P \angle 150^\circ$$

### Transformers (Ideal)



### Turns Ratio

$$a = N_1/N_2$$

$$a = \left| \frac{V_P}{V_S} \right| = \left| \frac{I_S}{I_P} \right|$$

The impedance seen at the input is

$$Z_P = a^2 Z_S$$

### AC Machines

The synchronous speed  $n_s$  for ac motors is given by

$$n_s = 120/fp, \text{ where}$$

$f$  = the line voltage frequency (Hz)

$p$  = the number of poles

The slip for an induction motor is

$$\text{slip} = (n_s - n)/n_s, \text{ where}$$

$n$  = the rotational speed (rpm)

### DC Machines

The armature circuit of a dc machine is approximated by a series connection of the armature resistance  $R_a$ , the armature inductance  $L_a$ , and a dependent voltage source of value

$$V_a = K_a n \phi \text{ volts, where}$$

$K_a$  = constant depending on the design

$n$  = armature speed (rpm)

$\phi$  = the magnetic flux generated by the field

The field circuit is approximated by the field resistance  $R_f$  in series with the field inductance  $L_f$ . Neglecting saturation, the magnetic flux generated by the field current  $I_f$  is

$$\phi = K_f I_f \text{ webers}$$

The mechanical power generated by the armature is

$$P_m = V_a I_a \text{ watts}$$

where  $I_a$  is the armature current. The mechanical torque produced is

$$T_m = (60/2\pi) K_a \phi I_a \text{ newton-meters.}$$

### Servomotors and Generators

Servomotors are electrical motors tied to a feedback system to obtain precise control. Smaller servomotors typically are dc motors.

A permanent magnet dc generator can be used to convert mechanical energy to electrical energy, as in a tachometer.

DC motor suppliers may provide data sheets with speed torque curves, motor torque constants ( $K_T$ ), and motor voltage constants ( $K_E$ ). An idealized dc motor at steady state exhibits the following relationships:

$$V = I R + K_E \omega$$

$$T = K_T I$$

where

$V$  = voltage at the motor terminals

$I$  = current through the motor

$T$  = torque applied by the motor

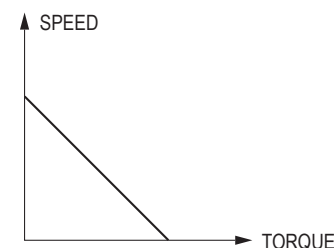
$R$  = resistance of the windings

$\omega$  = rotational speed

When using consistent SI units [ $\text{N}\cdot\text{m}/\text{A}$  and  $\text{V}/(\text{rad}/\text{s})$ ],

$$K_T = K_E.$$

An ideal speed-torque curve for a servomotor, with constant  $V$ , would look like this:



## ELECTROMAGNETIC DYNAMIC FIELDS

The integral and point form of Maxwell's equations are

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\iint_S (\partial \mathbf{B} / \partial t) \cdot d\mathbf{S}$$

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} + \iint_S (\partial \mathbf{D} / \partial t) \cdot d\mathbf{S}$$

$$\oiint_{S_V} \mathbf{D} \cdot d\mathbf{S} = \iiint_V \rho dv$$

$$\oiint_{S_V} \mathbf{B} \cdot d\mathbf{S} = 0$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

## LOSSLESS TRANSMISSION LINES

The wavelength,  $\lambda$ , of a sinusoidal signal is defined as the distance the signal will travel in one period.

$$\lambda = \frac{U}{f}$$

where  $U$  is the velocity of propagation and  $f$  is the frequency of the sinusoid.

The characteristic impedance,  $\mathbf{Z}_0$ , of a transmission line is the input impedance of an infinite length of the line and is given by

$$\mathbf{Z}_0 = \sqrt{L/C}$$

where  $L$  and  $C$  are the per unit length inductance and capacitance of the line.

The reflection coefficient at the load is defined as

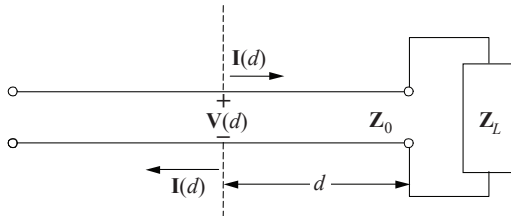
$$\Gamma = \frac{\mathbf{Z}_L - \mathbf{Z}_0}{\mathbf{Z}_L + \mathbf{Z}_0}$$

and the standing wave ratio SWR is

$$\text{SWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$\beta = \text{Propagation constant} = \frac{2\pi}{\lambda}$$

For sinusoidal voltages and currents:



Voltage across the transmission line:

$$\mathbf{V}(d) = \mathbf{V}^+ e^{j\beta d} + \mathbf{V}^- e^{-j\beta d}$$

Current along the transmission line:

$$\mathbf{I}(d) = \mathbf{I}^+ e^{j\beta d} + \mathbf{I}^- e^{-j\beta d}$$

where  $\mathbf{I}^+ = \mathbf{V}^+ / \mathbf{Z}_0$  and  $\mathbf{I}^- = -\mathbf{V}^- / \mathbf{Z}_0$

Input impedance at  $d$

$$\mathbf{Z}_{\text{in}}(d) = \mathbf{Z}_0 \frac{\mathbf{Z}_L + j\mathbf{Z}_0 \tan(\beta d)}{\mathbf{Z}_0 + j\mathbf{Z}_L \tan(\beta d)}$$

## DIFFERENCE EQUATIONS

Difference equations are used to model discrete systems. Systems which can be described by difference equations include computer program variables iteratively evaluated in a loop, sequential circuits, cash flows, recursive processes, systems with time-delay components, etc. Any system whose input  $x(t)$  and output  $y(t)$  are defined only at the equally spaced intervals  $t = kT$  can be described by a difference equation.

### First-Order Linear Difference Equation

A first-order difference equation is

$$y[k] + a_1 y[k-1] = x[k]$$

### Second-Order Linear Difference Equation

A second-order difference equation is

$$y[k] + a_1 y[k-1] + a_2 y[k-2] = x[k]$$

### z-Transforms

The transform definition is

$$F(z) = \sum_{k=0}^{\infty} f[k] z^{-k}$$

The inverse transform is given by the contour integral

$$f(k) = \frac{1}{2\pi j} \oint_{\Gamma} F(z) z^{k-1} dz$$

and it represents a powerful tool for solving linear shift-invariant difference equations. A limited unilateral list of z-transform pairs follows:

$f[k]$	$F(z)$
$\delta[k]$ , Impulse at $k = 0$	1
$u[k]$ , Step at $k = 0$	$1/(1 - z^{-1})$
$\beta^k$	$1/(1 - \beta z^{-1})$
$y[k-1]$	$z^{-1} Y(z) + y(-1)$
$y[k-2]$	$z^{-2} Y(z) + y(-2) + y(-1)z^{-1}$
$y[k+1]$	$z Y(z) - zy(0)$
$y[k+2]$	$z^2 Y(z) - z^2 y(0) - zy(1)$
$\sum_{m=0}^{\infty} X[k-m] h[m]$	$H(z)X(z)$
$\lim_{k \rightarrow 0} f[k]$	$\lim_{z \rightarrow \infty} F(z)$
$\lim_{k \rightarrow \infty} f[k]$	$\lim_{z \rightarrow 1} (1 - z^{-1}) F(z)$

[Note: The last two transform pairs represent the Initial Value Theorem (I.V.T.) and the Final Value Theorem (F.V.T.) respectively.]

## CONVOLUTION

Continuous-time convolution:

$$v(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau$$

Discrete-time convolution:

$$v[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n - k]$$

## DIGITAL SIGNAL PROCESSING

A discrete-time, linear, time-invariant (DTLTI) system with a single input  $x[n]$  and a single output  $y[n]$  can be described by a linear difference equation with constant coefficients of the form

$$y[n] + \sum_{i=1}^k b_i y[n - i] = \sum_{i=0}^l a_i x[n - i]$$

If all initial conditions are zero, taking a  $z$ -transform yields a transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^l a_i z^{-i}}{1 + \sum_{i=1}^k b_i z^{-i}}$$

Two common discrete inputs are the unit-step function  $u[n]$  and the unit impulse function  $\delta[n]$ , where

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases} \text{ and } \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

The impulse response  $h[n]$  is the response of a discrete-time system to  $x[n] = \delta[n]$ .

A finite impulse response (FIR) filter is one in which the impulse response  $h[n]$  is limited to a finite number of points:

$$h[n] = \sum_{i=0}^k a_i \delta[n - i]$$

The corresponding transfer function is given by

$$H(z) = \sum_{i=0}^k a_i z^{-i}$$

where  $k$  is the order of the filter.

An infinite impulse response (IIR) filter is one in which the impulse response  $h[n]$  has an infinite number of points:

$$h[n] = \sum_{i=0}^{\infty} a_i \delta[n - i]$$

## COMMUNICATION THEORY AND CONCEPTS

The following concepts and definitions are useful for communications systems analysis.

### Functions

Unit step, $u(t)$	$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$
Rectangular pulse, $\Pi(t/\tau)$	$\Pi(t/\tau) = \begin{cases} 1 &  t/\tau  < \frac{1}{2} \\ 0 &  t/\tau  > \frac{1}{2} \end{cases}$
Triangular pulse, $\Lambda(t/\tau)$	$\Lambda(t/\tau) = \begin{cases} 1 -  t/\tau  &  t/\tau  < 1 \\ 0 &  t/\tau  > 1 \end{cases}$
Sinc, $\text{sinc}(at)$	$\text{sinc}(at) = \frac{\sin(a\pi t)}{a\pi t}$
Unit impulse, $\delta(t)$	$\int_{-\infty}^{+\infty} x(t + t_0)\delta(t)dt = x(t_0)$ for every $x(t)$ defined and continuous at $t = t_0$ . This is equivalent to $\int_{-\infty}^{+\infty} x(t)\delta(t - t_0)dt = x(t_0)$

$$\begin{aligned} x(t) * h(t) &= \int_{-\infty}^{+\infty} x(\lambda)h(t - \lambda)d\lambda \\ &= h(t) * x(t) = \int_{-\infty}^{+\infty} h(\lambda)x(t - \lambda)d\lambda \end{aligned}$$

In particular,

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

### The Fourier Transform and its Inverse

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df$$

$x(t)$  and  $X(f)$  form a *Fourier transform pair*:

$$x(t) \leftrightarrow X(f)$$



## Frequency Response and Impulse Response

The *frequency response*  $H(f)$  of a system with input  $x(t)$  and output  $y(t)$  is given by

$$H(f) = \frac{Y(f)}{X(f)}$$

This gives

$$Y(f) = H(f)X(f)$$

The response  $h(t)$  of a linear time-invariant system to a unit-impulse input  $\delta(t)$  is called the *impulse response* of the system. The response  $y(t)$  of the system to any input  $x(t)$  is the convolution of the input  $x(t)$  with the impulse response  $h(t)$ :

$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda) h(t - \lambda) d\lambda \\ &= h(t) * x(t) = \int_{-\infty}^{+\infty} h(\lambda) x(t - \lambda) d\lambda \end{aligned}$$

Therefore, the impulse response  $h(t)$  and frequency response  $H(f)$  form a Fourier transform pair:

$$h(t) \leftrightarrow H(f)$$

## Parseval's Theorem

The total energy in an energy signal (finite energy)  $x(t)$  is given by

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

## Parseval's Theorem for Fourier Series

A periodic signal  $x(t)$  with period  $T_0$  and fundamental frequency  $f_0 = 1/T_0 = \omega_0/2\pi$  can be represented by a complex-exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn2\pi f_0 t}$$

The average power in the dc component and the first  $N$  harmonics is

$$P = \sum_{n=-N}^{n=+N} |X_n|^2 = X_0^2 + 2 \sum_{n=1}^N |X_n|^2$$

The total average power in the periodic signal  $x(t)$  is given by *Parseval's theorem*:

$$P = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{n=+\infty} |X_n|^2$$

## Decibels and Bode Plots

Decibels is a technique to measure the ratio of two powers:

$$\text{dB} = 10 \log_{10} (P_2/P_1)$$

The definition can be modified to measure the ratio of two voltages:

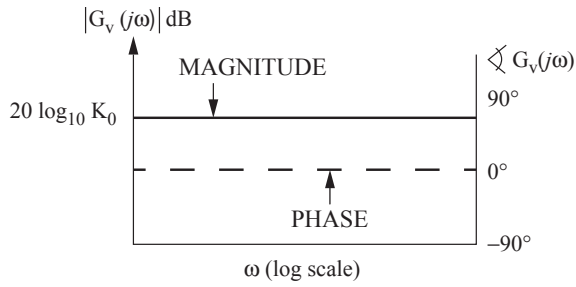
$$\text{dB} = 20 \log_{10} V_2/V_1$$

Bode plots use a logarithmic scale for the frequency when plotting magnitude and phase response, where the magnitude is plotted in dB using a straight-line (asymptotic) approximation.

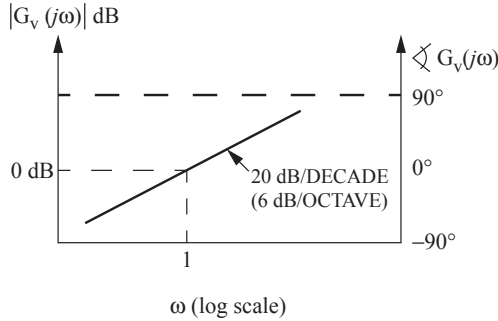
The information below summarizes Bode plots for several terms commonly encountered when determining voltage gain,  $G_v(j\omega)$ . Since logarithms are used to convert gain to decibels, the decibel response when these various terms are multiplied together can be added to determine the overall response.

Term	Magnitude Response $ G_v(j\omega) _{\text{dB}}$	Phase Response $\angle G_v(j\omega)$	Plot
$K_0$	$20 \log_{10}(K_0)$	$0^\circ$	a
$(j\omega)^{\pm 1}$	$\pm 20 \log_{10}(\omega)$	$\pm 90^\circ$	b & c
$(1 + j\omega/\omega_c)^{\pm 1}$	0 for $\omega \ll \omega_c$ $\pm 3 \text{ dB}$ for $\omega = \omega_c$ $\pm 20 \log_{10}(\omega)$ for $\omega \gg \omega_c$	$0^\circ$ for $\omega \ll \omega_c$ $\pm 45^\circ$ for $\omega = \omega_c$ $\pm 90^\circ$ for $\omega \gg \omega_c$	d & e

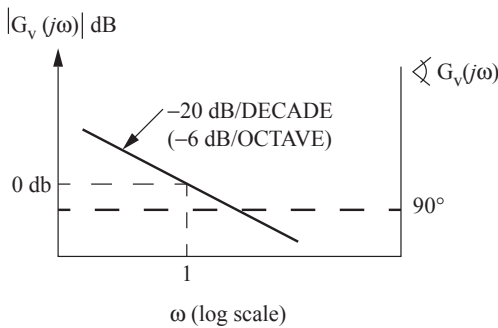
(a)  $K_0$



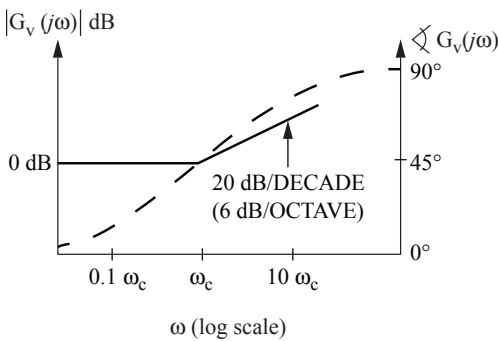
(b)  $j\omega$



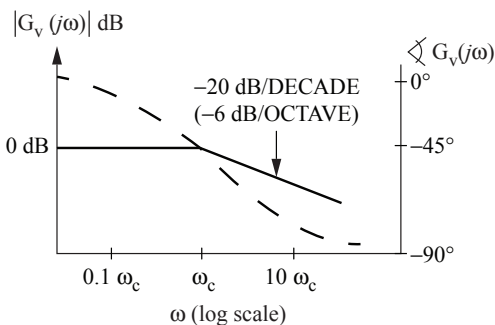
(c)  $(j\omega)^{-1}$



(d)  $(1 + j\omega/\omega_c)$



(e)  $(1 + j\omega/\omega_c)^{-1}$



## Amplitude Modulation (AM)

$$x_{AM}(t) = A_c [A + m(t)] \cos(2\pi f_c t)$$

$$= A'_c [1 + a m_n(t)] \cos(2\pi f_c t)$$

The *modulation index* is  $a$ , and the normalized message is

$$m_n(t) = \frac{m(t)}{\max|m(t)|}$$

The *efficiency*  $\eta$  is the percent of the total transmitted power that contains the message.

$$\eta = \frac{a^2 \langle m_n^2(t) \rangle}{1 + a^2 \langle m_n^2(t) \rangle} 100 \text{ percent}$$

where the mean-squared value or normalized average power in  $m_n(t)$  is

$$\langle m_n^2(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |m_n(t)|^2 dt$$

If  $M(f) = 0$  for  $|f| > W$ , then the *bandwidth* of  $x_{AM}(t)$  is  $2W$ . AM signals can be demodulated with an envelope detector or a synchronous demodulator.

## Double-Sideband Modulation (DSB)

$$x_{DSB}(t) = A_c m(t) \cos(2\pi f_c t)$$

If  $M(f) = 0$  for  $|f| > W$ , then the bandwidth of  $m(t)$  is  $W$  and the bandwidth of  $x_{DSB}(t)$  is  $2W$ . DSB signals must be demodulated with a synchronous demodulator. A Costas loop is often used.

## Single-Sideband Modulation (SSB)

Lower Sideband:

$$x_{LSB}(t) \leftrightarrow X_{LSB}(f) = X_{DSB}(f) \Pi\left(\frac{f}{2f_c}\right)$$

Upper Sideband:

$$x_{USB}(t) \leftrightarrow X_{USB}(f) = X_{DSB}(f) \left[1 - \Pi\left(\frac{f}{2f_c}\right)\right]$$

In either case, if  $M(f) = 0$  for  $|f| > W$ , then the bandwidth of  $x_{LSB}(t)$  or of  $x_{USB}(t)$  is  $W$ . SSB signals can be demodulated with a synchronous demodulator or by carrier reinsertion and envelope detection.

## Angle Modulation

$$x_{Ang}(t) = A_c \cos[2\pi f_c t + \phi(t)]$$

The *phase deviation*  $\phi(t)$  is a function of the message  $m(t)$ .

The *instantaneous phase* is

$$\phi_i(t) = 2\pi f_c t + \phi(t) \text{ rad}$$

The *instantaneous frequency* is

$$\omega_i(t) = \frac{d}{dt} \phi_i(t) = 2\pi f_c + \frac{d}{dt} \phi(t) \text{ rad/s}$$

The *frequency deviation* is

$$\Delta\omega(t) = \frac{d}{dt} \phi(t) \text{ rad/s}$$

### Phase Modulation (PM)

The *phase deviation* is

$$\phi(t) = k_{pm}(t) \text{ rad}$$

### Frequency Modulation (FM)

The *phase deviation* is

$$\phi(t) = k_F \int_{-\infty}^t m(\lambda) d\lambda \text{ rad}$$

The *frequency-deviation ratio* is

$$D = \frac{k_F \max |m(t)|}{2\pi W}$$

where  $W$  is the message bandwidth. If  $D \ll 1$  (narrowband FM), the 98% power bandwidth  $B$  is

$$B \cong 2W$$

If  $D > 1$ , (wideband FM) the 98% power bandwidth  $B$  is given by *Carson's rule*:

$$B \cong 2(D + 1)W$$

The *complete* bandwidth of an angle-modulated signal is infinite.

A discriminator or a phase-lock loop can demodulate angle-modulated signals.

### Sampled Messages

A low-pass message  $m(t)$  can be exactly reconstructed from uniformly spaced samples taken at a sampling frequency of  $f_s = 1/T_s$

$$f_s > 2W \text{ where } M(f) = 0 \text{ for } f > W$$

The frequency  $2W$  is called the *Nyquist frequency*. Sampled messages are typically transmitted by some form of pulse modulation. The minimum bandwidth  $B$  required for transmission of the modulated message is inversely proportional to the pulse length  $\tau$ .

$$B \propto \frac{1}{\tau}$$

Frequently, for approximate analysis

$$B \cong \frac{1}{2\tau}$$

is used as the *minimum* bandwidth of a pulse of length  $\tau$ .

### Ideal-Impulse Sampling

$$\begin{aligned} x_\delta(t) &= m(t) \sum_{n=-\infty}^{n=+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{n=+\infty} m(nT_s) \delta(t - nT_s) \\ X_\delta(f) &= M(f) * \left[ f_s \sum_{k=-\infty}^{k=+\infty} \delta(f - kf_s) \right] \\ &= f_s \sum_{k=-\infty}^{k=+\infty} M(f - kf_s) \end{aligned}$$

The message  $m(t)$  can be recovered from  $x_\delta(t)$  with an ideal low-pass filter of bandwidth  $W$  if  $f_s > 2W$ .

### (PAM) Pulse-Amplitude Modulation—Natural Sampling

A PAM signal can be generated by multiplying a message by a pulse train with pulses having duration  $\tau$  and period

$$T_s = 1/f_s$$

$$x_N(t) = m(t) \sum_{n=-\infty}^{n=+\infty} \Pi\left[\frac{t - nT_s}{\tau}\right] = \sum_{n=-\infty}^{n=+\infty} m(t) \Pi\left[\frac{t - nT_s}{\tau}\right]$$

$$X_N(f) = \tau f_s \sum_{k=-\infty}^{k=+\infty} \text{sinc}(k\tau f_s) M(f - kf_s)$$

The message  $m(t)$  can be recovered from  $x_N(t)$  with an ideal low-pass filter of bandwidth  $W$ .

### Pulse-Code Modulation (PCM)

PCM is formed by sampling a message  $m(t)$  and digitizing the sample values with an A/D converter. For an  $n$ -bit binary word length, transmission of a pulse-code-modulated low-pass message  $m(t)$ , with  $M(f) = 0$  for  $f \geq W$ , requires the transmission of at least  $2nW$  binary pulses per second. A binary word of length  $n$  bits can represent  $q$  quantization levels:

$$q = 2^n$$

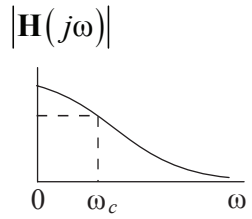
The minimum bandwidth required to transmit the PCM message will be

$$B \propto 2nW = 2W \log_2 q$$

## ANALOG FILTER CIRCUITS

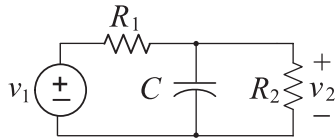
Analog filters are used to separate signals with different frequency content. The following circuits represent simple analog filters used in communications and signal processing.

### First-Order Low-Pass Filters



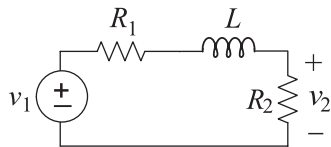
$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |H(0)|$$

**Frequency Response**



$$H(s) = \frac{V_2}{V_1} = \frac{R_P}{R_1} \cdot \frac{1}{1 + sR_PC}$$

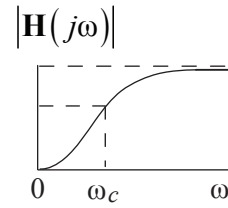
$$R_P = \frac{R_1 R_2}{R_1 + R_2} \quad \omega_c = \frac{1}{R_P C}$$



$$H(s) = \frac{V_2}{V_1} = \frac{R_2}{R_S} \cdot \frac{1}{1 + sL/R_S}$$

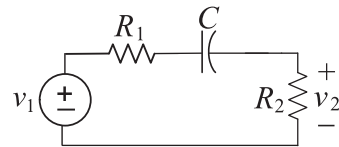
$$R_S = R_1 + R_2 \quad \omega_c = \frac{R_S}{L}$$

### First-Order High-Pass Filters



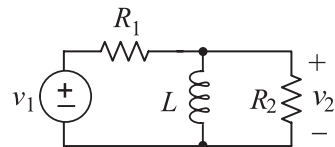
$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |H(j\infty)|$$

**Frequency Response**



$$H(s) = \frac{V_2}{V_1} = \frac{R_2}{R_S} \cdot \frac{sR_SC}{1 + sR_SC}$$

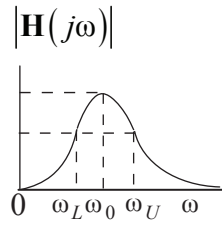
$$R_S = R_1 + R_2 \quad \omega_c = \frac{1}{R_S C}$$



$$H(s) = \frac{V_2}{V_1} = \frac{R_P}{R_1} \cdot \frac{sL/R_P}{1 + sL/R_P}$$

$$R_P = \frac{R_1 R_2}{R_1 + R_2} \quad \omega_c = \frac{R_P}{L}$$

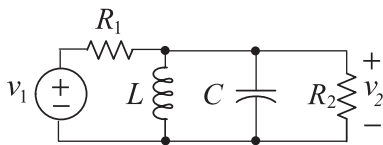
## Band-Pass Filters



$$|H(j\omega_L)| = |H(j\omega_U)| = \frac{1}{\sqrt{2}} |H(j\omega_0)|$$

$$\text{3-dB Bandwidth} = BW = \omega_U - \omega_L$$

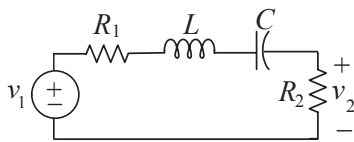
### Frequency Response



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{1}{R_1 C} \cdot \frac{s}{s^2 + s/R_P C + 1/LC}$$

$$R_P = \frac{R_1 R_2}{R_1 + R_2} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$|H(j\omega_0)| = \frac{R_2}{R_1 + R_2} = \frac{R_P}{R_1} \quad BW = \frac{1}{R_P C}$$

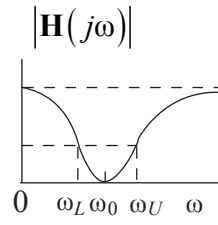


$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2}{L} \cdot \frac{s}{s^2 + sR_S/L + 1/LC}$$

$$R_S = R_1 + R_2 \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$|H(j\omega_0)| = \frac{R_2}{R_1 + R_2} = \frac{R_2}{R_S} \quad BW = \frac{R_S}{L}$$

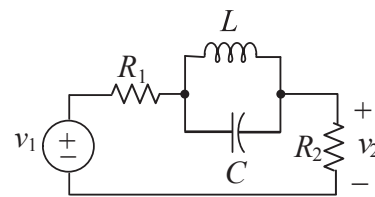
## Band-Reject Filters



$$|H(j\omega_L)| = |H(j\omega_U)| = \left[1 - \frac{1}{\sqrt{2}}\right] |H(0)|$$

$$\text{3-dB Bandwidth} = BW = \omega_U - \omega_L$$

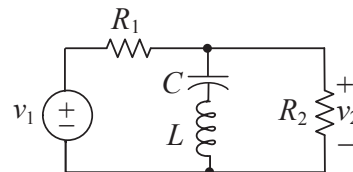
### Frequency Response



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2}{R_S} \cdot \frac{s^2 + 1/LC}{s^2 + s/R_S C + 1/LC}$$

$$R_S = R_1 + R_2 \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$|H(0)| = \frac{R_2}{R_1 + R_2} = \frac{R_2}{R_S} \quad BW = \frac{1}{R_S C}$$



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_P}{R_1} \cdot \frac{s^2 + 1/LC}{s^2 + sR_P/L + 1/LC}$$

$$R_P = \frac{R_1 R_2}{R_1 + R_2} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$|H(0)| = \frac{R_2}{R_1 + R_2} = \frac{R_P}{R_1} \quad BW = \frac{R_P}{L}$$

## OPERATIONAL AMPLIFIERS

### Ideal

$$v_0 = A(v_1 - v_2)$$

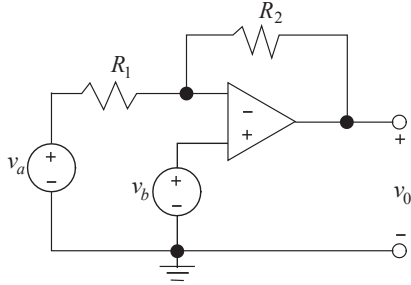
where

$A$  is large ( $> 10^4$ ), and

$v_1 - v_2$  is small enough so as not to saturate the amplifier.

For the ideal operational amplifier, assume that the input currents are zero and that the gain  $A$  is infinite so when operating linearly  $v_2 - v_1 = 0$ .

For the two-source configuration with an ideal operational amplifier,



$$v_0 = -\frac{R_2}{R_1} v_a + \left(1 + \frac{R_2}{R_1}\right) v_b$$

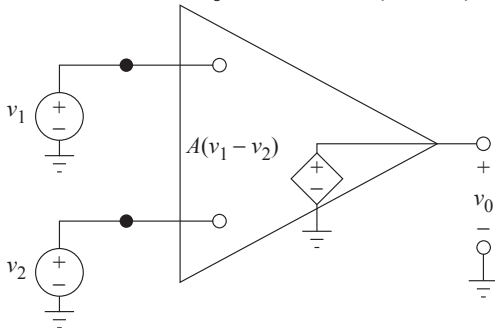
If  $v_a = 0$ , we have a non-inverting amplifier with

$$v_0 = \left(1 + \frac{R_2}{R_1}\right) v_b$$

If  $v_b = 0$ , we have an inverting amplifier with

$$v_0 = -\frac{R_2}{R_1} v_a$$

### Common Mode Rejection Ratio (CMRR)



Equivalent Circuit of an Ideal Op Amp

In the op amp circuit shown the differential input is defined as:

$$v_{id} = v_2 - v_1$$

The common-mode input voltage is defined as:

$$v_{icm} = (v_1 + v_2)/2$$

The output voltage is given by:

$$v_O = A v_{id} + A_{cm} v_{icm}$$

In an ideal op amp  $A_{cm} = 0$ . In a non-ideal op amp the CMRR is used to measure the relative degree of rejection between the differential gain and common-mode gain.

$$CMRR = \frac{|A|}{|A_{cm}|}$$

CMRR is usually expressed in decibels as:

$$CMRR = 20 \log_{10} \left[ \frac{|A|}{|A_{cm}|} \right]$$

## SOLID-STATE ELECTRONICS AND DEVICES

Conductivity of a semiconductor material:

$$\sigma = q (n\mu_n + p\mu_p), \text{ where}$$

$\mu_n$   $\equiv$  electron mobility

$\mu_p$   $\equiv$  hole mobility

$n$   $\equiv$  electron concentration

$p$   $\equiv$  hole concentration

$q$   $\equiv$  charge on an electron ( $1.6 \times 10^{-19} \text{C}$ )

Doped material:

$p$ -type material;  $p_p \approx N_a$

$n$ -type material;  $n_n \approx N_d$

Carrier concentrations at equilibrium

$$(p)(n) = n_i^2, \text{ where}$$

$n_i$   $\equiv$  intrinsic concentration.

Built-in potential (contact potential) of a  $p$ - $n$  junction:

$$V_0 = \frac{kT}{q} \ln \frac{N_a N_d}{n_i^2}$$

Thermal voltage

$$V_T = \frac{kT}{q} \approx 0.026 \text{ V at } 300\text{K}$$

$N_a$  = acceptor concentration

$N_d$  = donor concentration

$T$  = temperature (K)

$k$  = Boltzmann's Constant =  $1.38 \times 10^{-23} \text{ J/K}$

Capacitance of abrupt  $p$ - $n$  junction diode

$$C(V) = C_0 / \sqrt{1 - V/V_{bi}}, \text{ where:}$$

$C_0$  = junction capacitance at  $V = 0$

$V$  = potential of anode with respect to cathode

$V_{bi}$  = junction contact potential

Resistance of a diffused layer is

$$R = R_s (L/W), \text{ where:}$$

$R_s$  = sheet resistance =  $\rho/d$  in ohms per square

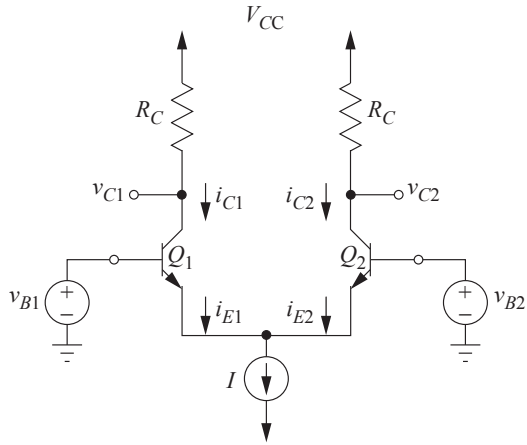
$\rho$  = resistivity

$d$  = thickness

$L$  = length of diffusion

$W$  = width of diffusion

## ◆ Differential Amplifier



A Basic BJT Differential Amplifier

A basic BJT differential amplifier consists of two matched transistors whose emitters are connected and that are biased by a constant-current source. The following equations govern the operation of the circuit given that neither transistor is operating in the saturation region.

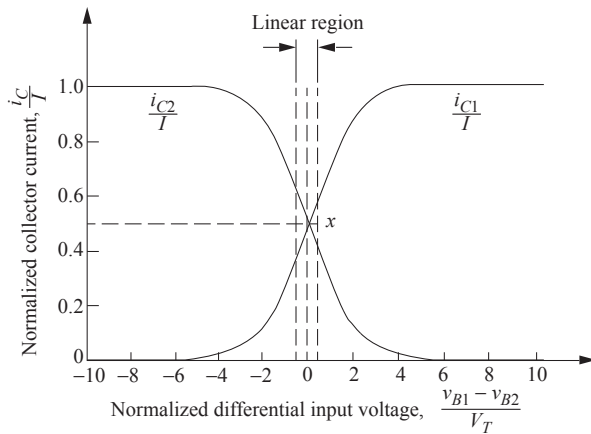
$$\frac{i_{E1}}{i_{E2}} = e^{(v_{B1} - v_{B2})/V_T}$$

$$i_{E1} + i_{E2} = I$$

$$i_{E1} = \frac{I}{1 + e^{(v_{B2} - v_{B1})/V_T}} \quad i_{E2} = \frac{I}{1 + e^{(v_{B1} - v_{B2})/V_T}}$$

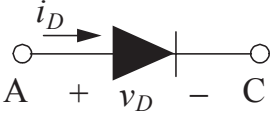
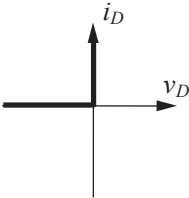
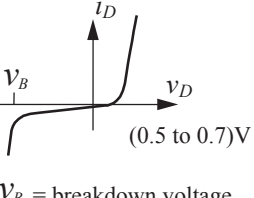
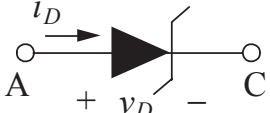
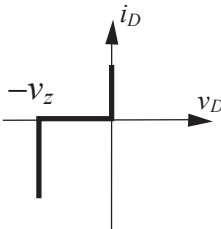
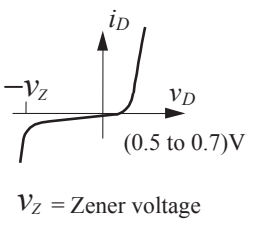
$$i_{C1} = \alpha i_{E1} \quad i_{C2} = \alpha i_{E2}$$

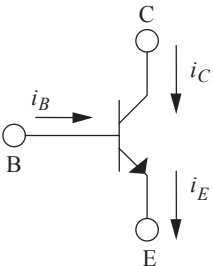
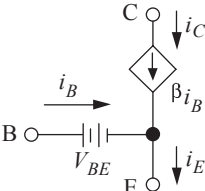
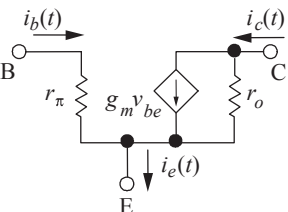
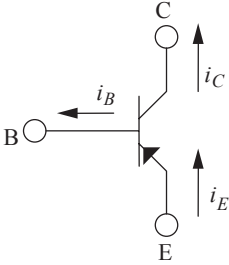
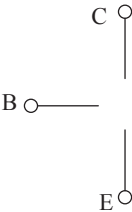
The following figure shows a plot of two normalized collector currents versus normalized differential input voltage for a circuit using transistors with  $\alpha \cong 1$ .



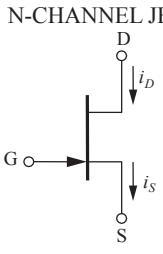
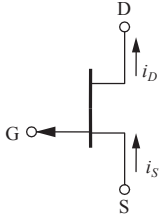
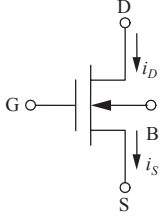
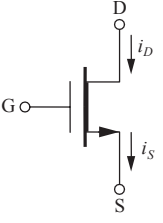
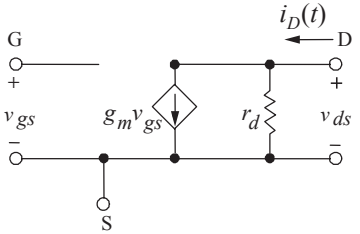
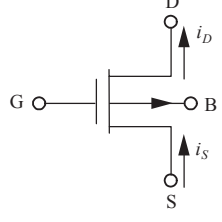
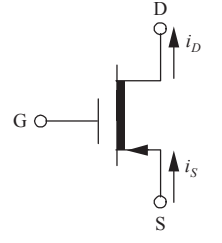
Transfer characteristics of the BJT differential amplifier with  $\alpha \cong 1$

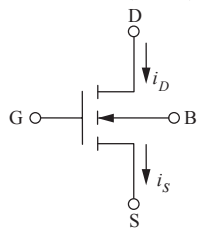
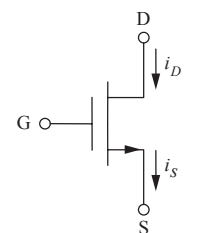
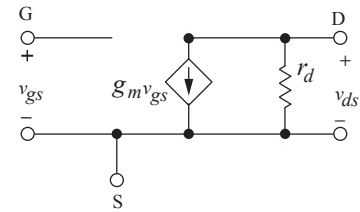
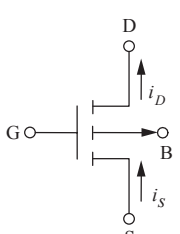
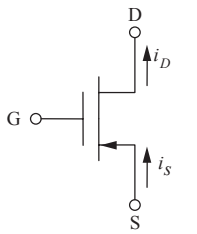
- ◆ Sedra, Adel, and Kenneth Smith, *Microelectronic Circuits*, 3rd ed., ©1991, pp. 408 and 412, by permission of Oxford University Press.

DIODES			
Device and Schematic Symbol	Ideal $I - V$ Relationship	Piecewise-Linear Approximation of the $I - V$ Relationship	Mathematical $I - V$ Relationship
(Junction Diode) 		 $v_B$ = breakdown voltage	Shockley Equation $i_D \approx I_s \left[ e^{(v_D/\eta V_T)} - 1 \right]$ where $I_s$ = saturation current $\eta$ = emission coefficient, typically 1 for Si $V_T$ = thermal voltage = $\frac{kT}{q}$
(Zener Diode) 		 $v_Z$ = Zener voltage	Same as above.

NPN Bipolar Junction Transistor (BJT)			
Schematic Symbol	Mathematical Relationships	Large-Signal (DC) Equivalent Circuit	Low-Frequency Small-Signal (AC) Equivalent Circuit
 NPN – Transistor	$i_E = i_B + i_C$ $i_C = \beta i_B$ $i_C = \alpha i_E$ $\alpha = \beta / (\beta + 1)$ $i_C \approx I_S e^{(v_{BE}/V_T)}$ $I_S = \text{emitter saturation current}$ $V_T = \text{thermal voltage}$ <b>Note:</b> These relationships are valid in the active mode of operation.	<u>Active Region:</u> base emitter junction forward biased; base collector junction reverse biased 	<u>Low Frequency:</u> $g_m \approx I_{CQ}/V_T$ $r_\pi \approx \beta/g_m,$ $r_o = \left[ \frac{\partial v_{CE}}{\partial i_c} \right]_{Q_{point}} \approx \frac{V_A}{I_{CQ}}$ where $I_{CQ}$ = dc collector current at the $Q_{point}$ $V_A$ = Early voltage 
 PNP – Transistor	Same as for NPN with current directions and voltage polarities reversed.	<u>Cutoff Region:</u> both junctions reverse biased  Same as NPN with current directions and voltage polarities reversed	Same as for NPN.



N-Channel Junction Field Effect Transistors (JFETs) and Depletion MOSFETs (Low and Medium Frequency)		
Schematic Symbol	Mathematical Relationships	Small-Signal (AC) Equivalent Circuit
<p>N-CHANNEL JFET</p>  <p>P-CHANNEL JFET</p>  <p>N-CHANNEL DEPLETION MOSFET (NMOS)</p>  <p>SIMPLIFIED SYMBOL</p> 	<p><u>Cutoff Region:</u> <math>v_{GS} &lt; V_p</math>  <math>i_D = 0</math></p> <p><u>Triode Region:</u> <math>v_{GS} &gt; V_p</math> and <math>v_{GD} &gt; V_p</math>  <math>i_D = (I_{DSS}/V_p^2)[2v_{DS}(v_{GS} - V_p) - v_{DS}^2]</math></p> <p><u>Saturation Region:</u> <math>v_{GS} &gt; V_p</math> and <math>v_{GD} &lt; V_p</math>  <math>i_D = I_{DSS}(1 - v_{GS}/V_p)^2</math></p> <p>where  <math>I_{DSS}</math> = drain current with <math>v_{GS} = 0</math>  (in the saturation region)  <math>= KV_p^2</math>,  <math>K</math> = conductivity factor, and  <math>V_p</math> = pinch-off voltage.</p>	<p><math>g_m = \frac{2\sqrt{I_{DSS}I_D}}{ V_p }</math> in saturation region</p>  <p>where</p> $r_d = \left. \frac{\partial v_{ds}}{\partial i_d} \right _{Q_{point}}$
<p>P-CHANNEL DEPLETION MOSFET (PMOS)</p>  <p>SIMPLIFIED SYMBOL</p> 	<p>Same as for N-Channel with current directions and voltage polarities reversed.</p>	<p>Same as for N-Channel.</p>

Enhancement MOSFET (Low and Medium Frequency)		
Schematic Symbol	Mathematical Relationships	Small-Signal (AC) Equivalent Circuit
<p>N-CHANNEL ENHANCEMENT MOSFET (NMOS)</p>  <p>SIMPLIFIED SYMBOL</p> 	<p><u>Cutoff Region:</u> <math>v_{GS} &lt; V_t</math>  <math>i_D = 0</math></p> <p><u>Triode Region:</u> <math>v_{GS} &gt; V_t</math> and <math>v_{GD} &gt; V_t</math>  <math>i_D = K [2v_{DS}(v_{GS} - V_t) - v_{DS}^2]</math></p> <p><u>Saturation Region:</u> <math>v_{GS} &gt; V_t</math> and <math>v_{GD} &lt; V_t</math>  <math>i_D = K (v_{GS} - V_t)^2</math>  where  <math>K =</math> conductivity factor  <math>V_t =</math> threshold voltage</p>	<p><math>g_m = 2K(v_{GS} - V_t)</math> in saturation region</p>  <p>where</p> $r_d = \left. \frac{\partial v_{ds}}{\partial i_d} \right _{Q_{point}}$
<p>P-CHANNEL ENHANCEMENT MOSFET (PMOS)</p>  <p>SIMPLIFIED SYMBOL</p> 	<p>Same as for N-channel with current directions and voltage polarities reversed.</p>	<p>Same as for N-channel.</p>

## NUMBER SYSTEMS AND CODES

An unsigned number of base- $r$  has a decimal equivalent  $D$  defined by

$$D = \sum_{k=0}^n a_k r^k + \sum_{i=1}^m a_i r^{-i}, \text{ where}$$

$a_k$  = the  $(k + 1)$  digit to the left of the radix point and

$a_i$  = the  $i$ th digit to the right of the radix point.

### Binary Number System

In digital computers, the base-2, or binary, number system is normally used. Thus the decimal equivalent,  $D$ , of a binary number is given by

$$D = a_k 2^k + a_{k-1} 2^{k-1} + \dots + a_0 + a_{-1} 2^{-1} + \dots$$

Since this number system is so widely used in the design of digital systems, we use a short-hand notation for some powers of two:

$2^{10} = 1,024$  is abbreviated "k" or "kilo"

$2^{20} = 1,048,576$  is abbreviated "M" or "mega"

Signed numbers of base- $r$  are often represented by the radix complement operation. If  $M$  is an  $N$ -digit value of base- $r$ , the radix complement  $R(M)$  is defined by

$$R(M) = r^N - M$$

The 2's complement of an  $N$ -bit binary integer can be written

$$2\text{'s Complement}(M) = 2^N - M$$

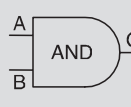
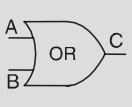
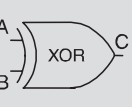
This operation is equivalent to taking the 1's complement (inverting each bit of  $M$ ) and adding one.

The following table contains equivalent codes for a four-bit binary value.

Binary Base-2	Decimal Base-10	Hexa-decimal Base-16	Octal Base-8	BCD Code	Gray Code
0000	0	0	0	0	0000
0001	1	1	1	1	0001
0010	2	2	2	2	0011
0011	3	3	3	3	0010
0100	4	4	4	4	0110
0101	5	5	5	5	0111
0110	6	6	6	6	0101
0111	7	7	7	7	0100
1000	8	8	10	8	1100
1001	9	9	11	9	1101
1010	10	A	12	---	1111
1011	11	B	13	---	1110
1100	12	C	14	---	1010
1101	13	D	15	---	1011
1110	14	E	16	---	1001
1111	15	F	17	---	1000

## LOGIC OPERATIONS AND BOOLEAN ALGEBRA

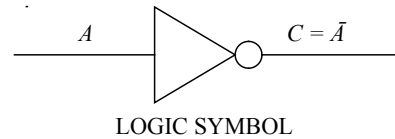
Three basic logic operations are the "AND ( $\bullet$ )," "OR ( $+$ )," and "Exclusive-OR ( $\oplus$ )" functions. The definition of each function, its logic symbol, and its Boolean expression are given in the following table.

Function			
Inputs	$A$ $B$	$A$ $B$	$A$ $B$
$A \ B$	$C = A \bullet B$	$C = A + B$	$C = A \oplus B$
0 0	0	0	0
0 1	0	1	1
1 0	0	1	1
1 1	1	1	0

As commonly used,  $A$  AND  $B$  is often written  $AB$  or  $A \bullet B$ .

The not operator inverts the sense of a binary value ( $0 \rightarrow 1, 1 \rightarrow 0$ )

NOT OPERATOR	
Input	Output
$A$	$C = \bar{A}$
0	1
1	0



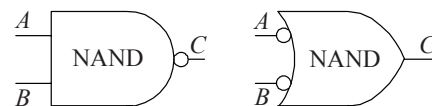
### De Morgan's Theorems

first theorem:  $\overline{A + B} = \bar{A} \bullet \bar{B}$

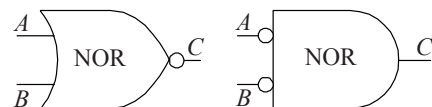
second theorem:  $\overline{A \bullet B} = \bar{A} + \bar{B}$

These theorems define the NAND gate and the NOR gate. Logic symbols for these gates are shown below.

NAND Gates:  $\overline{A \bullet B} = \bar{A} + \bar{B}$

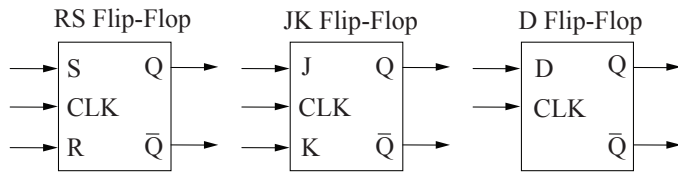


NOR Gates:  $\overline{A + B} = \bar{A} \bullet \bar{B}$



## FLIP-FLOPS

A flip-flop is a device whose output can be placed in one of two states, 0 or 1. The flip-flop output is synchronized with a clock (CLK) signal.  $Q_n$  represents the value of the flip-flop output before CLK is applied, and  $Q_{n+1}$  represents the output after CLK has been applied. Three basic flip-flops are described below.



SR	$Q_{n+1}$	JK	$Q_{n+1}$	D	$Q_{n+1}$
00	$Q_n$ no change	00	$Q_n$ no change	0	0
01	0	01	0	1	1
10	1	10	1		
11	x invalid	11	$\overline{Q_n}$ toggle		

Composite Flip-Flop State Transition						
$Q_n$	$Q_{n+1}$	S	R	J	K	D
0	0	0	x	0	x	0
0	1	1	0	1	x	1
1	0	0	1	x	1	0
1	1	x	0	x	0	1

## Switching Function Terminology

**Minterm**,  $m_i$  – A product term which contains an occurrence of every variable in the function.

**Maxterm**,  $M_i$  – A sum term which contains an occurrence of every variable in the function.

**Implicant** – A Boolean algebra term, either in sum or product form, which contains one or more minterms or maxterms of a function.

**Prime Implicant** – An implicant which is not entirely contained in any other implicant.

**Essential Prime Implicant** – A prime implicant which contains a minterm or maxterm which is not contained in any other prime implicant.

A function can be described as a sum of minterms using the notation

$$F(ABCD) = \sum m(h, i, j, \dots) \\ = m_h + m_i + m_j + \dots$$

A function can be described as a product of maxterms using the notation

$$G(ABCD) = \prod M(h, i, j, \dots) \\ = M_h \bullet M_i \bullet M_j \bullet \dots$$

A function represented as a sum of minterms only is said to be in *canonical sum of products* (SOP) form. A function represented as a product of maxterms only is said to be in *canonical product of sums* (POS) form. A function in canonical SOP form is often represented as a *minterm list*, while a function in canonical POS form is often represented as a *maxterm list*.

A *Karnaugh Map* (K-Map) is a graphical technique used to represent a truth table. Each square in the K-Map represents one minterm, and the squares of the K-Map are arranged so that the adjacent squares differ by a change in exactly one variable. A four-variable K-Map with its corresponding minterms is shown below. K-Maps are used to simplify switching functions by visually identifying all essential prime implicants.

AB \ CD				
	00	01	11	10
00	$m_0$	$m_1$	$m_3$	$m_2$
01	$m_4$	$m_5$	$m_7$	$m_6$
11	$m_{12}$	$m_{13}$	$m_{15}$	$m_{14}$
10	$m_8$	$m_9$	$m_{11}$	$m_{10}$

# INDUSTRIAL ENGINEERING

## LINEAR PROGRAMMING

The general linear programming (LP) problem is:

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

$\vdots$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$x_1, \dots, x_n \geq 0$$

An LP problem is frequently reformulated by inserting non-negative slack and surplus variables. Although these variables usually have zero costs (depending on the application), they can have non-zero cost coefficients in the objective function. A slack variable is used with a "less than" inequality and transforms it into an equality. For example, the inequality  $5x_1 + 3x_2 + 2x_3 \leq 5$  could be changed to  $5x_1 + 3x_2 + 2x_3 + s_1 = 5$  if  $s_1$  were chosen as a slack variable. The inequality  $3x_1 + x_2 - 4x_3 \geq 10$  might be transformed into  $3x_1 + x_2 - 4x_3 - s_2 = 10$  by the addition of the surplus variable  $s_2$ . Computer printouts of the results of processing an LP usually include values for all slack and surplus variables, the dual prices, and the reduced costs for each variable.

### Dual Linear Program

Associated with the above linear programming problem is another problem called the dual linear programming problem. If we take the previous problem and call it the primal problem, then in matrix form the primal and dual problems are respectively:

<u>Primal</u>	<u>Dual</u>
Maximize $Z = \mathbf{c}\mathbf{x}$	Minimize $W = \mathbf{y}\mathbf{b}$
Subject to: $\mathbf{A}\mathbf{x} \leq \mathbf{b}$	Subject to: $\mathbf{y}\mathbf{A} \geq \mathbf{c}$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

It is assumed that if  $\mathbf{A}$  is a matrix of size  $[m \times n]$ , then  $\mathbf{y}$  is a  $[1 \times m]$  vector,  $\mathbf{c}$  is a  $[1 \times n]$  vector,  $\mathbf{b}$  is an  $[m \times 1]$  vector, and  $\mathbf{x}$  is an  $[n \times 1]$  vector.

### Network Optimization

Assume we have a graph  $G(N, A)$  with a finite set of nodes  $N$  and a finite set of arcs  $A$ . Furthermore, let

$$N = \{1, 2, \dots, n\}$$

$$x_{ij} = \text{flow from node } i \text{ to node } j$$

$$c_{ij} = \text{cost per unit flow from } i \text{ to } j$$

$$u_{ij} = \text{capacity of arc } (i, j)$$

$$b_i = \text{net flow generated at node } i$$

We wish to minimize the total cost of sending the available supply through the network to satisfy the given demand. The minimal cost flow model is formulated as follows:

$$\text{Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^n x_{ij} - \sum_{j=1}^n x_{ji} = b_i \text{ for each node } i \in N$$

and

$$0 \leq x_{ij} \leq u_{ij} \text{ for each arc } (i, j) \in A$$

The constraints on the nodes represent a conservation of flow relationship. The first summation represents total flow out of node  $i$ , and the second summation represents total flow into node  $i$ . The net difference generated at node  $i$  is equal to  $b_i$ .

Many models, such as shortest-path, maximal-flow, assignment and transportation models, can be reformulated as minimal-cost network flow models.

## PROCESS CAPABILITY

### Actual Capability

$$PCR_k = C_{pk} = \min\left(\frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma}\right)$$

### Potential Capability (i.e., Centered Process)

$$PCR = C_p = \frac{USL - LSL}{6\sigma}, \text{ where}$$

$\mu$  and  $\sigma$  are the process mean and standard deviation, respectively, and  $LSL$  and  $USL$  are the lower and upper specification limits, respectively.

## QUEUEING MODELS

### Definitions

$P_n$  = probability of  $n$  units in system

$L$  = expected number of units in the system

$L_q$  = expected number of units in the queue

$W$  = expected waiting time in system

$W_q$  = expected waiting time in queue

$\lambda$  = mean arrival rate (constant)

$\tilde{\lambda}$  = effective arrival rate

$\mu$  = mean service rate (constant)

$\rho$  = server utilization factor

$s$  = number of servers

Kendall notation for describing a queueing system:

$$A / B / s / M$$

$A$  = the arrival process

$B$  = the service time distribution

$s$  = the number of servers

$M$  = the total number of customers including those in service

## Fundamental Relationships

$$L = \lambda W$$

$$L_q = \lambda W_q$$

$$W = W_q + 1/\mu$$

$$\rho = \lambda/(s\mu)$$

## Single Server Models ( $s = 1$ )

Poisson Input—Exponential Service Time:  $M = \infty$

$$P_0 = 1 - \lambda/\mu = 1 - \rho$$

$$P_n = (1 - \rho)\rho^n = P_0\rho^n$$

$$L = \rho/(1 - \rho) = \lambda/(\mu - \lambda)$$

$$L_q = \lambda^2/[\mu(\mu - \lambda)]$$

$$W = 1/[\mu(1 - \rho)] = 1/(\mu - \lambda)$$

$$W_q = W - 1/\mu = \lambda/[\mu(\mu - \lambda)]$$

Finite queue:  $M < \infty$

$$\tilde{\lambda} = \lambda(1 - P_m)$$

$$P_0 = (1 - \rho)/(1 - \rho^{M+1})$$

$$P_n = [(1 - \rho)/(1 - \rho^{M+1})]\rho^n$$

$$L = \rho/(1 - \rho) - (M + 1)\rho^{M+1}/(1 - \rho^{M+1})$$

$$L_q = L - (1 - P_0)$$

$$W = L/\tilde{\lambda}$$

$$W = W_q + 1/\mu$$

Poisson Input—Arbitrary Service Time

Variance  $\sigma^2$  is known. For constant service time,  $\sigma^2 = 0$ .

$$P_0 = 1 - \rho$$

$$L_q = (\lambda^2\sigma^2 + \rho^2)/[2(1 - \rho)]$$

$$L = \rho + L_q$$

$$W_q = L_q/\lambda$$

$$W = W_q + 1/\mu$$

Poisson Input—Erlang Service Times,  $\sigma^2 = 1/(k\mu^2)$

$$L_q = [(1 + k)/(2k)][(\lambda^2)/(\mu(\mu - \lambda))]$$

$$= [\lambda^2/(k\mu^2) + \rho^2]/[2(1 - \rho)]$$

$$W_q = [(1 + k)/(2k)]\{\lambda/[\mu(\mu - \lambda)]\}$$

$$W = W_q + 1/\mu$$

## Multiple Server Model ( $s > 1$ )

Poisson Input—Exponential Service Times

$$P_0 = \left[ \sum_{n=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \left( \frac{1}{1 - \frac{\lambda}{s\mu}} \right) \right]^{-1}$$

$$= 1 / \left[ \sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1 - \rho)} \right]$$

$$L_q = \frac{P_0 \left(\frac{\lambda}{\mu}\right)^s \rho}{s!(1 - \rho)^2}$$

$$= \frac{P_0 s^s \rho^{s+1}}{s!(1 - \rho)^2}$$

$$P_n = P_0 (\lambda/\mu)^n / n! \quad 0 \leq n \leq s$$

$$P_n = P_0 (\lambda/\mu)^n / (s! s^{n-s}) \quad n \geq s$$

$$W_q = L_q/\lambda$$

$$W = W_q + 1/\mu$$

$$L = L_q + \lambda/\mu$$

Calculations for  $P_0$  and  $L_q$  can be time consuming; however, the following table gives formulas for 1, 2, and 3 servers.

$s$	$P_0$	$L_q$
1	$1 - \rho$	$\rho^2/(1 - \rho)$
2	$(1 - \rho)/(1 + \rho)$	$2\rho^3/(1 - \rho^2)$
3	$\frac{2(1 - \rho)}{2 + 4\rho + 3\rho^2}$	$\frac{9\rho^4}{2 + 2\rho - \rho^2 - 3\rho^3}$

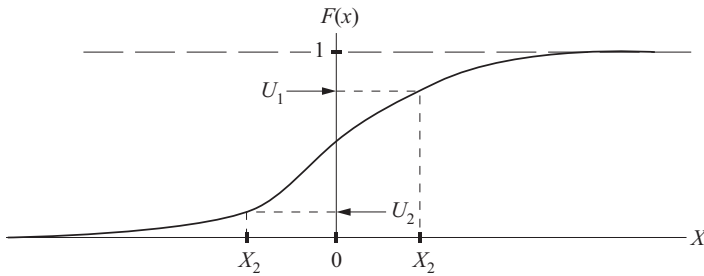
## SIMULATION

### 1. Random Variate Generation

The linear congruential method of generating pseudo-random numbers  $U_i$  between 0 and 1 is obtained using  $Z_n = (aZ_{n-1} + C) \pmod{m}$  where  $a$ ,  $C$ ,  $m$ , and  $Z_0$  are given nonnegative integers and where  $U_i = Z_i/m$ . Two integers are equal  $\pmod{m}$  if their remainders are the same when divided by  $m$ .

### 2. Inverse Transform Method

If  $X$  is a continuous random variable with cumulative distribution function  $F(x)$ , and  $U_i$  is a random number between 0 and 1, then the value of  $X_i$  corresponding to  $U_i$  can be calculated by solving  $U_i = F(x_i)$  for  $x_i$ . The solution obtained is  $x_i = F^{-1}(U_i)$ , where  $F^{-1}$  is the inverse function of  $F(x)$ .



Inverse Transform Method for Continuous Random Variables

## FORECASTING

### Moving Average

$$\hat{d}_t = \frac{\sum_{i=1}^n d_{t-i}}{n}, \text{ where}$$

$\hat{d}_t$  = forecasted demand for period  $t$

$d_{t-i}$  = actual demand for  $i$ th period preceding  $t$

$n$  = number of time periods to include in the moving average

### Exponentially Weighted Moving Average

$$\hat{d}_t = \alpha d_{t-1} + (1 - \alpha) \hat{d}_{t-1}, \text{ where}$$

$\hat{d}_t$  = forecasted demand for  $t$

$\alpha$  = smoothing constant,  $0 \leq \alpha \leq 1$

## RANDOMIZED BLOCK DESIGN

The experimental material is divided into  $n$  randomized blocks. One observation is taken at random for every treatment within the same block. The total number of observations is  $N = nk$ . The total value of these observations is equal to  $T$ . The total value of observations for treatment  $i$  is  $T_i$ . The total value of observations in block  $j$  is  $B_j$ .

$$C = T^2/N$$

$$SS_{\text{total}} = \sum_{i=1}^k \sum_{j=1}^n x_{ij}^2 - C$$

$$SS_{\text{blocks}} = \sum_{j=1}^n (B_j^2/k) - C$$

$$SS_{\text{treatments}} = \sum_{i=1}^k (T_i^2/n) - C$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{blocks}} - SS_{\text{treatments}}$$

## 2<sup>n</sup> FACTORIAL EXPERIMENTS

Factors:  $X_1, X_2, \dots, X_n$

Levels of each factor: 1, 2 (sometimes these levels are represented by the symbols – and +, respectively)

$r$  = number of observations for each experimental condition (treatment)

$E_i$  = estimate of the effect of factor  $X_i$ ,  $i = 1, 2, \dots, n$

$E_{ij}$  = estimate of the effect of the interaction between factors  $X_i$  and  $X_j$

$\bar{Y}_{ik}$  = average response value for all  $r2^{n-1}$  observations having  $X_i$  set at level  $k$ ,  $k = 1, 2$

$\bar{Y}_{ij}^{km}$  = average response value for all  $r2^{n-2}$  observations having  $X_i$  set at level  $k$ ,  $k = 1, 2$ , and  $X_j$  set at level  $m$ ,  $m = 1, 2$ .

$$E_i = \bar{Y}_{i2} - \bar{Y}_{i1}$$

$$E_{ij} = \frac{(\bar{Y}_{ij}^{22} - \bar{Y}_{ij}^{21}) - (\bar{Y}_{ij}^{12} - \bar{Y}_{ij}^{11})}{2}$$

## ANALYSIS OF VARIANCE FOR 2<sup>n</sup> FACTORIAL DESIGNS

### Main Effects

Let  $E$  be the estimate of the effect of a given factor, let  $L$  be the orthogonal contrast belonging to this effect. It can be proved that

$$E = \frac{L}{2^{n-1}}$$

$$L = \sum_{c=1}^m a_{(c)} \bar{Y}_{(c)}$$

$$SS_L = \frac{rL^2}{2^n}, \text{ where}$$

$m$  = number of experimental conditions  
( $m = 2^n$  for  $n$  factors)

$a_{(c)} = -1$  if the factor is set at its low level (level 1) in experimental condition  $c$

$a_{(c)} = +1$  if the factor is set at its high level (level 2) in experimental condition  $c$

$r$  = number of replications for each experimental condition

$\bar{Y}_{(c)}$  = average response value for experimental condition  $c$

$SS_L$  = sum of squares associated with the factor

### Interaction Effects

Consider any group of two or more factors.

$a_{(c)} = +1$  if there is an even number (or zero) of factors in the group set at the low level (level 1) in experimental condition  $c = 1, 2, \dots, m$

$a_{(c)} = -1$  if there is an odd number of factors in the group set at the low level (level 1) in experimental condition  $c = 1, 2, \dots, m$

It can be proved that the interaction effect  $E$  for the factors in the group and the corresponding sum of squares  $SS_L$  can be determined as follows:

$$E = \frac{L}{2^{n-1}}$$

$$L = \sum_{c=1}^m a_{(c)} \bar{Y}_{(c)}$$

$$SS_L = \frac{rL^2}{2^n}$$

### Sum of Squares of Random Error

The sum of the squares due to the random error can be computed as

$$SS_{\text{error}} = SS_{\text{total}} - \sum_i SS_i - \sum_i \sum_j SS_{ij} - \dots - SS_{12\dots n}$$

where  $SS_i$  is the sum of squares due to factor  $X_i$ ,  $SS_{ij}$  is the sum of squares due to the interaction of factors  $X_i$  and  $X_j$ , and so on. The total sum of squares is equal to

$$SS_{\text{total}} = \sum_{c=1}^m \sum_{k=1}^r Y_{ck}^2 - \frac{T^2}{N}$$

where  $Y_{ck}$  is the  $k$ th observation taken for the  $c$ th experimental condition,  $m = 2^n$ ,  $T$  is the grand total of all observations, and  $N = r2^n$ .

## RELIABILITY

If  $P_i$  is the probability that component  $i$  is functioning, a reliability function  $R(P_1, P_2, \dots, P_n)$  represents the probability that a system consisting of  $n$  components will work.

For  $n$  independent components connected in series,

$$R(P_1, P_2, \dots, P_n) = \prod_{i=1}^n P_i$$

For  $n$  independent components connected in parallel,

$$R(P_1, P_2, \dots, P_n) = 1 - \prod_{i=1}^n (1 - P_i)$$

## LEARNING CURVES

The time to do the repetition  $N$  of a task is given by

$$T_N = KN^s, \text{ where}$$

$K$  = constant

$s$  =  $\ln$  (learning rate, as a decimal)/ $\ln 2$ ; or, learning rate =  $2^s$

If  $N$  units are to be produced, the average time per unit is given by

$$T_{\text{avg}} = \frac{K}{N(1+s)} [(N+0.5)^{(1+s)} - 0.5^{(1+s)}]$$

## INVENTORY MODELS

For instantaneous replenishment (with constant demand rate, known holding and ordering costs, and an infinite stockout cost), the economic order quantity is given by

$$EOQ = \sqrt{\frac{2AD}{h}}, \text{ where}$$

$A$  = cost to place one order

$D$  = number of units used per year

$h$  = holding cost per unit per year

Under the same conditions as above with a finite replenishment rate, the economic manufacturing quantity is given by

$$EMQ = \sqrt{\frac{2AD}{h(1-D/R)}}, \text{ where}$$

$R$  = the replenishment rate

## FACILITY PLANNING

### Equipment Requirements

$$M_j = \sum_{i=1}^n \frac{P_{ij}T_{ij}}{C_{ij}} \text{ where}$$

$M_j$  = number of machines of type  $j$  required per production period

$P_{ij}$  = desired production rate for product  $i$  on machine  $j$ , measured in pieces per production period

$T_{ij}$  = production time for product  $i$  on machine  $j$ , measured in hours per piece

$C_{ij}$  = number of hours in the production period available for the production of product  $i$  on machine  $j$

$n$  = number of products



## People Requirements

$$A_j = \sum_{i=1}^n \frac{P_{ij}T_{ij}}{C_{ij}}, \text{ where}$$

$A_j$  = number of crews required for assembly operation  $j$

$P_{ij}$  = desired production rate for product  $i$  and assembly operation  $j$  (pieces per day)

$T_{ij}$  = standard time to perform operation  $j$  on product  $i$  (minutes per piece)

$C_{ij}$  = number of minutes available per day for assembly operation  $j$  on product  $i$

$n$  = number of products

## Standard Time Determination

$$ST = NT \times AF$$

where

$NT$  = normal time

$AF$  = allowance factor

Case 1: Allowances are based on the *job time*.

$$AF_{\text{job}} = 1 + A_{\text{job}}$$

$A_{\text{job}}$  = allowance fraction (percentage/100) based on *job time*.

Case 2: Allowances are based on *workday*.

$$AF_{\text{time}} = 1/(1 - A_{\text{day}})$$

$A_{\text{day}}$  = allowance fraction (percentage/100) based on *workday*.

- ◆ Predetermined time systems are useful in cases where either (1) the task does not yet exist or (2) changes to a task are being designed and normal times have not yet been established for all elements of the new task or changed task. In such cases no opportunity exists to measure the element time. Unfortunately, there is no scientific basis for predicting element times without breaking them down into motion-level parts. A task consists of elements. An organization may develop its own database of normal element durations, and normal times for new or changed tasks may be predicted if the tasks consist entirely of elements whose normal times are already in the database. But new elements can be decomposed into motions, for which scientifically predetermined times exist in databases called MTM-1, MTM-2, and MTM-3. These databases and software to manipulate them are commercially available. To use one of them effectively requires about 50 hours of training.

◆ Kennedy, W.J., and Daniel P. Rogers, *Review for the Professional Engineers' Examination in Industrial Engineering*, 2012.

## Plant Location

The following is one formulation of a discrete plant location problem.

Minimize

$$z = \sum_{i=1}^m \sum_{j=1}^n c_{ij}y_{ij} + \sum_{j=1}^n f_jx_j$$

subject to

$$\sum_{i=1}^m y_{ij} \leq mx_j, \quad j = 1, \dots, n$$

$$\sum_{j=1}^n y_{ij} = 1, \quad i = 1, \dots, m$$

$$y_{ij} \geq 0, \text{ for all } i, j$$

$$x_j = (0, 1), \text{ for all } j, \text{ where}$$

$m$  = number of customers

$n$  = number of possible plant sites

$y_{ij}$  = fraction or proportion of the demand of customer  $i$  which is satisfied by a plant located at site  $j$ ;  $i = 1, \dots, m$ ;  $j = 1, \dots, n$

$x_j = 1$ , if a plant is located at site  $j$

$x_j = 0$ , otherwise

$c_{ij}$  = cost of supplying the entire demand of customer  $i$  from a plant located at site  $j$

$f_j$  = fixed cost resulting from locating a plant at site  $j$

## Material Handling

Distances between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  under different metrics:

Euclidean:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Rectilinear (or Manhattan):

$$D = |x_1 - x_2| + |y_1 - y_2|$$

Chebyshev (simultaneous  $x$  and  $y$  movement):

$$D = \max(|x_1 - x_2|, |y_1 - y_2|)$$

## Line Balancing

$$N_{\min} = \left( OR \times \sum_i t_i / OT \right)$$

= theoretical minimum number of stations

$$\text{Idle Time/Station} = CT - ST$$

$$\text{Idle Time/Cycle} = \sum (CT - ST)$$

$$\text{Percent Idle Time} = \frac{\text{Idle Time/Cycle}}{N_{\text{actual}} \times CT} \times 100, \text{ where}$$

$CT$  = cycle time (time between units)

$OT$  = operating time/period

$OR$  = output rate/period

$ST$  = station time (time to complete task at each station)

$t_i$  = individual task times

$N$  = number of stations

## Job Sequencing

Two Work Centers—Johnson's Rule

1. Select the job with the shortest time, from the list of jobs, and its time at each work center.
2. If the shortest job time is the time at the first work center, schedule it first, otherwise schedule it last. Break ties arbitrarily.
3. Eliminate that job from consideration.
4. Repeat 1, 2, and 3 until all jobs have been scheduled.

## CRITICAL PATH METHOD (CPM)

$d_{ij}$  = duration of activity ( $i, j$ )

CP = critical path (longest path)

$T$  = duration of project

$$T = \sum_{(i,j) \in CP} d_{ij}$$

## PERT

$(a_{ij}, b_{ij}, c_{ij})$  = (optimistic, most likely, pessimistic) durations for activity ( $i, j$ )

$\mu_{ij}$  = mean duration of activity ( $i, j$ )

$\sigma_{ij}$  = standard deviation of the duration of activity ( $i, j$ )

$\mu$  = project mean duration

$\sigma$  = standard deviation of project duration

$$\mu_{ij} = \frac{a_{ij} + 4b_{ij} + c_{ij}}{6}$$

$$\sigma_{ij} = \frac{c_{ij} - a_{ij}}{6}$$

$$\mu = \sum_{(i,j) \in CP} \mu_{ij}$$

$$\sigma^2 = \sum_{(i,j) \in CP} \sigma_{ij}^2$$

## TAYLOR TOOL LIFE FORMULA

$VT^n = C$ , where

$V$  = speed in surface feet per minute

$T$  = tool life in minutes

$C, n$  = constants that depend on the material and on the tool

## WORK SAMPLING FORMULAS

$$D = Z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} \quad \text{and} \quad R = Z_{\alpha/2} \sqrt{\frac{1-p}{pn}}, \text{ where}$$

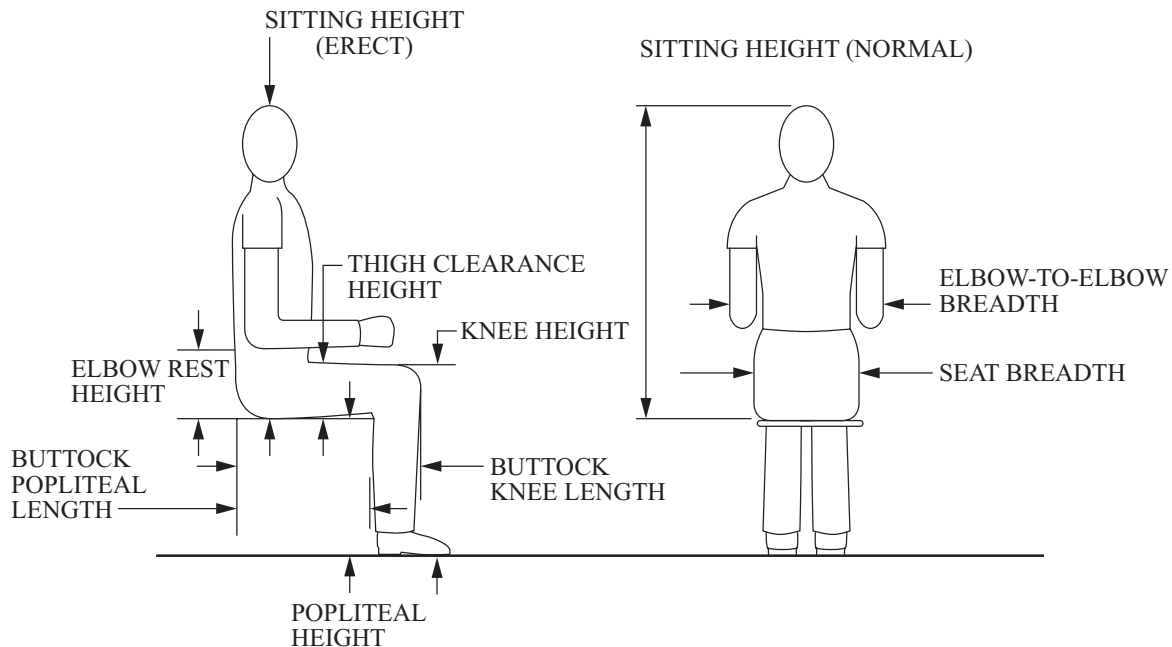
$p$  = proportion of observed time in an activity

$D$  = absolute error

$R$  = relative error ( $R = D/p$ )

$n$  = sample size

## ANTHROPOMETRIC MEASUREMENTS

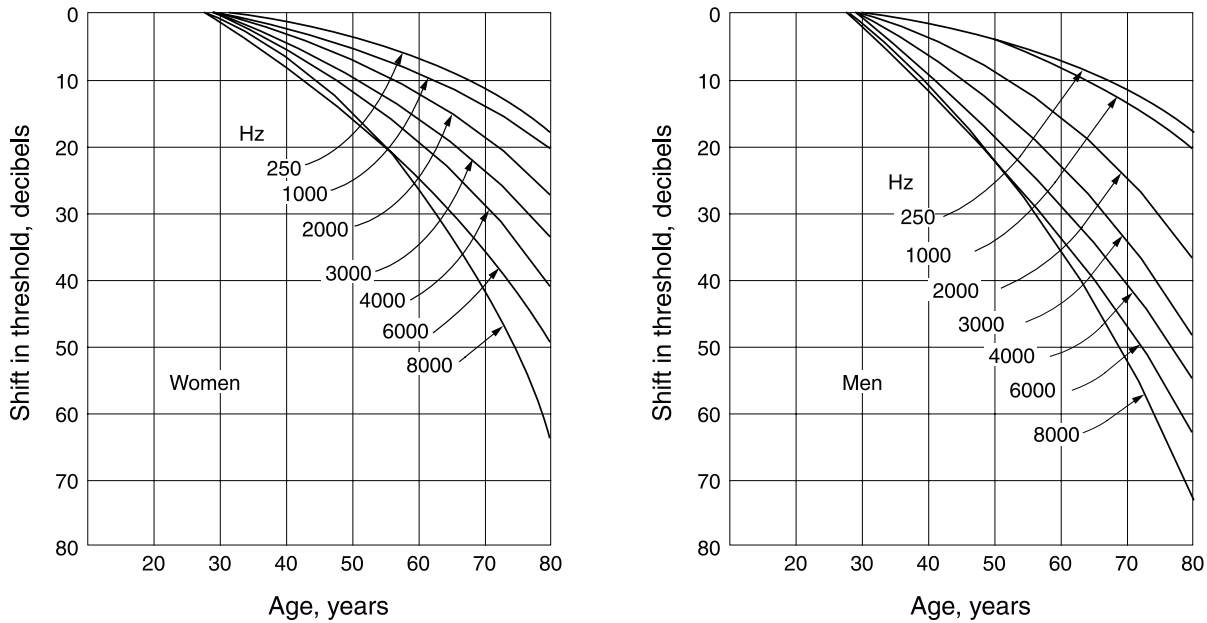


After Sanders and McCormick, *Human Factors In Design*, McGraw-Hill, 1987.

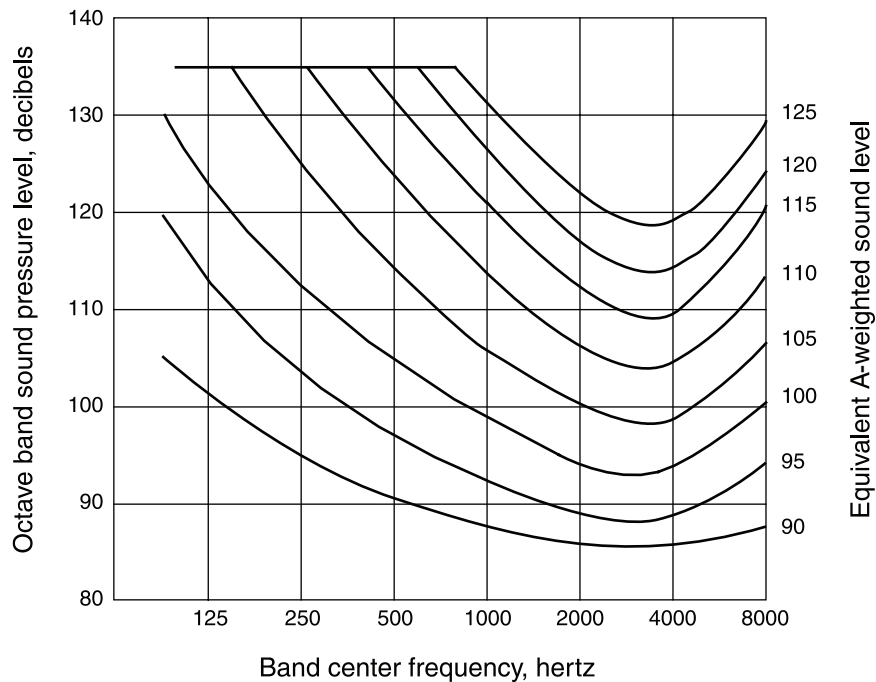
U.S. Civilian Body Dimensions, Female/Male, for Ages 20 to 60 Years (Centimeters)				
(See Anthropometric Measurements Figure)	Percentiles			
	5th	50th	95th	Std. Dev.
<b>HEIGHTS</b>				
Stature (height)	149.5 / 161.8	160.5 / 173.6	171.3 / 184.4	6.6 / 6.9
Eye height	138.3 / 151.1	148.9 / 162.4	159.3 / 172.7	6.4 / 6.6
Shoulder (acromion) height	121.1 / 132.3	131.1 / 142.8	141.9 / 152.4	6.1 / 6.1
Elbow height	93.6 / 100.0	101.2 / 109.9	108.8 / 119.0	4.6 / 5.8
<b>Knuckle height</b>	<b>64.3 / 69.8</b>	<b>70.2 / 75.4</b>	<b>75.9 / 80.4</b>	<b>3.5 / 3.2</b>
Height, sitting (erect)	78.6 / 84.2	85.0 / 90.6	90.7 / 96.7	3.5 / 3.7
Eye height, sitting	67.5 / 72.6	73.3 / 78.6	78.5 / 84.4	3.3 / 3.6
Shoulder height, sitting	49.2 / 52.7	55.7 / 59.4	61.7 / 65.8	3.8 / 4.0
Elbow rest height, sitting	18.1 / 19.0	23.3 / 24.3	28.1 / 29.4	2.9 / 3.0
<b>Knee height, sitting</b>	<b>45.2 / 49.3</b>	<b>49.8 / 54.3</b>	<b>54.5 / 59.3</b>	<b>2.7 / 2.9</b>
Popliteal height, sitting	35.5 / 39.2	39.8 / 44.2	44.3 / 48.8	2.6 / 2.8
Thigh clearance height	10.6 / 11.4	13.7 / 14.4	17.5 / 17.7	1.8 / 1.7
<b>DEPTHS</b>				
Chest depth	21.4 / 21.4	24.2 / 24.2	29.7 / 27.6	2.5 / 1.9
Elbow-fingertip distance	38.5 / 44.1	42.1 / 47.9	46.0 / 51.4	2.2 / 2.2
Buttock-knee length, sitting	51.8 / 54.0	56.9 / 59.4	62.5 / 64.2	3.1 / 3.0
Buttock-popliteal length, sitting	43.0 / 44.2	48.1 / 49.5	53.5 / 54.8	3.1 / 3.0
Forward reach, functional	64.0 / 76.3	71.0 / 82.5	79.0 / 88.3	4.5 / 5.0
<b>BREADTHS</b>				
Elbow-to-elbow breadth	31.5 / 35.0	38.4 / 41.7	49.1 / 50.6	5.4 / 4.6
Seat (hip) breadth, sitting	31.2 / 30.8	36.4 / 35.4	43.7 / 40.6	3.7 / 2.8
<b>HEAD DIMENSIONS</b>				
Head breadth	13.6 / 14.4	14.54 / 15.42	15.5 / 16.4	0.57 / 0.59
Head circumference	52.3 / 53.8	54.9 / 56.8	57.7 / 59.3	1.63 / 1.68
Interpupillary distance	5.1 / 5.5	5.83 / 6.20	6.5 / 6.8	0.4 / 0.39
<b>HAND DIMENSIONS</b>				
Hand length	16.4 / 17.6	17.95 / 19.05	19.8 / 20.6	1.04 / 0.93
Breadth, metacarpal	7.0 / 8.2	7.66 / 8.88	8.4 / 9.8	0.41 / 0.47
Circumference, metacarpal	16.9 / 19.9	18.36 / 21.55	19.9 / 23.5	0.89 / 1.09
Thickness, metacarpal III	2.5 / 2.4	2.77 / 2.76	3.1 / 3.1	0.18 / 0.21
Digit 1				
Breadth, interphalangeal	1.7 / 2.1	1.98 / 2.29	2.1 / 2.5	0.12 / 0.13
Crotch-tip length	4.7 / 5.1	5.36 / 5.88	6.1 / 6.6	0.44 / 0.45
Digit 2				
Breadth, distal joint	1.4 / 1.7	1.55 / 1.85	1.7 / 2.0	0.10 / 0.12
Crotch-tip length	6.1 / 6.8	6.88 / 7.52	7.8 / 8.2	0.52 / 0.46
Digit 3				
Breadth, distal joint	1.4 / 1.7	1.53 / 1.85	1.7 / 2.0	0.09 / 0.12
Crotch-tip length	7.0 / 7.8	7.77 / 8.53	8.7 / 9.5	0.51 / 0.51
Digit 4				
Breadth, distal joint	1.3 / 1.6	1.42 / 1.70	1.6 / 1.9	0.09 / 0.11
Crotch-tip length	6.5 / 7.4	7.29 / 7.99	8.2 / 8.9	0.53 / 0.47
Digit 5				
Breadth, distal joint	1.2 / 1.4	1.32 / 1.57	1.5 / 1.8	0.09/0.12
Crotch-tip length	4.8 / 5.4	5.44 / 6.08	6.2 / 6.99	0.44/0.47
<b>FOOT DIMENSIONS</b>				
Foot length	22.3 / 24.8	24.1 / 26.9	26.2 / 29.0	1.19 / 1.28
Foot breadth	8.1 / 9.0	8.84 / 9.79	9.7 / 10.7	0.50 / 0.53
Lateral malleolus height	5.8 / 6.2	6.78 / 7.03	7.8 / 8.0	0.59 / 0.54
Weight (kg)	46.2 / 56.2	61.1 / 74.0	89.9 / 97.1	13.8 / 12.6

## ERGONOMICS—HEARING

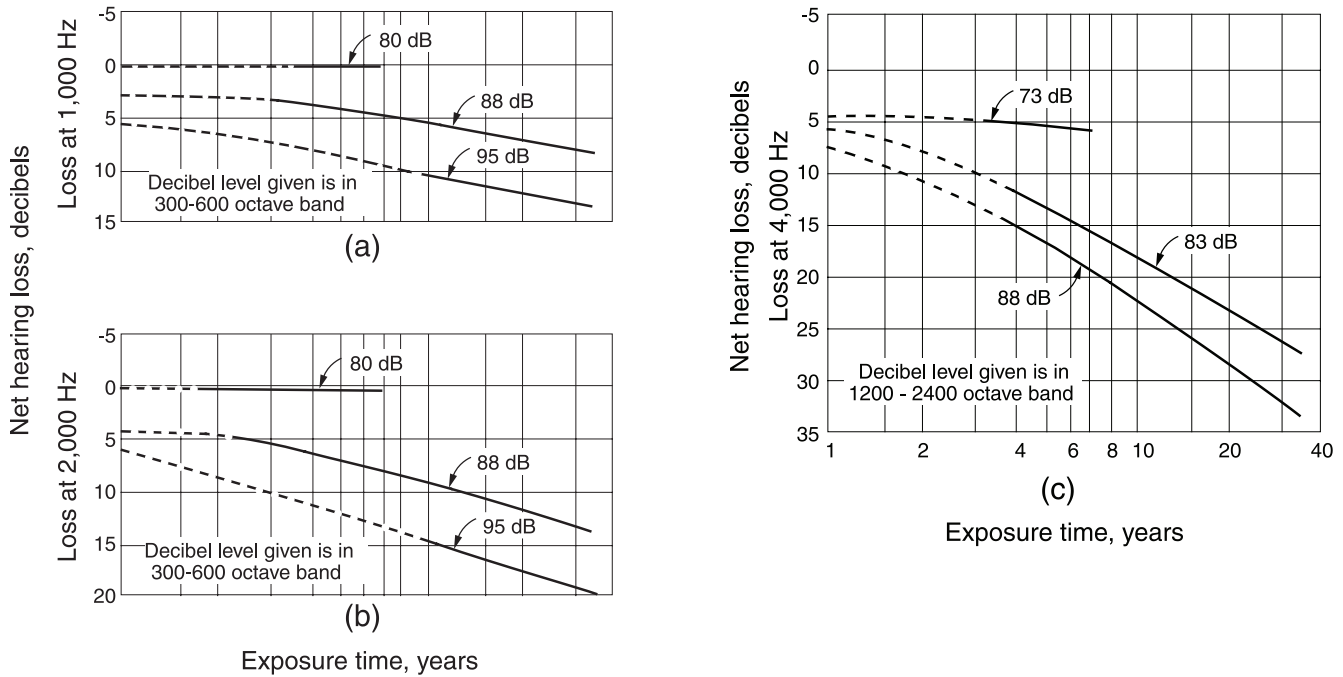
The average shifts with age of the threshold of hearing for pure tones of persons with “normal” hearing, using a 25-year-old group as a reference group.



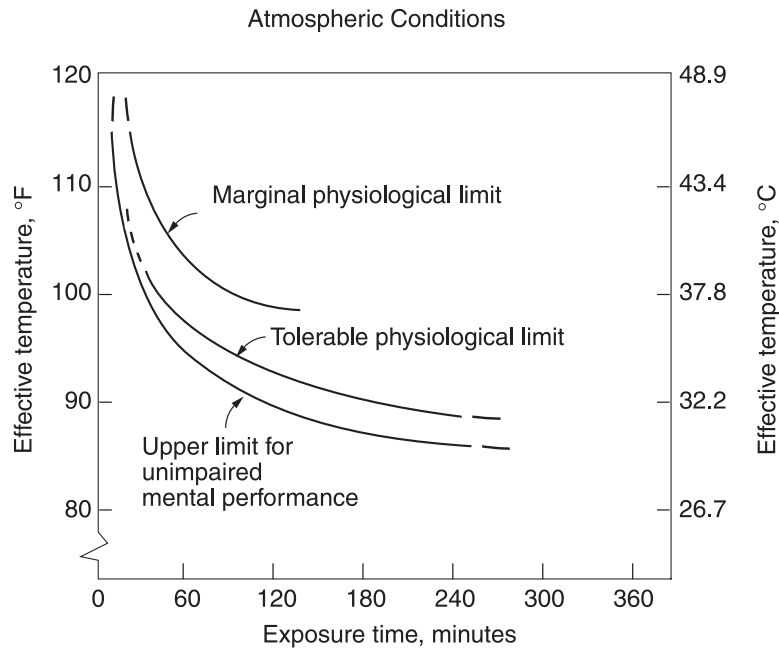
Equivalent sound-level contours used in determining the A-weighted sound level on the basis of an octave-band analysis. The curve at the point of the highest penetration of the noise spectrum reflects the A-weighted sound level.



Estimated average trend curves for net hearing loss at 1,000, 2,000, and 4,000 Hz after continuous exposure to steady noise. Data are corrected for age, but not for temporary threshold shift. Dotted portions of curves represent extrapolation from available data.



Tentative upper limit of effective temperature (ET) for unimpaired mental performance as related to exposure time; data are based on an analysis of 15 studies. Comparative curves of tolerable and marginal physiological limits are also given.



Effective temperature (ET) is the dry bulb temperature at 50% relative humidity, which results in the same physiological effect as the present conditions.

# MECHANICAL ENGINEERING

## MECHANICAL DESIGN AND ANALYSIS

### Springs

#### Mechanical Springs

*Helical Linear Springs:* The shear stress in a helical linear spring is

$$\tau = K_s \frac{8FD}{\pi d^3}, \text{ where}$$

$d$  = wire diameter

$F$  = applied force

$D$  = mean spring diameter

$$K_s = (2C + 1)/(2C)$$

$$C = D/d$$

The deflection and force are related by  $F = kx$  where the spring rate (spring constant)  $k$  is given by

$$k = \frac{d^4 G}{8D^3 N}$$

where  $G$  is the shear modulus of elasticity and  $N$  is the number of active coils.

*Spring Material:* The minimum tensile strength of common spring steels may be determined from

$$S_{ut} = A/d^m$$

where  $S_{ut}$  is the tensile strength in MPa,  $d$  is the wire diameter in millimeters, and  $A$  and  $m$  are listed in the following table:

Material	ASTM	$m$	$A$
Music wire	A228	0.163	2060
Oil-tempered wire	A229	0.193	1610
Hard-drawn wire	A227	0.201	1510
Chrome vanadium	A232	0.155	1790
Chrome silicon	A401	0.091	1960

Maximum allowable torsional stress for static applications may be approximated as

$$S_{sy} = \tau = 0.45S_{ut} \text{ cold-drawn carbon steel (A227, A228, A229)}$$

$$S_{sy} = \tau = 0.50S_{ut} \text{ hardened and tempered carbon and low-alloy steels (A232, A401)}$$

### Compression Spring Dimensions

Type of Spring Ends		
Term	Plain	Plain and Ground
End coils, $N_e$	0	1
Total coils, $N_t$	$N$	$N + 1$
Free length, $L_0$	$pN + d$	$p(N + 1)$
Solid length, $L_s$	$d(N_t + 1)$	$dN_t$
Pitch, $p$	$(L_0 - d)/N$	$L_0/(N + 1)$

Term	Squared or Closed	Squared and Ground
End coils, $N_e$	2	2
Total coils, $N_t$	$N + 2$	$N + 2$
Free length, $L_0$	$pN + 3d$	$pN + 2d$
Solid length, $L_s$	$d(N_t + 1)$	$dN_t$
Pitch, $p$	$(L_0 - 3d)/N$	$(L_0 - 2d)/N$

*Helical Torsion Springs:* The bending stress is given as

$$\sigma = K_i [32Fr/(\pi d^3)]$$

where  $F$  is the applied load and  $r$  is the radius from the center of the coil to the load.

$K_i$  = correction factor

$$= (4C^2 - C - 1) / [4C(C - 1)]$$

$$C = D/d$$

The deflection  $\theta$  and moment  $Fr$  are related by

$$Fr = k\theta$$

where the spring rate  $k$  is given by

$$k = \frac{d^4 E}{64DN}$$

where  $k$  has units of N•m/rad and  $\theta$  is in radians.

*Spring Material:* The strength of the spring wire may be found as shown in the section on linear springs. The allowable stress  $\sigma$  is then given by

$$S_y = \sigma = 0.78S_{ut} \text{ cold-drawn carbon steel (A227, A228, A229)}$$

$$S_y = \sigma = 0.87S_{ut} \text{ hardened and tempered carbon and low-alloy steel (A232, A401)}$$

## Bearings

### Ball/Roller Bearing Selection

The minimum required *basic load rating* (load for which 90% of the bearings from a given population will survive 1 million revolutions) is given by

$$C = PL^{1/a}, \text{ where}$$

$C$  = minimum required basic load rating

$P$  = design radial load

$L$  = design life (in millions of revolutions)

$a$  = 3 for ball bearings, 10/3 for roller bearings

When a ball bearing is subjected to both radial and axial loads, an equivalent radial load must be used in the equation above.

The equivalent radial load is

$$P_{eq} = XVF_r + YF_a, \text{ where}$$

$P_{eq}$  = equivalent radial load

$F_r$  = applied constant radial load

$F_a$  = applied constant axial (thrust) load

For radial contact, deep-groove ball bearings:

$V$  = 1 if inner ring rotating, 1.2 if outer ring rotating,

$$\text{If } F_a/(VF_r) > e,$$

$$X = 0.56, \text{ and } Y = 0.840 \left( \frac{F_a}{C_0} \right)^{-0.247}$$

$$\text{where } e = 0.513 \left( \frac{F_a}{C_0} \right)^{0.236}$$

$C_0$  = basic static load rating from bearing catalog

If  $F_a/(VF_r) \leq e$ ,  $X = 1$  and  $Y = 0$ .

## Power Screws

**Square Thread Power Screws:** The torque required to raise,  $T_R$ , or to lower,  $T_L$ , a load is given by

$$T_R = \frac{Fd_m}{2} \left( \frac{l + \pi \mu d_m}{\pi d_m - \mu l} \right) + \frac{F \mu_c d_c}{2}$$
$$T_L = \frac{Fd_m}{2} \left( \frac{\pi \mu d_m - l}{\pi d_m + \mu l} \right) + \frac{F \mu_c d_c}{2}, \text{ where}$$

$d_c$  = mean collar diameter

$d_m$  = mean thread diameter

$l$  = lead

$F$  = load

$\mu$  = coefficient of friction for thread

$\mu_c$  = coefficient of friction for collar

The efficiency of a power screw may be expressed as

$$\eta = Fl/(2\pi T)$$

## Power Transmission

### Shafts and Axles

**Static Loading:** The maximum shear stress and the von Mises stress may be calculated in terms of the loads from

$$\tau_{\max} = \frac{2}{\pi d^3} [(8M + Fd)^2 + (8T)^2]^{1/2},$$

$$\sigma' = \frac{4}{\pi d^3} [(8M + Fd)^2 + 48T^2]^{1/2}, \text{ where}$$

$M$  = the bending moment

$F$  = the axial load

$T$  = the torque

$d$  = the diameter

**Fatigue Loading:** Using the maximum-shear-stress theory combined with the Soderberg line for fatigue, the diameter and safety factor are related by

$$\frac{\pi d^3}{32} = n \left[ \left( \frac{M_m}{S_y} + \frac{K_f M_a}{S_e} \right)^2 + \left( \frac{T_m}{S_y} + \frac{K_{fs} T_a}{S_e} \right)^2 \right]^{1/2}$$

where

$d$  = diameter

$n$  = safety factor

$M_a$  = alternating moment

$M_m$  = mean moment

$T_a$  = alternating torque

$T_m$  = mean torque

$S_e$  = fatigue limit

$S_y$  = yield strength

$K_f$  = fatigue strength reduction factor

$K_{fs}$  = fatigue strength reduction factor for shear

### Gearing

#### Involute Gear Tooth Nomenclature

Circular pitch  $p_c = \pi d/N$

Base pitch  $p_b = p_c \cos \phi$

Module  $m = d/N$

Center distance  $C = (d_1 + d_2)/2$

where

$N$  = number of teeth on pinion or gear

$d$  = pitch circle diameter

$\phi$  = pressure angle

**Gear Trains:** *Velocity ratio*,  $m_v$ , is the ratio of the output velocity to the input velocity. Thus,  $m_v = \omega_{\text{out}}/\omega_{\text{in}}$ . For a two-gear train,  $m_v = -N_{\text{in}}/N_{\text{out}}$  where  $N_{\text{in}}$  is the number of teeth on the input gear and  $N_{\text{out}}$  is the number of teeth on the output gear. The negative sign indicates that the output gear rotates in the opposite sense with respect to the input gear. In a *compound gear train*, at least one shaft carries more than one gear (rotating at the same speed). The velocity ratio for a compound train is:

$$m_v = \pm \frac{\text{product of number of teeth on driver gears}}{\text{product of number of teeth on driven gears}}$$



A *simple planetary gearset* has a sun gear, an arm that rotates about the sun gear axis, one or more gears (planets) that rotate about a point on the arm, and a ring (internal) gear that is concentric with the sun gear. The planet gear(s) mesh with the sun gear on one side and with the ring gear on the other. A planetary gearset has two independent inputs and one output (or two outputs and one input, as in a differential gearset).

Often one of the inputs is zero, which is achieved by grounding either the sun or the ring gear. The velocities in a planetary set are related by

$$\frac{\omega_L - \omega_{\text{arm}}}{\omega_f - \omega_{\text{arm}}} = \pm m_v, \text{ where}$$

$\omega_f$  = speed of the first gear in the train

$\omega_L$  = speed of the last gear in the train

$\omega_{\text{arm}}$  = speed of the arm

Neither the first nor the last gear can be one that has planetary motion. In determining  $m_v$ , it is helpful to invert the mechanism by grounding the arm and releasing any gears that are grounded.

## Dynamics of Mechanisms

### Gearing

*Loading on Straight Spur Gears:* The load,  $W$ , on straight spur gears is transmitted along a plane that, in edge view, is called the *line of action*. This line makes an angle with a tangent line to the pitch circle that is called the *pressure angle*  $\phi$ . Thus, the contact force has two components: one in the tangential direction,  $W_t$ , and one in the radial direction,  $W_r$ . These components are related to the pressure angle by

$$W_r = W_t \tan(\phi).$$

Only the tangential component  $W_t$  transmits torque from one gear to another. Neglecting friction, the transmitted force may be found if either the transmitted torque or power is known:

$$W_t = \frac{2T}{d} = \frac{2T}{mN},$$

$$W_t = \frac{2H}{d\omega} = \frac{2H}{mN\omega}, \text{ where}$$

$W_t$  = transmitted force (newtons)

$T$  = torque on the gear (newton-mm)

$d$  = pitch diameter of the gear (mm)

$N$  = number of teeth on the gear

$m$  = gear module (mm) (same for both gears in mesh)

$H$  = power (kW)

$\omega$  = speed of gear (rad/s)

*Stresses in Spur Gears:* Spur gears can fail in either bending (as a cantilever beam, near the root) or by surface fatigue due to contact stresses near the pitch circle. AGMA Standard 2001 gives equations for bending stress and surface stress. They are:

$$\sigma_b = \frac{W_t}{FmJ} \frac{K_a K_m}{K_v} K_s K_B K_I, \text{ bending and}$$

$$\sigma_c = C_p \sqrt{\frac{W_t}{FId} \frac{C_a C_m}{C_v} C_s C_f}, \text{ surface stress, where}$$

$\sigma_b$  = bending stress

$\sigma_c$  = surface stress

$W_t$  = transmitted load

$F$  = face width

$m$  = module

$J$  = bending strength geometry factor

$K_a$  = application factor

$K_B$  = rim thickness factor

$K_I$  = idler factor

$K_m$  = load distribution factor

$K_s$  = size factor

$K_v$  = dynamic factor

$C_p$  = elastic coefficient

$I$  = surface geometry factor

$d$  = pitch diameter of gear being analyzed

$C_f$  = surface finish factor

$C_a$ ,  $C_m$ ,  $C_s$ , and  $C_v$  are the same as  $K_a$ ,  $K_m$ ,  $K_s$ , and  $K_v$ , respectively.

## Joining Methods

*Threaded Fasteners:* The load carried by a bolt in a threaded connection is given by

$$F_b = CP + F_i \quad F_m < 0$$

while the load carried by the members is

$$F_m = (1 - C)P - F_i \quad F_m < 0, \text{ where}$$

$C$  = joint coefficient

$$= k_b / (k_b + k_m)$$

$F_b$  = total bolt load

$F_i$  = bolt preload

$F_m$  = total material load

$P$  = externally applied load

$k_b$  = effective stiffness of the bolt or fastener in the grip

$k_m$  = effective stiffness of the members in the grip



*Bolt stiffness* may be calculated from

$$k_b = \frac{A_d A_t E}{A_d l_t + A_t l_d}, \text{ where}$$

$A_d$  = major-diameter area

$A_t$  = tensile-stress area

$E$  = modulus of elasticity

$l_d$  = length of unthreaded shank

$l_t$  = length of threaded shank contained within the grip

If all members within the grip are of the same material, *member stiffness* may be obtained from

$$k_m = dEAe^{b(d/l)}, \text{ where}$$

$d$  = bolt diameter

$E$  = modulus of elasticity of members

$l$  = grip length

Coefficients  $A$  and  $b$  are given in the table below for various joint member materials.

Material	$A$	$b$
Steel	0.78715	0.62873
Aluminum	0.79670	0.63816
Copper	0.79568	0.63553
Gray cast iron	0.77871	0.61616

The approximate tightening torque required for a given preload  $F_i$  and for a steel bolt in a steel member is given by  $T = 0.2 F_i d$ .

Threaded Fasteners – Design Factors: The bolt load factor is

$$n_b = (S_p A_t - F_i) / CP, \text{ where}$$

$S_p$  = proof strength

The factor of safety guarding against joint separation is

$$n_s = F_i / [P(1 - C)]$$

Threaded Fasteners – Fatigue Loading: If the externally applied load varies between zero and  $P$ , the alternating stress is

$$\sigma_a = CP / (2A_t)$$

and the mean stress is

$$\sigma_m = \sigma_a + F_i / A_t$$

Bolted and Riveted Joints Loaded in Shear:

*Failure by Pure Shear*

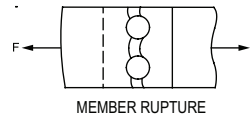


$$\tau = F/A, \text{ where}$$

$F$  = shear load

$A$  = cross-sectional area of bolt or rivet

*Failure by Rupture*

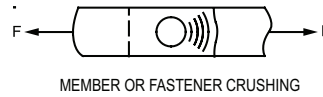


$$\sigma = F/A, \text{ where}$$

$F$  = load

$A$  = net cross-sectional area of thinnest member

*Failure by Crushing of Rivet or Member*

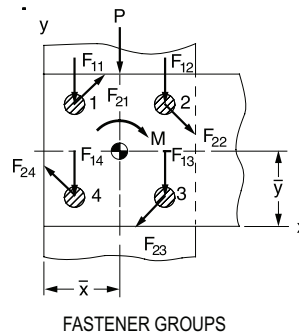


$$\sigma = F/A, \text{ where}$$

$F$  = load

$A$  = projected area of a single rivet

*Fastener Groups in Shear*



The location of the centroid of a fastener group with respect to any convenient coordinate frame is:

$$\bar{x} = \frac{\sum_{i=1}^n A_i x_i}{\sum_{i=1}^n A_i}, \quad \bar{y} = \frac{\sum_{i=1}^n A_i y_i}{\sum_{i=1}^n A_i}, \text{ where}$$

$n$  = total number of fasteners

$i$  = the index number of a particular fastener

$A_i$  = cross-sectional area of the  $i$ th fastener

$x_i$  =  $x$ -coordinate of the center of the  $i$ th fastener

$y_i$  =  $y$ -coordinate of the center of the  $i$ th fastener

The total shear force on a fastener is the **vector** sum of the force due to direct shear  $P$  and the force due to the moment  $M$  acting on the group at its centroid.

The magnitude of the direct shear force due to  $P$  is

$$|F_{1i}| = \frac{P}{n}.$$

This force acts in the same direction as  $P$ .

The magnitude of the shear force due to  $M$  is

$$|F_{2i}| = \frac{M r_i}{\sum_{i=1}^n r_i^2}.$$

This force acts perpendicular to a line drawn from the group centroid to the center of a particular fastener. Its sense is such that its moment is in the same direction (CW or CCW) as  $M$ .

## Press/Shrink Fits

The interface pressure induced by a press/shrink fit is

$$p = \frac{0.5\delta}{\frac{r}{E_o} \left( \frac{r_o^2 + r^2}{r_o^2 - r^2} + \nu_o \right) + \frac{r}{E_i} \left( \frac{r^2 + r_i^2}{r^2 - r_i^2} - \nu_i \right)}$$

where the subscripts  $i$  and  $o$  stand for the inner and outer member, respectively, and

$p$  = inside pressure on the outer member and outside pressure on the inner member

$\delta$  = the diametral interference

$r$  = nominal interference radius

$r_i$  = inside radius of inner member

$r_o$  = outside radius of outer member

$E$  = Young's modulus of respective member

$\nu$  = Poisson's ratio of respective member

The *maximum torque* that can be transmitted by a press fit joint is approximately

$$T = 2\pi r^2 \mu p l,$$

where  $r$  and  $p$  are defined above,

$T$  = torque capacity of the joint

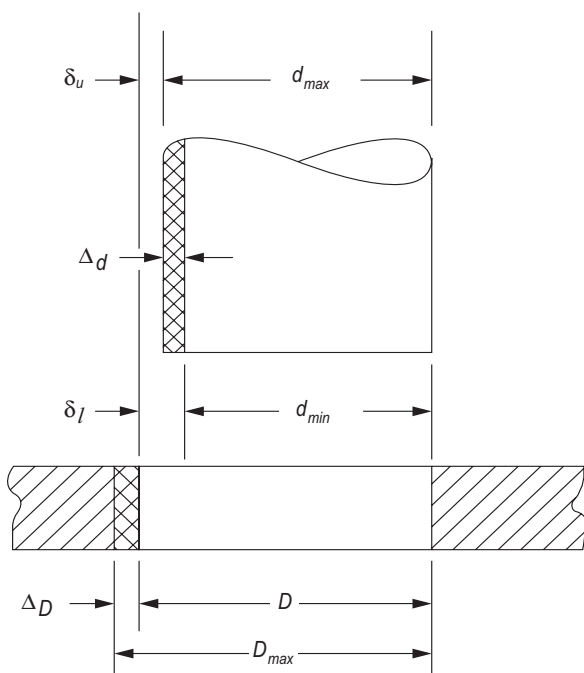
$\mu$  = coefficient of friction at the interface

$l$  = length of hub engagement

## MANUFACTURABILITY

### Limits and Fits

The designer is free to adopt any geometry of fit for shafts and holes that will ensure intended function. Over time, sufficient experience with common situations has resulted in the development of a standard. The metric version of the standard is newer and will be presented. The standard specifies that uppercase letters always refer to the hole, while lowercase letters always refer to the shaft.



## Definitions

*Basic Size* or *nominal size*,  $D$  or  $d$ , is the size to which the limits or deviations are applied. It is the same for both components.

*Deviation* is the algebraic difference between the actual size and the corresponding basic size.

*Upper Deviation*,  $\delta_u$ , is the algebraic difference between the maximum limit and the corresponding basic size.

*Lower Deviation*,  $\delta_l$ , is the algebraic difference between the minimum limit and the corresponding basic size.

*Fundamental Deviation*,  $\delta_F$ , is the upper or lower deviation, depending on which is closer to the basic size.

*Tolerance*,  $\Delta_D$  or  $\Delta_d$ , is the difference between the maximum and minimum size limits of a part.

*International tolerance (IT)* grade numbers designate groups of tolerances such that the tolerance for a particular IT number will have the same relative accuracy for a basic size.

*Hole basis* represents a system of fits corresponding to a basic hole size. The fundamental deviation is H.

## Some Preferred Fits

Clearance	
<i>Free running fit</i> : not used where accuracy is essential but good for large temperature variations, high running speeds, or heavy journal loads.	H9/d9
<i>Sliding fit</i> : where parts are not intended to run freely but must move and turn freely and locate accurately.	H7/g6
<i>Locational clearance fit</i> : provides snug fit for location of stationary parts, but can be freely assembled and disassembled.	H7/h6
<i>Loose running fit</i> : for wide commercial tolerances or allowances on external members	H11/c11
<i>Close running fit</i> : for running on accurate machines and for accurate location at moderate speeds and journal pressures	H8/f7
Transition	
<i>Locational transition fit</i> : for accurate location, a compromise between clearance and interference	H7/k6
<i>Locational transition fit</i> : for more accurate location where greater interface is permissible	H7/n6
Interference	
<i>Location interference fit</i> : for parts requiring rigidity and alignment with prime accuracy of location but without special bore pressure requirements.	H7/p6
<i>Medium drive fit</i> : for ordinary steel parts or shrink fits on light sections. The tightest fit usable on cast iron.	H7/s6
<i>Force fit</i> : suitable for parts which can be highly stressed or for shrink fits where the heavy pressing forces required are impractical.	H7/u6

For the hole

$$D_{\max} = D + \Delta_D$$

$$D_{\min} = D$$

For a shaft with clearance fits  $d$ ,  $g$ ,  $h$ ,  $c$ , or  $f$

$$d_{\max} = d + \delta_F$$

$$d_{\min} = d_{\max} - \Delta_d$$

For a shaft with transition or interference fits  $k$ ,  $p$ ,  $s$ ,  $u$ , or  $n$

$$d_{\min} = d + \delta_F$$

$$d_{\max} = d_{\min} + \Delta_d$$

where

$D$  = basic size of hole

$d$  = basic size of shaft

$\delta_u$  = upper deviation

$\delta_l$  = lower deviation

$\delta_F$  = fundamental deviation

$\Delta_D$  = tolerance grade for hole

$\Delta_d$  = tolerance grade for shaft

### International Tolerance (IT) Grades

Lower limit < Basic Size ≤ Upper Limit  
All values in mm

Basic Size	Tolerance Grade, ( $\Delta_D$ or $\Delta_d$ )		
	IT6	IT7	IT9
0–3	0.006	0.010	0.025
3–6	0.008	0.012	0.030
6–10	0.009	0.015	0.036
10–18	0.011	0.018	0.043
18–30	0.013	0.021	0.052
30–50	0.016	0.025	0.062

Source: Preferred Metric Limits and Fits, ANSI B4.2-1978

### Deviations for shafts

Lower limit < Basic Size ≤ Upper Limit  
All values in mm

Basic Size	Upper Deviation Letter, ( $\delta_u$ )					Lower Deviation Letter, ( $\delta_l$ )				
	c	d	f	g	h	k	n	p	s	u
0–3	–0.060	–0.020	–0.006	–0.002	0	0	+0.004	+0.006	+0.014	+0.018
3–6	–0.070	–0.030	–0.010	–0.004	0	+0.001	+0.008	+0.012	+0.019	+0.023
6–10	–0.080	–0.040	–0.013	–0.005	0	+0.001	+0.010	+0.015	+0.023	+0.028
10–14	–0.095	–0.050	–0.016	–0.006	0	+0.001	+0.012	+0.018	+0.028	+0.033
14–18	–0.095	–0.050	–0.016	–0.006	0	+0.001	+0.012	+0.018	+0.028	+0.033
18–24	–0.110	–0.065	–0.020	–0.007	0	+0.002	+0.015	+0.022	+0.035	+0.041
24–30	–0.110	–0.065	–0.020	–0.007	0	+0.002	+0.015	+0.022	+0.035	+0.048
30–40	–0.120	–0.080	–0.025	–0.009	0	+0.002	+0.017	+0.026	+0.043	+0.060
40–50	–0.130	–0.080	–0.025	–0.009	0	+0.002	+0.017	+0.026	+0.043	+0.070

Source: ASME B4.2:2009

As an example, 34H7/s6 denotes a basic size of  $D = d = 34$  mm, an IT class of 7 for the hole, and an IT class of 6 and an “s” fit class for the shaft.

### Maximum Material Condition (MMC)

The maximum material condition defines the dimension of a part such that the part weighs the most. The MMC of a shaft is at the maximum size of the shaft while the MMC of a hole is at the minimum size of the hole.

### Least Material Condition (LMC)

The least material condition or minimum material condition defines the dimensions of a part such that the part weighs the least. The LMC of a shaft is the minimum size of the shaft while the LMC of a hole is at the maximum size of the hole.

### Intermediate- and Long-Length Columns

The slenderness ratio of a column is  $S_r = l/r$ , where  $l$  is the length of the column and  $r$  is the radius of gyration. The radius of gyration of a column cross-section is,  $r = \sqrt{I/A}$  where  $I$  is the area moment of inertia and  $A$  is the cross-sectional area of the column. A column is considered to be intermediate if its slenderness ratio is less than or equal to  $(S_r)_D$ , where

$$(S_r)_D = \pi \sqrt{\frac{2E}{S_y}}, \text{ and}$$

$E$  = Young's modulus of respective member

$S_y$  = yield strength of the column material

For intermediate columns, the critical load is

$$P_{cr} = A \left[ S_y - \frac{1}{E} \left( \frac{S_y S_r}{2\pi} \right)^2 \right], \text{ where}$$

$P_{cr}$  = critical buckling load

$A$  = cross-sectional area of the column

$S_y$  = yield strength of the column material

$E$  = Young's modulus of respective member

$S_r$  = slenderness ratio

For long columns, the critical load is

$$P_{cr} = \frac{\pi^2 EA}{S_r^2}$$

where the variables are as defined above.

For both intermediate and long columns, the effective column length depends on the end conditions. The AISC recommended design values for the effective lengths of columns are, for: rounded-rounded or pinned-pinned ends,  $l_{eff} = l$ ; fixed-free,  $l_{eff} = 2.1l$ ; fixed-pinned,  $l_{eff} = 0.80l$ ; fixed-fixed,  $l_{eff} = 0.65l$ . The effective column length should be used when calculating the slenderness ratio.

## STATIC LOADING FAILURE THEORIES

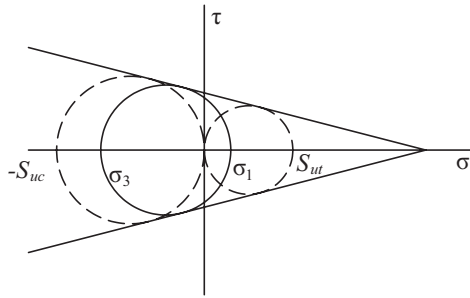
### Brittle Materials

#### Maximum-Normal-Stress Theory

The maximum-normal-stress theory states that failure occurs when one of the three principal stresses equals the strength of the material. If  $\sigma_1 \geq \sigma_2 \geq \sigma_3$ , then the theory predicts that failure occurs whenever  $\sigma_1 \geq S_{ut}$  or  $\sigma_3 \leq -S_{uc}$  where  $S_{ut}$  and  $S_{uc}$  are the tensile and compressive strengths, respectively.

#### Coulomb-Mohr Theory

The Coulomb-Mohr theory is based upon the results of tensile and compression tests. On the  $\sigma, \tau$  coordinate system, one circle is plotted for  $S_{ut}$  and one for  $S_{uc}$ . As shown in the figure, lines are then drawn tangent to these circles. The Coulomb-Mohr theory then states that fracture will occur for any stress situation that produces a circle that is either tangent to or crosses the envelope defined by the lines tangent to the  $S_{ut}$  and  $S_{uc}$  circles.



If  $\sigma_1 \geq \sigma_2 \geq \sigma_3$  and  $\sigma_3 < 0$ , then the theory predicts that yielding will occur whenever

$$\frac{\sigma_1}{S_{ut}} - \frac{\sigma_3}{S_{uc}} \geq 1$$

### Ductile Materials

#### Maximum-Shear-Stress Theory

The maximum-shear-stress theory states that yielding begins when the maximum shear stress equals the maximum shear stress in a tension-test specimen of the same material when that specimen begins to yield. If  $\sigma_1 \geq \sigma_2 \geq \sigma_3$ , then the theory predicts that yielding will occur whenever  $\tau_{\max} \geq S_y/2$  where  $S_y$  is the yield strength.

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

#### Distortion-Energy Theory

The distortion-energy theory states that yielding begins whenever the distortion energy in a unit volume equals the distortion energy in the same volume when uniaxially stressed to the yield strength. The theory predicts that yielding will occur whenever

$$\left[ \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2} \right]^{1/2} \geq S_y$$

The term on the left side of the inequality is known as the effective or Von Mises stress. For a biaxial stress state the effective stress becomes

$$\sigma' = (\sigma_A^2 - \sigma_A \sigma_B + \sigma_B^2)^{1/2}$$

or

$$\sigma' = (\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2)^{1/2}$$

where  $\sigma_A$  and  $\sigma_B$  are the two nonzero principal stresses and  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  are the stresses in orthogonal directions.

## VARIABLE LOADING FAILURE THEORIES

**Modified Goodman Theory:** The modified Goodman criterion states that a fatigue failure will occur whenever

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} \geq 1 \quad \text{or} \quad \frac{\sigma_{\max}}{S_y} \geq 1, \quad \sigma_m \geq 0,$$

where

$S_e$  = fatigue strength

$S_{ut}$  = ultimate strength

$S_y$  = yield strength

$\sigma_a$  = alternating stress

$\sigma_m$  = mean stress

$\sigma_{\max} = \sigma_m + \sigma_a$

**Soderberg Theory:** The Soderberg theory states that a fatigue failure will occur whenever

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} \geq 1 \quad \sigma_m \geq 0$$

**Endurance Limit for Steels:** When test data is unavailable, the endurance limit for steels may be estimated as

$$S'_e = \begin{cases} 0.5 S_{ut}, & S_{ut} \leq 1,400 \text{ MPa} \\ 700 \text{ MPa}, & S_{ut} > 1,400 \text{ MPa} \end{cases}$$

**Endurance Limit Modifying Factors:** Endurance limit modifying factors are used to account for the differences between the endurance limit as determined from a rotating beam test,  $S'_e$ , and that which would result in the real part,  $S_e$ .

$$S_e = k_a k_b k_c k_d k_e S'_e$$

where

Surface Factor,  $k_a = a S_{ut}^b$

Surface Finish	Factor $a$		Exponent $b$
	kpsi	MPa	
Ground	1.34	1.58	-0.085
Machined or CD	2.70	4.51	-0.265
Hot rolled	14.4	57.7	-0.718
As forged	39.9	272.0	-0.995

*Size Factor,  $k_b$ :*

For bending and torsion:

$$d \leq 8 \text{ mm}; \quad k_b = 1$$

$$8 \text{ mm} \leq d \leq 250 \text{ mm}; \quad k_b = 1.189d_{\text{eff}}^{-0.097}$$

$$d > 250 \text{ mm}; \quad 0.6 \leq k_b \leq 0.75$$

$$\text{For axial loading:} \quad k_b = 1$$

*Load Factor,  $k_c$ :*

$$k_c = 0.923 \quad \text{axial loading, } S_{ut} \leq 1,520 \text{ MPa}$$

$$k_c = 1 \quad \text{axial loading, } S_{ut} > 1,520 \text{ MPa}$$

$$k_c = 1 \quad \text{bending}$$

$$k_c = 0.577 \quad \text{torsion}$$

*Temperature Factor,  $k_d$ :*

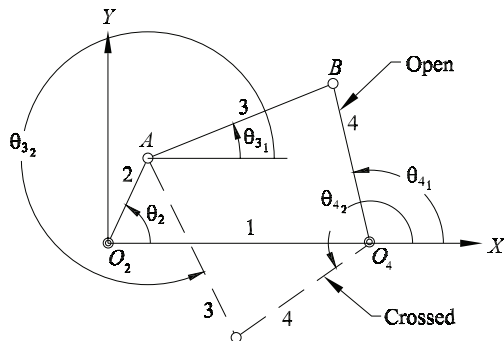
$$\text{for } T \leq 450^\circ\text{C}, k_d = 1$$

*Miscellaneous Effects Factor,  $k_e$ :* Used to account for strength reduction effects such as corrosion, plating, and residual stresses. In the absence of known effects, use  $k_e = 1$ .

## KINEMATICS, DYNAMICS, AND VIBRATIONS

### Kinematics of Mechanisms

#### Four-Bar Linkage



The four-bar linkage shown above consists of a reference (usually grounded) link (1), a crank (input) link (2), a coupler link (3), and an output link (4). Links 2 and 4 rotate about the fixed pivots  $O_2$  and  $O_4$ , respectively. Link 3 is joined to link 2 at the moving pivot  $A$  and to link 4 at the moving pivot  $B$ . The lengths of links 2, 3, 4, and 1 are  $a$ ,  $b$ ,  $c$ , and  $d$ , respectively.

Taking link 1 (ground) as the reference ( $X$ -axis), the angles that links 2, 3, and 4 make with the axis are  $\theta_2$ ,  $\theta_3$ , and  $\theta_4$ , respectively. It is possible to assemble a four-bar in two different configurations for a given position of the input link (2). These are known as the “open” and “crossed” positions or circuits.

*Position Analysis.* Given  $a$ ,  $b$ ,  $c$ , and  $d$ , and  $\theta_2$

$$\theta_{4,2} = 2 \arctan \left( \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right)$$

where  $A = \cos \theta_2 - K_1 - K_2 \cos \theta_2 + K_3$

$$B = -2 \sin \theta_2$$

$$C = K_1 - (K_2 + 1) \cos \theta_2 + K_3$$

$$K_1 = \frac{d}{a}, K_2 = \frac{d}{c}, K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

In the equation for  $\theta_4$ , using the minus sign in front of the radical yields the open solution. Using the plus sign yields the crossed solution.

$$\theta_{3,2} = 2 \arctan \left( \frac{-E \pm \sqrt{E^2 - 4DF}}{2D} \right)$$

where  $D = \cos \theta_2 - K_1 + K_4 \cos \theta_2 + K_5$

$$E = -2 \sin \theta_2$$

$$F = K_1 + (K_4 - 1) \cos \theta_2 + K_5$$

$$K_4 = \frac{d}{b}, K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2ab}$$

In the equation for  $\theta_3$ , using the minus sign in front of the radical yields the open solution. Using the plus sign yields the crossed solution.

*Velocity Analysis.* Given  $a$ ,  $b$ ,  $c$ , and  $d$ ,  $\theta_2$ ,  $\theta_3$ ,  $\theta_4$ , and  $\omega_2$

$$\omega_3 = \frac{a\omega_2}{b} \frac{\sin(\theta_4 - \theta_2)}{\sin(\theta_3 - \theta_4)}$$

$$\omega_4 = \frac{a\omega_2}{c} \frac{\sin(\theta_2 - \theta_3)}{\sin(\theta_4 - \theta_3)}$$

$$V_{Ax} = -a\omega_2 \sin \theta_2, \quad V_{Ay} = a\omega_2 \cos \theta_2$$

$$V_{Bx} = -b\omega_3 \sin \theta_3, \quad V_{By} = b\omega_3 \cos \theta_3$$

$$V_{Bx} = -c\omega_4 \sin \theta_4, \quad V_{By} = c\omega_4 \cos \theta_4$$

*Acceleration Analysis.* Given  $a$ ,  $b$ ,  $c$ , and  $d$ ,  $\theta_2$ ,  $\theta_3$ ,  $\theta_4$ , and  $\omega_2$ ,  $\omega_3$ ,  $\omega_4$ , and  $\alpha_2$

$$\alpha_3 = \frac{CD - AF}{AE - BD}, \quad \alpha_4 = \frac{CE - BF}{AE - BD}, \text{ where}$$

$$A = c \sin \theta_4, B = b \sin \theta_3$$

$$C = a\alpha_2 \sin \theta_2 + a\omega_2^2 \cos \theta_2 + b\omega_3^2 \cos \theta_3 - c\omega_4^2 \cos \theta_4$$

$$D = c \cos \theta_4, E = b \cos \theta_3$$

$$F = a\alpha_2 \cos \theta_2 - a\omega_2^2 \sin \theta_2 - b\omega_3^2 \sin \theta_3 + c\omega_4^2 \sin \theta_4$$

## HVAC

### Nomenclature

$h$  = specific enthalpy

$h_f$  = specific enthalpy of saturated liquid

$\dot{m}_a$  = mass flow rate of dry air

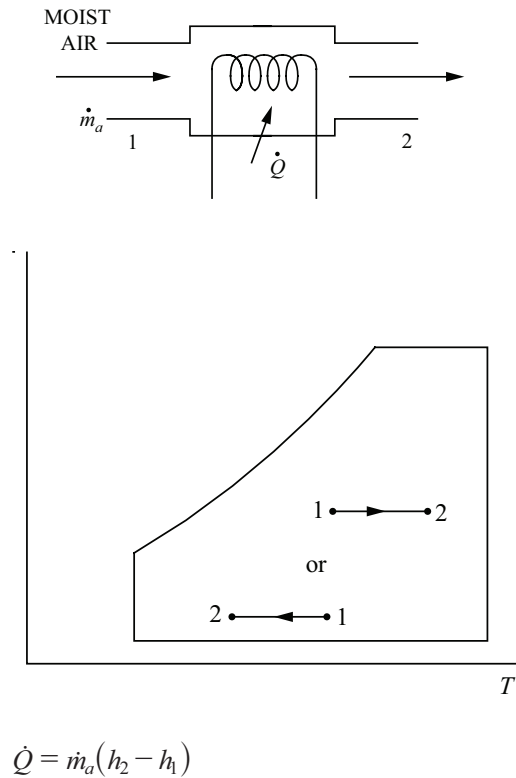
$\dot{m}_w$  = mass flow rate of water

$\dot{Q}$  = rate of heat transfer

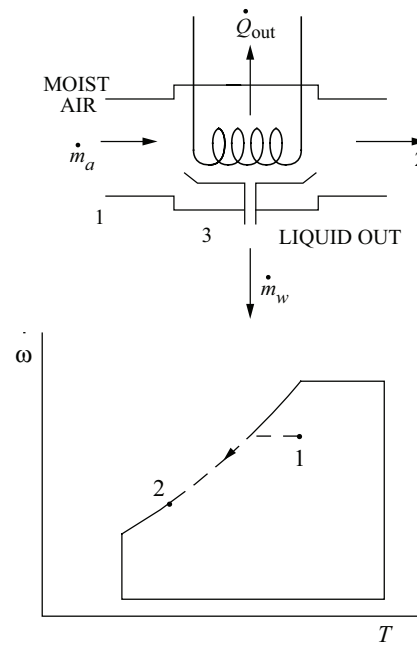
$T_{wb}$  = wet bulb temperature

$\omega$  = specific humidity (absolute humidity, humidity ratio)

### HVAC—Pure Heating and Cooling



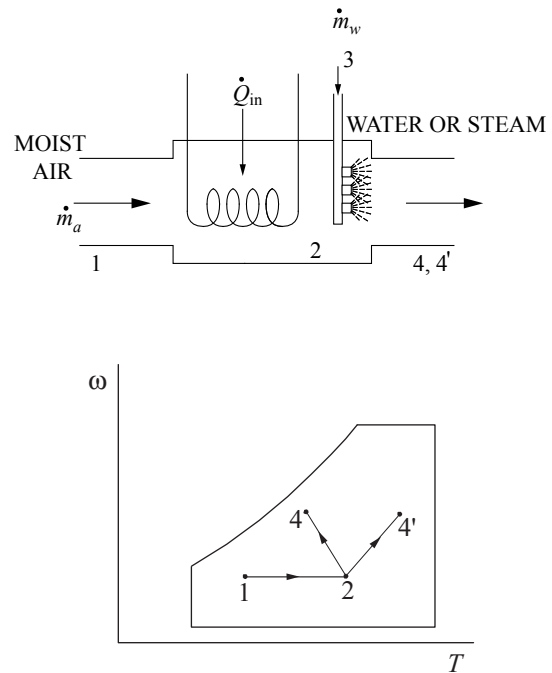
### Cooling and Dehumidification



$$\dot{Q}_{out} = \dot{m}_a[(h_1 - h_2) - h_{f3}(\omega_1 - \omega_2)]$$

$$\dot{m}_w = \dot{m}_a(\omega_1 - \omega_2)$$

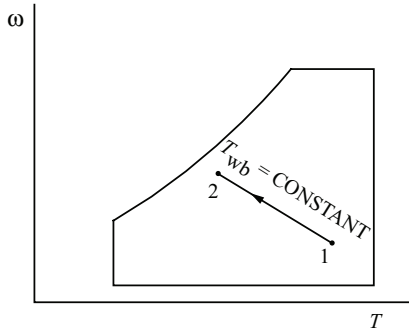
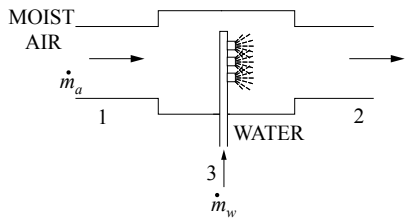
### Heating and Humidification



$$\dot{Q}_{in} = \dot{m}_a(h_2 - h_1)$$

$$\dot{m}_w = \dot{m}_a(\omega_{4'} - \omega_2) \text{ or } \dot{m}_w = \dot{m}_a(\omega_4 - \omega_2)$$

### Adiabatic Humidification (evaporative cooling)

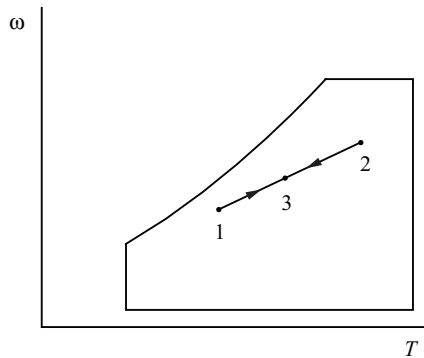
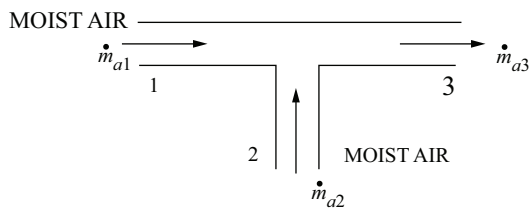


$$h_2 = h_1 + h_3(\omega_2 - \omega_1)$$

$$\dot{m}_w = \dot{m}_a(\omega_2 - \omega_1)$$

$$h_3 = h_f \text{ at } T_{wb}$$

### Adiabatic Mixing



$$\dot{m}_{a3} = \dot{m}_{a1} + \dot{m}_{a2}$$

$$h_3 = \frac{\dot{m}_{a1}h_1 + \dot{m}_{a2}h_2}{\dot{m}_{a3}}$$

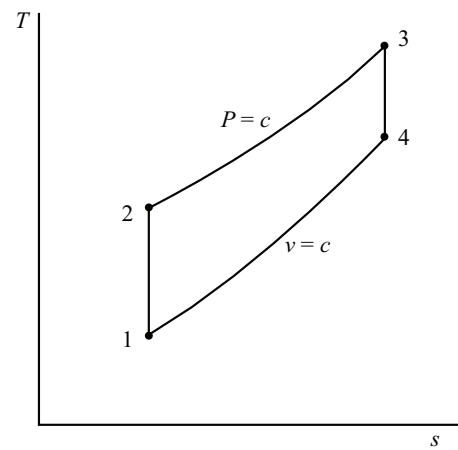
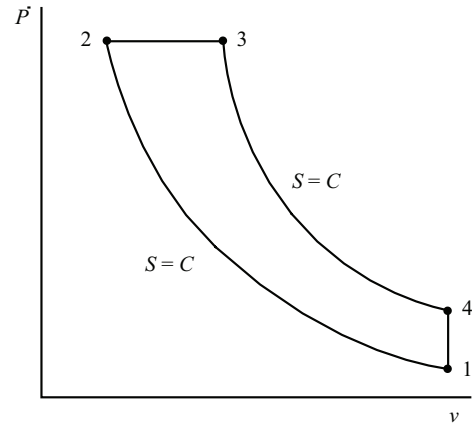
$$\omega_3 = \frac{\dot{m}_{a1}\omega_1 + \dot{m}_{a2}\omega_2}{\dot{m}_{a3}}$$

distance  $\overline{13} = \frac{\dot{m}_{a2}}{\dot{m}_{a3}} \times \text{distance } \overline{12} \text{ measured on psychrometric chart}$

### Cycles and Processes

#### Internal Combustion Engines

##### Diesel Cycle



$$r = V_1/V_2$$

$$r_c = V_3/V_2$$

$$\eta = 1 - \frac{1}{r^{k-1}} \left[ \frac{r_c^k - 1}{k(r_c - 1)} \right]$$

$$k = c_p/c_v$$



### Brake Power

$$\dot{W}_b = 2\pi TN = 2\pi FRN, \text{ where}$$

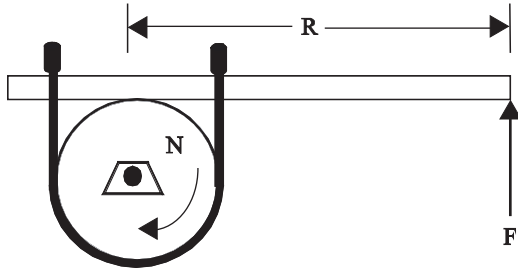
$\dot{W}_b$  = brake power (W)

$T$  = torque (N•m)

$N$  = rotation speed (rev/s)

$F$  = force at end of brake arm (N)

$R$  = length of brake arm (m)



### Indicated Power

$$\dot{W}_i = \dot{W}_b + \dot{W}_f, \text{ where}$$

$\dot{W}_i$  = indicated power (W)

$\dot{W}_f$  = friction power (W)

### Brake Thermal Efficiency

$$\eta_b = \frac{\dot{W}_b}{\dot{m}_f (HV)}, \text{ where}$$

$\eta_b$  = brake thermal efficiency

$\dot{m}_f$  = fuel consumption rate (kg/s)

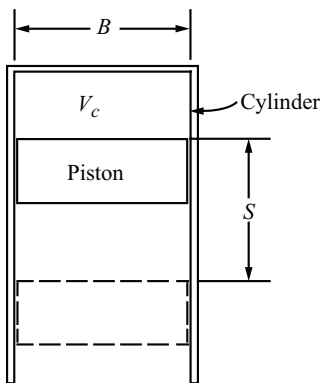
$HV$  = heating value of fuel (J/kg)

### Indicated Thermal Efficiency

$$\eta_i = \frac{\dot{W}_i}{\dot{m}_f (HV)}$$

### Mechanical Efficiency

$$\eta_i = \frac{\dot{W}_b}{\dot{W}_i} = \frac{\eta_b}{\eta_i}$$



### Displacement Volume

$$V_d = \frac{\pi B^2 S}{4}, \text{ m}^3 \text{ for each cylinder}$$

$$\text{Total volume} = V_t = V_d + V_c \text{ (m}^3\text{)}$$

$$V_c = \text{clearance volume (m}^3\text{)}$$

### Compression Ratio

$$r_c = V_t / V_c$$

### Mean Effective Pressure (mep)

$$mep = \frac{\dot{W} n_s}{V_d n_c N}, \text{ where}$$

$n_s$  = number of crank revolutions per power stroke

$n_c$  = number of cylinders

$V_d$  = displacement volume per cylinder

$mep$  can be based on brake power ( $bme_p$ ), indicated power ( $ime_p$ ), or friction power ( $fme_p$ ).

### Volumetric Efficiency

$$\eta_v = \frac{2\dot{m}_a}{\rho_a V_d n_c N} \quad (\text{four-stroke cycles only})$$

where

$\dot{m}_a$  = mass flow rate of air into engine (kg/s)

$\rho_a$  = density of air (kg/m<sup>3</sup>)

### Specific Fuel Consumption (SFC)

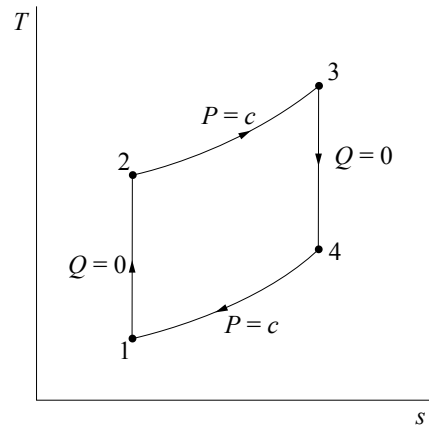
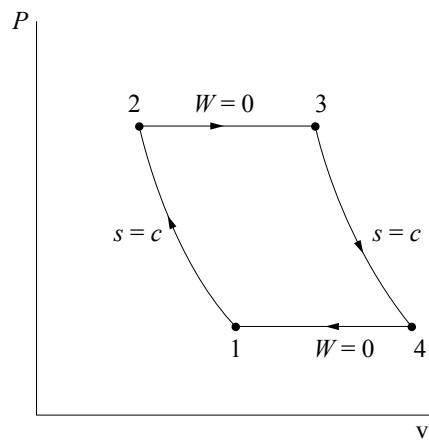
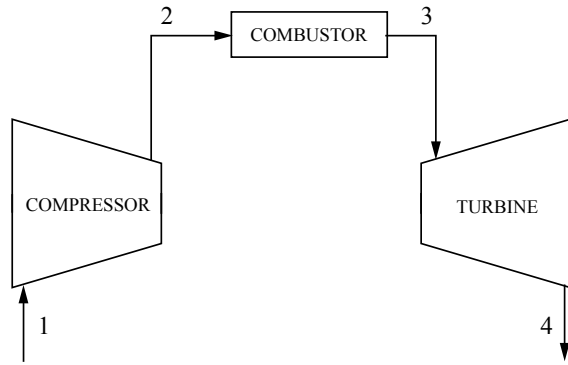
$$sfc = \frac{\dot{m}_f}{\dot{W}} = \frac{1}{\eta HV}, \text{ kg/J}$$

Use  $\eta_b$  and  $\dot{W}_b$  for  $bsfc$  and  $\eta_i$  and  $\dot{W}_i$  for  $isfc$ .



## Gas Turbines

### Brayton Cycle (Steady-Flow Cycle)



$$w_{12} = h_1 - h_2 = c_p (T_1 - T_2)$$

$$w_{34} = h_3 - h_4 = c_p (T_3 - T_4)$$

$$w_{\text{net}} = w_{12} + w_{34}$$

$$q_{23} = h_3 - h_2 = c_p (T_3 - T_2)$$

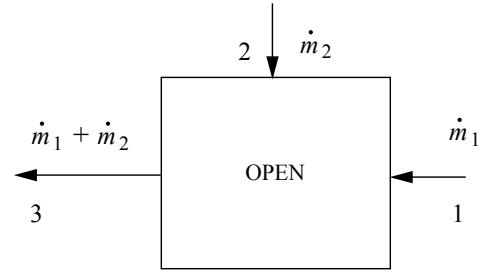
$$q_{41} = h_1 - h_4 = c_p (T_1 - T_4)$$

$$q_{\text{net}} = q_{23} + q_{41}$$

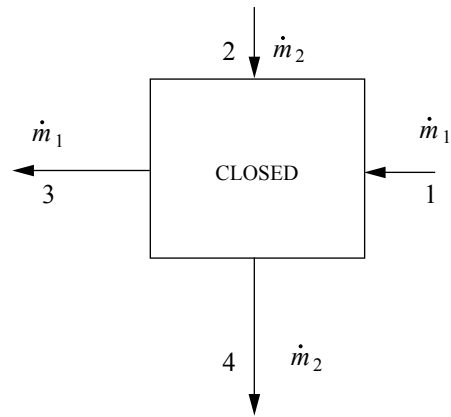
$$\eta = w_{\text{net}} / q_{23}$$

## Steam Power Plants

### Feedwater Heaters

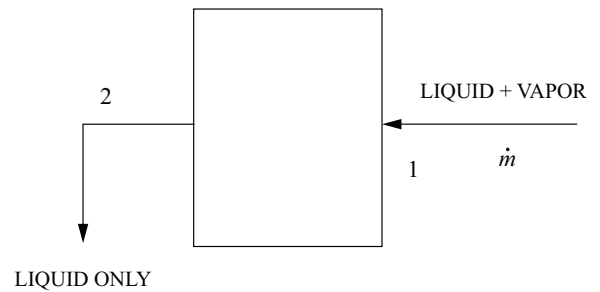


$$\dot{m}_1 h_1 + \dot{m}_2 h_2 = h_3 (\dot{m}_1 + \dot{m}_2)$$



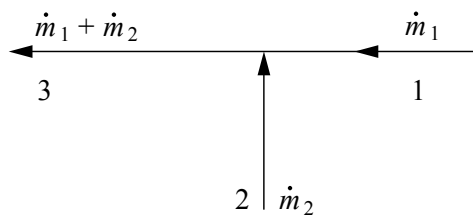
$$\dot{m}_1 h_1 + \dot{m}_2 h_2 = \dot{m}_1 h_3 + \dot{m}_2 h_4$$

### Steam Trap



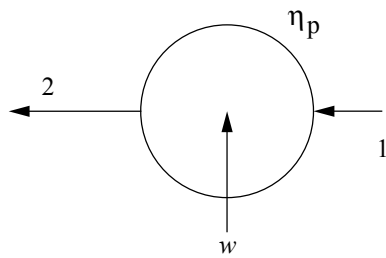
$$h_2 = h_1$$

### *Junction*



$$\dot{m}_1 h_1 + \dot{m}_2 h_2 = h_3 (\dot{m}_1 + \dot{m}_2)$$

### *Pump*



$$w = h_1 - h_2 = (h_1 - h_{2S}) / \eta_P$$

$$h_{2S} - h_1 = v(P_2 - P_1)$$

$$w = - \frac{v(P_2 - P_1)}{\eta_P}$$

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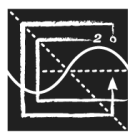
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## Fundamentals of Engineering (FE) CHEMICAL CBT Exam Specifications

Effective Beginning with the January 2014 Examinations

- The FE exam is a computer-based test (CBT). It is closed book with an electronic reference.
- Examinees have 6 hours to complete the exam, which contains 110 multiple-choice questions. The 6-hour time also includes a tutorial, a break, and a brief survey at the conclusion.
- The FE exam uses both the International System of Units (SI) and the US Customary System (USCS).

Knowledge	Number of Questions
<b>1. Mathematics</b>	<b>8–12</b>
A. Analytic geometry	
B. Roots of equations	
C. Calculus	
D. Differential equations	
<b>2. Probability and Statistics</b>	<b>4–6</b>
A. Probability distributions (e.g., discrete, continuous, normal, binomial)	
B. Expected value (weighted average) in decision making	
C. Hypothesis testing	
D. Measures of central tendencies and dispersions (e.g., mean, mode, standard deviation)	
E. Estimation for a single mean (e.g., point, confidence intervals)	
F. Regression and curve fitting	
<b>3. Engineering Sciences</b>	<b>4–6</b>
A. Applications of vector analysis (e.g., statics)	
B. Basic dynamics (e.g., friction, force, mass, acceleration, momentum)	
C. Work, energy, and power (as applied to particles or rigid bodies)	
D. Electricity and current and voltage laws (e.g., charge, energy, current, voltage, power, Kirchhoff, Ohm)	
<b>4. Computational Tools</b>	<b>4–6</b>
A. Numerical methods and concepts (e.g., convergence, tolerance)	
B. Spreadsheets for chemical engineering calculations	
C. Simulators	
<b>5. Materials Science</b>	<b>4–6</b>
A. Chemical, electrical, mechanical, and physical properties (e.g., effect of temperature, pressure, stress, strain)	
B. Material types and compatibilities (e.g., engineered materials, ferrous and nonferrous metals)	
C. Corrosion mechanisms and control	

- |   |             |
|---|-------------|
| <b>6. Chemistry</b>   | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Inorganic chemistry (e.g., molarity, normality, molality, acids, bases, redox reactions, valence, solubility product, pH, pK, electrochemistry, periodic table)</li> <li>B. Organic chemistry (e.g., nomenclature, structure, qualitative and quantitative analyses, balanced equations, reactions, synthesis, basic biochemistry)</li> </ul>   |             |
| <b>7. Fluid Mechanics/Dynamics</b>  | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Fluid properties</li> <li>B. Dimensionless numbers (e.g., Reynolds number)</li> <li>C. Mechanical energy balance (e.g., pipes, valves, fittings, pressure losses across packed beds, pipe networks)</li> <li>D. Bernoulli equation (hydrostatic pressure, velocity head)</li> <li>E. Laminar and turbulent flow</li> <li>F. Flow measurement (e.g., orifices, Venturi meters)</li> <li>G. Pumps, turbines, and compressors</li> <li>H. Compressible flow and non-Newtonian fluids</li> </ul>  |             |
| <b>8. Thermodynamics</b>  | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Thermodynamic properties (e.g. specific volume, internal energy, enthalpy, entropy, free energy)</li> <li>B. Properties data and phase diagrams (e.g. steam tables, psychrometric charts, T-s, P-h, x-y, T-x-y)</li> <li>C. Thermodynamic laws (e.g., 1st law, 2nd law)</li> <li>D. Thermodynamic processes (e.g., isothermal, adiabatic, isentropic)</li> <li>E. Cyclic processes and efficiency (e.g., power, refrigeration, heat pump)</li> <li>F. Phase equilibrium (e.g., fugacity, activity coefficient)</li> <li>G. Chemical equilibrium</li> <li>H. Heats of reaction and mixing</li> </ul> |             |
| <b>9. Material/Energy Balances</b>  | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Mass balance (steady and unsteady state)</li> <li>B. Energy balance (steady and unsteady state)</li> <li>C. Recycle/bypass processes</li> <li>D. Reactive systems (e.g., combustion)</li> </ul>   |             |
| <b>10. Heat Transfer</b>  | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Conductive heat transfer</li> <li>B. Convective heat transfer (natural and forced)</li> <li>C. Radiation heat transfer</li> <li>D. Heat transfer coefficients (e.g., overall, local, fouling)</li> <li>E. Heat transfer equipment, operation, and design (e.g., double pipe, shell and tube, fouling, number of transfer units, log-mean temperature difference, flow configuration)</li> </ul>   |             |
| <b>11. Mass Transfer and Separation</b>   | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Molecular diffusion (e.g., steady and unsteady state, physical property estimation)</li> <li>B. Convective mass transfer (e.g., mass transfer coefficient, eddy diffusion)</li> <li>C. Separation systems (e.g., distillation, absorption, extraction, membrane processes)</li> </ul>   |             |

- D. Equilibrium stage methods (e.g., graphical methods, McCabe-Thiele, efficiency)
- E. Continuous contact methods (e.g., number of transfer units, height equivalent to a theoretical plate, height of transfer unit, number of theoretical plates)
- F. Humidification and drying

**12. Chemical Reaction Engineering**

**8–12**

- A. Reaction rates and order
- B. Rate constant (e.g., Arrhenius function)
- C. Conversion, yield, and selectivity
- D. Type of reactions (e.g., series, parallel, forward, reverse, homogeneous, heterogeneous, catalysis, biocatalysis)
- E. Reactor types (e.g., batch, semibatch, continuous stirred tank, plug flow, gas phase, liquid phase)

**13. Process Design and Economics**

**8–12**

- A. Process flow diagrams and piping and instrumentation diagrams
- B. Equipment selection (e.g., sizing and scale-up)
- C. Cost estimation
- D. Comparison of economic alternatives (e.g., net present value, discounted cash flow, rate of return, expected value and risk)
- E. Process design and optimization (e.g., sustainability, efficiency, green engineering, inherently safer design, evaluation of specifications)

**14. Process Control**

**5–8**

- A. Dynamics (e.g., time constants and 2nd order, underdamped, and transfer functions)
- B. Control strategies (e.g., feedback, feed-forward, cascade, ratio, and PID)
- C. Control loop design and hardware (e.g., matching measured and manipulated variables, sensors, control valves, and conceptual process control)

**15. Safety, Health, and Environment**

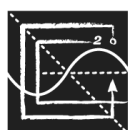
**5–8**

- A. Hazardous properties of materials (e.g., corrosivity, flammability, toxicity, reactivity, handling and storage), including MSDS
- B. Industrial hygiene (e.g., noise, PPE, ergonomics)
- C. Process safety and hazard analysis [e.g., layer of protection analysis, hazard and operability studies (HazOps), fault-tree analysis or event tree]
- D. Overpressure and underpressure protection (e.g., relief, redundant control, intrinsically safe)
- E. Waste minimization, waste treatment, and regulation (e.g., air, water, solids, RCRA, CWA, EPA, OSHA)

**16. Ethics and Professional Practice**

**2–3**

- A. Codes of ethics (professional and technical societies)
- B. Agreements and contracts
- C. Ethical and legal considerations
- D. Professional liability
- E. Public protection issues (e.g., licensing boards)



# NCEES

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## Fundamentals of Engineering (FE) CIVIL CBT Exam Specifications

Effective Beginning with the January 2014 Examinations

- The FE exam is a computer-based test (CBT). It is closed book with an electronic reference.
- Examinees have 6 hours to complete the exam, which contains 110 multiple-choice questions. The 6-hour time also includes a tutorial, a break, and a brief survey at the conclusion.
- The FE exam uses both the International System of Units (SI) and the US Customary System (USCS).

Knowledge	Number of Questions
<b>1. Mathematics</b>	<b>7–11</b>
A. Analytic geometry	
B. Calculus	
C. Roots of equations	
D. Vector analysis	
<b>2. Probability and Statistics</b>	<b>4–6</b>
A. Measures of central tendencies and dispersions (e.g., mean, mode, standard deviation)	
B. Estimation for a single mean (e.g., point, confidence intervals)	
C. Regression and curve fitting	
D. Expected value (weighted average) in decision making	
<b>3. Computational Tools</b>	<b>4–6</b>
A. Spreadsheet computations	
B. Structured programming (e.g., if-then, loops, macros)	
<b>4. Ethics and Professional Practice</b>	<b>4–6</b>
A. Codes of ethics (professional and technical societies)	
B. Professional liability	
C. Licensure	
D. Sustainability and sustainable design	
E. Professional skills (e.g., public policy, management, and business)	
F. Contracts and contract law	
<b>5. Engineering Economics</b>	<b>4–6</b>
A. Discounted cash flow (e.g., equivalence, PW, equivalent annual worth, FW, rate of return)	
B. Cost (e.g., incremental, average, sunk, estimating)	
C. Analyses (e.g., breakeven, benefit-cost, life cycle)	
D. Uncertainty (e.g., expected value and risk)	
<b>6. Statics</b>	<b>7–11</b>
A. Resultants of force systems	
B. Equivalent force systems	
C. Equilibrium of rigid bodies	
D. Frames and trusses	

- E. Centroid of area
- F. Area moments of inertia
- G. Static friction

**7. Dynamics 4–6**

- A. Kinematics (e.g., particles and rigid bodies)
- B. Mass moments of inertia
- C. Force acceleration (e.g., particles and rigid bodies)
- D. Impulse momentum (e.g., particles and rigid bodies)
- E. Work, energy, and power (e.g., particles and rigid bodies)

**8. Mechanics of Materials 7–11**

- A. Shear and moment diagrams
- B. Stresses and strains (e.g., axial, torsion, bending, shear, thermal)
- C. Deformations (e.g., axial, torsion, bending, thermal)
- D. Combined stresses
- E. Principal stresses
- F. Mohr's circle
- G. Column analysis (e.g., buckling, boundary conditions)
- H. Composite sections
- I. Elastic and plastic deformations
- J. Stress-strain diagrams

**9. Materials 4–6**

- A. Mix design (e.g., concrete and asphalt)
- B. Test methods and specifications (e.g., steel, concrete, aggregates, asphalt, wood)
- C. Physical and mechanical properties of concrete, ferrous and nonferrous metals, masonry, wood, engineered materials (e.g., FRP, laminated lumber, wood/plastic composites), and asphalt

**10. Fluid Mechanics 4–6**

- A. Flow measurement
- B. Fluid properties
- C. Fluid statics
- D. Energy, impulse, and momentum equations

**11. Hydraulics and Hydrologic Systems 8–12**

- A. Basic hydrology (e.g., infiltration, rainfall, runoff, detention, flood flows, watersheds)
- B. Basic hydraulics (e.g., Manning equation, Bernoulli theorem, open-channel flow, pipe flow)
- C. Pumping systems (water and wastewater)
- D. Water distribution systems
- E. Reservoirs (e.g., dams, routing, spillways)
- F. Groundwater (e.g., flow, wells, drawdown)
- G. Storm sewer collection systems

**12. Structural Analysis 6–9**

- A. Analysis of forces in statically determinant beams, trusses, and frames
- B. Deflection of statically determinant beams, trusses, and frames
- C. Structural determinacy and stability analysis of beams, trusses, and frames



- D. Loads and load paths (e.g., dead, live, lateral, influence lines and moving loads, tributary areas)
- E. Elementary statically indeterminate structures

**13. Structural Design** **6–9**

- A. Design of steel components (e.g., codes and design philosophies, beams, columns, beam-columns, tension members, connections)
- B. Design of reinforced concrete components (e.g., codes and design philosophies, beams, slabs, columns, walls, footings)

**14. Geotechnical Engineering** **9–14**

- A. Geology
- B. Index properties and soil classifications
- C. Phase relations (air-water-solid)
- D. Laboratory and field tests
- E. Effective stress (buoyancy)
- F. Stability of retaining walls (e.g., active pressure/passive pressure)
- G. Shear strength
- H. Bearing capacity (cohesive and noncohesive)
- I. Foundation types (e.g., spread footings, deep foundations, wall footings, mats)
- J. Consolidation and differential settlement
- K. Seepage/flow nets
- L. Slope stability (e.g., fills, embankments, cuts, dams)
- M. Soil stabilization (e.g., chemical additives, geosynthetics)
- N. Drainage systems
- O. Erosion control

**15. Transportation Engineering** **8–12**

- A. Geometric design of streets and highways
- B. Geometric design of intersections
- C. Pavement system design (e.g., thickness, subgrade, drainage, rehabilitation)
- D. Traffic safety
- E. Traffic capacity
- F. Traffic flow theory
- G. Traffic control devices
- H. Transportation planning (e.g., travel forecast modeling)

**16. Environmental Engineering** **6–9**

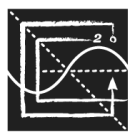
- A. Water quality (ground and surface)
- B. Basic tests (e.g., water, wastewater, air)
- C. Environmental regulations
- D. Water supply and treatment
- E. Wastewater collection and treatment

**17. Construction****4–6**

- A. Construction documents
- B. Procurement methods (e.g., competitive bid, qualifications-based)
- C. Project delivery methods (e.g., design-bid-build, design build, construction management, multiple prime)
- D. Construction operations and methods (e.g., lifting, rigging, dewatering and pumping, equipment production, productivity analysis and improvement, temporary erosion control)
- E. Project scheduling (e.g., CPM, allocation of resources)
- F. Project management (e.g., owner/contractor/client relations)
- G. Construction safety
- H. Construction estimating

**18. Surveying****4–6**

- A. Angles, distances, and trigonometry
- B. Area computations
- C. Earthwork and volume computations
- D. Closure
- E. Coordinate systems (e.g., state plane, latitude/longitude)
- F. Leveling (e.g., differential, elevations, percent grades)



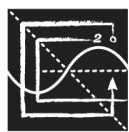
## Fundamentals of Engineering (FE) ELECTRICAL AND COMPUTER CBT Exam Specifications Effective Beginning with the January 2014 Examinations

- The FE exam is a computer-based test (CBT). It is closed book with an electronic reference.
- Examinees have 6 hours to complete the exam, which contains 110 multiple-choice questions. The 6-hour time also includes a tutorial, a break, and a brief survey at the conclusion.
- The FE exam uses both the International System of Units (SI) and the US Customary System (USCS).

Knowledge	Number of Questions
<b>1. Mathematics</b>	<b>11–17</b>
A. Algebra and trigonometry	
B. Complex numbers	
C. Discrete mathematics	
D. Analytic geometry	
E. Calculus	
F. Differential equations	
G. Linear algebra	
H. Vector analysis	
<b>2. Probability and Statistics</b>	<b>4–6</b>
A. Measures of central tendencies and dispersions (e.g., mean, mode, standard deviation)	
B. Probability distributions (e.g., discrete, continuous, normal, binomial)	
C. Expected value (weighted average) in decision making	
D. Estimation for a single mean (e.g., point, confidence intervals, conditional probability)	
<b>3. Ethics and Professional Practice</b>	<b>3–5</b>
A. Codes of ethics (professional and technical societies)	
B. NCEES Model Law and Model Rules	
C. Intellectual property (e.g., copyright, trade secrets, patents)	
<b>4. Engineering Economics</b>	<b>3–5</b>
A. Time value of money (e.g., present value, future value, annuities)	
B. Cost estimation	
C. Risk identification	
D. Analysis (e.g., cost-benefit, trade-off, breakeven)	
<b>5. Properties of Electrical Materials</b>	<b>4–6</b>
A. Chemical (e.g., corrosion, ions, diffusion)	
B. Electrical (e.g., conductivity, resistivity, permittivity, magnetic permeability)	
C. Mechanical (e.g., piezoelectric, strength)	
D. Thermal (e.g., conductivity, expansion)	

- |  |              |
|--|--------------|
| <b>6. Engineering Sciences</b>   | <b>6–9</b>   |
| <ul style="list-style-type: none"> <li>A. Work, energy, power, heat</li> <li>B. Charge, energy, current, voltage, power</li> <li>C. Forces (e.g., between charges, on conductors)</li> <li>D. Work done in moving a charge in an electric field (relationship between voltage and work)</li> <li>E. Capacitance</li> <li>F. Inductance</li> </ul>  |              |
| <b>7. Circuit Analysis (DC and AC Steady State)</b>  | <b>10–15</b> |
| <ul style="list-style-type: none"> <li>A. KCL, KVL</li> <li>B. Series/parallel equivalent circuits</li> <li>C. Thevenin and Norton theorems</li> <li>D. Node and loop analysis</li> <li>E. Waveform analysis (e.g., RMS, average, frequency, phase, wavelength)</li> <li>F. Phasors</li> <li>G. Impedance</li> </ul>   |              |
| <b>8. Linear Systems</b>   | <b>5–8</b>   |
| <ul style="list-style-type: none"> <li>A. Frequency/transient response</li> <li>B. Resonance</li> <li>C. Laplace transforms</li> <li>D. Transfer functions</li> <li>E. 2-port theory</li> </ul>  |              |
| <b>9. Signal Processing</b>  | <b>5–8</b>   |
| <ul style="list-style-type: none"> <li>A. Convolution (continuous and discrete)</li> <li>B. Difference equations</li> <li>C. Z-transforms</li> <li>D. Sampling (e.g., aliasing, Nyquist theorem)</li> <li>E. Analog filters</li> <li>F. Digital filters</li> </ul>   |              |
| <b>10. Electronics</b>   | <b>7–11</b>  |
| <ul style="list-style-type: none"> <li>A. Solid-state fundamentals (e.g., tunneling, diffusion/drift current, energy bands, doping bands, p-n theory)</li> <li>B. Discrete devices (diodes, transistors, BJT, CMOS) and models and their performance</li> <li>C. Bias circuits</li> <li>D. Amplifiers (e.g., single-stage/common emitter, differential)</li> <li>E. Operational amplifiers (ideal, non-ideal)</li> <li>F. Instrumentation (e.g., measurements, data acquisition, transducers)</li> <li>G. Power electronics</li> </ul> |              |
| <b>11. Power</b>   | <b>8–12</b>  |
| <ul style="list-style-type: none"> <li>A. Single phase and three phase</li> <li>B. Transmission and distribution</li> <li>C. Voltage regulation</li> <li>D. Transformers</li> <li>E. Motors and generators</li> <li>F. Power factor (pf)</li> </ul>  |              |

<b>12. Electromagnetics</b>	<b>5–8</b>
A. Maxwell equations	
B. Electrostatics/magnetostatics (e.g., measurement of spatial relationships, vector analysis)	
C. Wave propagation	
D. Transmission lines (high frequency)	
E. Electromagnetic compatibility	
<b>13. Control Systems</b>	<b>6–9</b>
A. Block diagrams (feed-forward, feedback)	
B. Bode plots	
C. Closed-loop and open-loop response	
D. Controller performance (gain, PID), steady-state errors	
E. Root locus	
F. Stability	
G. State variables	
<b>14. Communications</b>	<b>5–8</b>
A. Basic modulation/demodulation concepts (e.g., AM, FM, PCM)	
B. Fourier transforms/Fourier series	
C. Multiplexing (e.g., time division, frequency division)	
D. Digital communications	
<b>15. Computer Networks</b>	<b>3–5</b>
A. Routing and switching	
B. Network topologies/frameworks/models	
C. Local area networks	
<b>16. Digital Systems</b>	<b>7–11</b>
A. Number systems	
B. Boolean logic	
C. Logic gates and circuits	
D. Logic minimization (e.g., SOP, POS, Karnaugh maps)	
E. Flip-flops and counters	
F. Programmable logic devices and gate arrays	
G. State machine design	
H. Data path/controller design	
I. Timing (diagrams, asynchronous inputs, races, hazards)	
<b>17. Computer Systems</b>	<b>4–6</b>
A. Architecture (e.g., pipelining, cache memory)	
B. Microprocessors	
C. Memory technology and systems	
D. Interfacing	
<b>18. Software Development</b>	<b>4–6</b>
A. Algorithms	
B. Data structures	
C. Software design methods (structured, object-oriented)	
D. Software implementation (e.g., procedural, scripting languages)	
E. Software testing	



## Fundamentals of Engineering (FE) ENVIRONMENTAL CBT Exam Specifications

Effective Beginning with the January 2014 Examinations

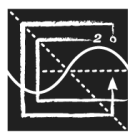
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- Examinees have 6 hours to complete the exam, which contains 110 multiple-choice questions. The 6-hour time also includes a tutorial, a break, and a brief survey at the conclusion.
- The FE exam uses both the International System of Units (SI) and the US Customary System (USCS).

Knowledge	Number of Questions
<b>1. Mathematics</b>	<b>4–6</b>
A. Analytic geometry	
B. Numerical methods	
C. Roots of equations	
D. Calculus	
E. Differential equations	
<b>2. Probability and Statistics</b>	<b>3–5</b>
A. Measures of central tendencies and dispersions (e.g., mean, mode, standard deviation)	
B. Probability distributions (e.g., discrete, continuous, normal, binomial)	
C. Estimation (point, confidence intervals) for a single mean	
D. Regression and curve fitting	
E. Expected value (weighted average) in decision making	
F. Hypothesis testing	
<b>3. Ethics and Professional Practice</b>	<b>5–8</b>
A. Codes of ethics (professional and technical societies)	
B. Agreements and contracts	
C. Ethical and legal considerations	
D. Professional liability	
E. Public protection issues (e.g., licensing boards)	
F. Regulations (e.g., water, wastewater, air, solid/hazardous waste, groundwater/soils)	
<b>4. Engineering Economics</b>	<b>4–6</b>
A. Discounted cash flow (e.g., life cycle, equivalence, PW, equivalent annual worth, FW, rate of return)	
B. Cost (e.g., incremental, average, sunk, estimating)	
C. Analyses (e.g., breakeven, benefit-cost)	
D. Uncertainty (expected value and risk)	
<b>5. Materials Science</b>	<b>3–5</b>
A. Properties (e.g., chemical, electrical, mechanical, physical)	
B. Corrosion mechanisms and controls	
C. Material selection and compatibility	

<b>6. Environmental Science and Chemistry</b>	<b>11–17</b>
A. Reactions (e.g., equilibrium, acid base, oxidation-reduction, precipitation)	
B. Stoichiometry	
C. Kinetics (chemical, microbiological)	
D. Organic chemistry (e.g., nomenclature, functional group reactions)	
E. Ecology (e.g., Streeter-Phelps, fluviology, limnology, eutrophication)	
F. Multimedia equilibrium partitioning (e.g., Henry's law, octonal partitioning coefficient)	
<b>7. Risk Assessment</b>	<b>5–8</b>
A. Dose-response toxicity (carcinogen, noncarcinogen)	
B. Exposure routes	
<b>8. Fluid Mechanics</b>	<b>9–14</b>
A. Fluid statics	
B. Closed conduits (e.g., Darcy-Weisbach, Hazen-Williams, Moody)	
C. Open channel (Manning)	
D. Pumps (e.g., power, operating point, parallel and series)	
E. Flow measurement (e.g., weirs, orifices, flowmeters)	
F. Blowers (e.g., power, operating point, parallel, and series)	
<b>9. Thermodynamics</b>	<b>3–5</b>
A. Thermodynamic laws (e.g., 1st law, 2nd law)	
B. Energy, heat, and work	
C. Ideal gases	
D. Mixture of nonreacting gases	
E. Heat transfer	
<b>10. Water Resources</b>	<b>10–15</b>
A. Demand calculations	
B. Population estimations	
C. Runoff calculations (e.g., land use, land cover, time of concentration, duration, intensity, frequency)	
D. Reservoir sizing	
E. Routing (e.g., channel, reservoir)	
F. Water quality and modeling (e.g., erosion, channel stability, stormwater quality management)	
<b>11. Water and Wastewater</b>	<b>14–21</b>
A. Water and wastewater characteristics	
B. Mass and energy balances	
C. Conventional water treatment processes (e.g., clarification, disinfection, filtration, flocculation, softening, rapid mix)	
D. Conventional wastewater treatment processes (e.g., activated sludge, decentralized wastewater systems, fixed-film system, disinfection, flow equalization, headworks, lagoons)	
E. Alternative treatment process (e.g., conservation and reuse, membranes, nutrient removal, ion exchange, activated carbon, air stripping)	
F. Sludge treatment and handling (e.g., land application, sludge digestion, sludge dewatering)	

- 12. Air Quality** **10–15**
- A. Chemical principles (e.g., ideal gas, mole fractions, stoichiometry, Henry's law)
  - B. Mass balances
  - C. Emissions (factors, rates)
  - D. Atmospheric sciences (e.g., stability classes, dispersion modeling, lapse rates)
  - E. Gas handling and treatment technologies (e.g., hoods, ducts, coolers, biofiltration, scrubbers, adsorbers, incineration)
  - F. Particle handling and treatment technologies (e.g., baghouses, cyclones, electrostatic precipitators, settling velocity)
- 13. Solid and Hazardous Waste** **10–15**
- A. Composting
  - B. Mass balances
  - C. Compatibility
  - D. Landfilling (e.g., siting, design, leachate, material and energy recovery)
  - E. Site characterization and remediation
  - F. Hazardous waste treatment (e.g., physical, chemical, thermal)
  - G. Radioactive waste treatment and disposal
- 14. Groundwater and Soils** **9–14**
- A. Basic hydrogeology (e.g., aquifers, permeability, water table, hydraulic conductivity, saturation, soil characteristics)
  - B. Drawdown (e.g., Jacob, Theis, Thiem)
  - C. Groundwater flow (e.g., Darcy's law, specific capacity, velocity, gradient)
  - D. Soil and groundwater remediation





## Fundamentals of Engineering (FE) INDUSTRIAL CBT Exam Specifications

Effective Beginning with the January 2014 Examinations

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- The FE exam uses both the International System of Units (SI) and the US Customary System (USCS).

Knowledge	Number of Questions
<b>1. Mathematics</b>	<b>6–9</b>
A. Analytic geometry	
B. Calculus	
C. Matrix operations	
D. Vector analysis	
E. Linear algebra	
<b>2. Engineering Sciences</b>	<b>5–8</b>
A. Work, energy, and power	
B. Material properties and selection	
C. Charge, energy, current, voltage, and power	
<b>3. Ethics and Professional Practice</b>	<b>5–8</b>
A. Codes of ethics and licensure	
B. Agreements and contracts	
C. Professional, ethical, and legal responsibility	
D. Public protection and regulatory issues	
<b>4. Engineering Economics</b>	<b>10–15</b>
A. Discounted cash flows (PW, EAC, FW, IRR, amortization)	
B. Types and breakdown of costs (e.g., fixed, variable, direct and indirect labor)	
C. Cost analyses (e.g., benefit-cost, breakeven, minimum cost, overhead)	
D. Accounting (financial statements and overhead cost allocation)	
E. Cost estimation	
F. Depreciation and taxes	
G. Capital budgeting	
<b>5. Probability and Statistics</b>	<b>10–15</b>
A. Combinatorics (e.g., combinations, permutations)	
B. Probability distributions (e.g., normal, binomial, empirical)	
C. Conditional probabilities	
D. Sampling distributions, sample sizes, and statistics (e.g., central tendency, dispersion)	
E. Estimation (e.g., point, confidence intervals)	
F. Hypothesis testing	
G. Regression (linear, multiple)	

- H. System reliability (e.g., single components, parallel and series systems)
- I. Design of experiments (e.g., ANOVA, factorial designs)

**6. Modeling and Computations 8–12**

- A. Algorithm and logic development (e.g., flow charts, pseudocode)
- B. Databases (e.g., types, information content, relational)
- C. Decision theory (e.g., uncertainty, risk, utility, decision trees)
- D. Optimization modeling (e.g., decision variables, objective functions, and constraints)
- E. Linear programming (e.g., formulation, primal, dual, graphical solutions)
- F. Mathematical programming (e.g., network, integer, dynamic, transportation, assignment)
- G. Stochastic models (e.g., queuing, Markov, reliability)
- H. Simulation

**7. Industrial Management 8–12**

- A. Principles (e.g., planning, organizing, motivational theory)
- B. Tools of management (e.g., MBO, reengineering, organizational structure)
- C. Project management (e.g., scheduling, PERT, CPM)
- D. Productivity measures

**8. Manufacturing, Production, and Service Systems 8–12**

- A. Manufacturing processes
- B. Manufacturing systems (e.g., cellular, group technology, flexible)
- C. Process design (e.g., resources, equipment selection, line balancing)
- D. Inventory analysis (e.g., EOQ, safety stock)
- E. Forecasting
- F. Scheduling (e.g., sequencing, cycle time, material control)
- G. Aggregate planning
- H. Production planning (e.g., JIT, MRP, ERP)
- I. Lean enterprises
- J. Automation concepts (e.g., robotics, CIM)
- K. Sustainable manufacturing (e.g., energy efficiency, waste reduction)
- L. Value engineering

**9. Facilities and Logistics 8–12**

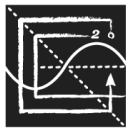
- A. Flow measurements and analysis (e.g., from/to charts, flow planning)
- B. Layouts (e.g., types, distance metrics, planning, evaluation)
- C. Location analysis (e.g., single- and multiple-facility location, warehouses)
- D. Process capacity analysis (e.g., number of machines and people, trade-offs)
- E. Material handling capacity analysis
- F. Supply chain management and design

**10. Human Factors, Ergonomics, and Safety 8–12**

- A. Hazard identification and risk assessment
- B. Environmental stress assessment (e.g., noise, vibrations, heat)
- C. Industrial hygiene
- D. Design for usability (e.g., tasks, tools, displays, controls, user interfaces)
- E. Anthropometry
- F. Biomechanics
- G. Cumulative trauma disorders (e.g., low back injuries, carpal tunnel syndrome)

- H. Systems safety
- I. Cognitive engineering (e.g., information processing, situation awareness, human error, mental models)

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|-----|---|------|
| 11. | <b>Work Design</b>  | 8–12 |
|     | <ul style="list-style-type: none"> <li>A. Methods analysis (e.g., charting, workstation design, motion economy)</li> <li>B. Time study (e.g., time standards, allowances)</li> <li>C. Predetermined time standard systems (e.g., MOST, MTM)</li> <li>D. Work sampling</li> <li>E. Learning curves</li> </ul>                                      |      |
| 12. | <b>Quality</b>  | 8–12 |
|     | <ul style="list-style-type: none"> <li>A. Six sigma</li> <li>B. Management and planning tools (e.g., fishbone, Pareto, QFD, TQM)</li> <li>C. Control charts</li> <li>D. Process capability and specifications</li> <li>E. Sampling plans</li> <li>F. Design of experiments for quality improvement</li> <li>G. Reliability engineering</li> </ul> |      |
| 13. | <b>Systems Engineering</b>  | 8–12 |
|     | <ul style="list-style-type: none"> <li>A. Requirements analysis</li> <li>B. System design</li> <li>C. Human systems integration</li> <li>D. Functional analysis and allocation</li> <li>E. Configuration management</li> <li>F. Risk management</li> <li>G. Verification and assurance</li> <li>H. System life-cycle engineering</li> </ul>       |      |



# NCEES

*advancing licensure for  
engineers and surveyors*

## Fundamentals of Engineering (FE) MECHANICAL CBT Exam Specifications

Effective Beginning with the January 2014 Examinations

- The FE exam is a computer-based test (CBT). It is closed book with an electronic reference.
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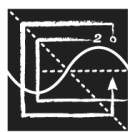
Knowledge	Number of Questions
<b>1. Mathematics</b>	<b>6–9</b>
A. Analytic geometry	
B. Calculus	
C. Linear algebra	
D. Vector analysis	
E. Differential equations	
F. Numerical methods	
<b>2. Probability and Statistics</b>	<b>4–6</b>
A. Probability distributions	
B. Regression and curve fitting	
<b>3. Computational Tools</b>	<b>3–5</b>
A. Spreadsheets	
B. Flow charts	
<b>4. Ethics and Professional Practice</b>	<b>3–5</b>
A. Codes of ethics	
B. Agreements and contracts	
C. Ethical and legal considerations	
D. Professional liability	
E. Public health, safety, and welfare	
<b>5. Engineering Economics</b>	<b>3–5</b>
A. Time value of money	
B. Cost, including incremental, average, sunk, and estimating	
C. Economic analyses	
D. Depreciation	
<b>6. Electricity and Magnetism</b>	<b>3–5</b>
A. Charge, current, voltage, power, and energy	
B. Current and voltage laws (Kirchhoff, Ohm)	
C. Equivalent circuits (series, parallel)	
D. AC circuits	
E. Motors and generators	

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|--|-------------|
| <b>7. Statics</b>  | <b>8–12</b> |
| A. Resultants of force systems   |             |
| B. Concurrent force systems  |             |
| C. Equilibrium of rigid bodies   |             |
| D. Frames and trusses  |             |
| E. Centroids   |             |
| F. Moments of inertia  |             |
| G. Static friction   |             |
| <b>8. Dynamics, Kinematics, and Vibrations</b>                                   | <b>9–14</b> |
| A. Kinematics of particles   |             |
| B. Kinetic friction  |             |
| C. Newton's second law for particles   |             |
| D. Work-energy of particles  |             |
| E. Impulse-momentum of particles   |             |
| F. Kinematics of rigid bodies  |             |
| G. Kinematics of mechanisms  |             |
| H. Newton's second law for rigid bodies  |             |
| I. Work-energy of rigid bodies   |             |
| J. Impulse-momentum of rigid bodies  |             |
| K. Free and forced vibrations  |             |
| <b>9. Mechanics of Materials</b>   | <b>8–12</b> |
| A. Shear and moment diagrams   |             |
| B. Stress types (axial, bending, torsion, shear)                                 |             |
| C. Stress transformations  |             |
| D. Mohr's circle   |             |
| E. Stress and strain caused by axial loads                                       |             |
| F. Stress and strain caused by bending loads                                     |             |
| G. Stress and strain caused by torsion   |             |
| H. Stress and strain caused by shear   |             |
| I. Combined loading  |             |
| J. Deformations  |             |
| K. Columns   |             |
| <b>10. Material Properties and Processing</b>                                    | <b>8–12</b> |
| A. Properties, including chemical, electrical, mechanical, physical, and thermal |             |
| B. Stress-strain diagrams  |             |
| C. Engineered materials  |             |
| D. Ferrous metals  |             |
| E. Nonferrous metals   |             |
| F. Manufacturing processes   |             |
| G. Phase diagrams  |             |
| H. Phase transformation, equilibrium, and heat treating                          |             |
| I. Materials selection   |             |
| J. Surface conditions  |             |
| K. Corrosion mechanisms and control  |             |
| L. Thermal failure   |             |

- M. Ductile or brittle behavior
- N. Fatigue
- O. Crack propagation

- |     |   |       |
|-----|---|-------|
| 11. | <b>Fluid Mechanics</b><br>A. Fluid properties<br>B. Fluid statics<br>C. Energy, impulse, and momentum<br>D. Internal flow<br>E. External flow<br>F. Incompressible flow<br>G. Compressible flow<br>H. Power and efficiency<br>I. Performance curves<br>J. Scaling laws for fans, pumps, and compressors   | 9–14  |
| 12. | <b>Thermodynamics</b><br>A. Properties of ideal gases and pure substances<br>B. Energy transfers<br>C. Laws of thermodynamics<br>D. Processes<br>E. Performance of components<br>F. Power cycles, thermal efficiency, and enhancements<br>G. Refrigeration and heat pump cycles and coefficients of performance<br>H. Nonreacting mixtures of gases<br>I. Psychrometrics<br>J. Heating, ventilating, and air-conditioning (HVAC) processes<br>K. Combustion and combustion products | 13–20 |
| 13. | <b>Heat Transfer</b><br>A. Conduction<br>B. Convection<br>C. Radiation<br>D. Thermal resistance<br>E. Transient processes<br>F. Heat exchangers<br>G. Boiling and condensation  | 9–14  |
| 14. | <b>Measurements, Instrumentation, and Controls</b><br>A. Sensors<br>B. Block diagrams<br>C. System response<br>D. Measurement uncertainty   | 5–8   |
| 15. | <b>Mechanical Design and Analysis</b><br>A. Stress analysis of machine elements<br>B. Failure theories and analysis<br>C. Deformation and stiffness<br>D. Springs<br>E. Pressure vessels<br>F. Beams<br>G. Piping   | 9–14  |

- H. Bearings
- I. Power screws
- J. Power transmission
- K. Joining methods
- L. Manufacturability
- M. Quality and reliability
- N. Hydraulic components
- O. Pneumatic components
- P. Electromechanical components



## Fundamentals of Engineering (FE) OTHER DISCIPLINES CBT Exam Specifications

Effective Beginning with the January 2014 Examinations

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Knowledge	Number of Questions
<b>1. Mathematics and Advanced Engineering Mathematics</b>	<b>12–18</b>
A. Analytic geometry and trigonometry	
B. Calculus	
C. Differential equations (e.g., homogeneous, nonhomogeneous, Laplace transforms)	
D. Numerical methods (e.g., algebraic equations, roots of equations, approximations, precision limits)	
E. Linear algebra (e.g., matrix operations)	
<b>2. Probability and Statistics</b>	<b>6–9</b>
A. Measures of central tendencies and dispersions (e.g., mean, mode, variance, standard deviation)	
B. Probability distributions (e.g., discrete, continuous, normal, binomial)	
C. Estimation (e.g., point, confidence intervals)	
D. Expected value (weighted average) in decision making	
E. Sample distributions and sizes	
F. Goodness of fit (e.g., correlation coefficient, least squares)	
<b>3. Chemistry</b>	<b>7–11</b>
A. Periodic table (e.g., nomenclature, metals and nonmetals, atomic structure of matter)	
B. Oxidation and reduction	
C. Acids and bases	
D. Equations (e.g., stoichiometry, equilibrium)	
E. Gas laws (e.g., Boyle's and Charles' Laws, molar volume)	
<b>4. Instrumentation and Data Acquisition</b>	<b>4–6</b>
A. Sensors (e.g., temperature, pressure, motion, pH, chemical constituents)	
B. Data acquisition (e.g., logging, sampling rate, sampling range, filtering, amplification, signal interface)	
C. Data processing (e.g., flow charts, loops, branches)	
<b>5. Ethics and Professional Practice</b>	<b>3–5</b>
A. Codes of ethics	
B. NCEES <i>Model Law</i>	
C. Public protection issues (e.g., licensing boards)	



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| <b>6. Safety, Health, and Environment</b>  | <b>4–6</b>  |
| <ul style="list-style-type: none"> <li>A. Industrial hygiene (e.g., carcinogens, toxicology, MSDS, lower exposure limits)</li> <li>B. Basic safety equipment (e.g., pressure relief valves, emergency shut-offs, fire prevention and control, personal protective equipment)</li> <li>C. Gas detection and monitoring (e.g., O<sub>2</sub>, CO, CO<sub>2</sub>, CH<sub>4</sub>, H<sub>2</sub>S, Radon)</li> <li>D. Electrical safety</li> </ul>  |             |
| <b>7. Engineering Economics</b>  | <b>7–11</b> |
| <ul style="list-style-type: none"> <li>A. Time value of money (e.g., present worth, annual worth, future worth, rate of return)</li> <li>B. Cost (e.g., incremental, average, sunk, estimating)</li> <li>C. Economic analyses (e.g., breakeven, benefit-cost, optimal economic life)</li> <li>D. Uncertainty (e.g., expected value and risk)</li> <li>E. Project selection (e.g., comparison of unequal life projects, lease/buy/make, depreciation, discounted cash flow)</li> </ul>  |             |
| <b>8. Statics</b>  | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Resultants of force systems and vector analysis</li> <li>B. Concurrent force systems</li> <li>C. Force couple systems</li> <li>D. Equilibrium of rigid bodies</li> <li>E. Frames and trusses</li> <li>F. Area properties (e.g., centroids, moments of inertia, radius of gyration)</li> <li>G. Static friction</li> </ul>  |             |
| <b>9. Dynamics</b>   | <b>7–11</b> |
| <ul style="list-style-type: none"> <li>A. Kinematics</li> <li>B. Linear motion (e.g., force, mass, acceleration)</li> <li>C. Angular motion (e.g., torque, inertia, acceleration)</li> <li>D. Mass moment of inertia</li> <li>E. Impulse and momentum (linear and angular)</li> <li>F. Work, energy, and power</li> <li>G. Dynamic friction</li> <li>H. Vibrations</li> </ul>  |             |
| <b>10. Strength of Materials</b>   | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Stress types (e.g., normal, shear, bending, torsion)</li> <li>B. Combined stresses</li> <li>C. Stress and strain caused by axial loads, bending loads, torsion, or shear</li> <li>D. Shear and moment diagrams</li> <li>E. Analysis of beams, trusses, frames, and columns</li> <li>F. Deflection and deformations (e.g., axial, bending, torsion)</li> <li>G. Elastic and plastic deformation</li> <li>H. Failure theory and analysis (e.g., static/dynamic, creep, fatigue, fracture, buckling)</li> </ul> |             |

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| <b>11. Materials Science</b>  | <b>6–9</b>  |
| <ul style="list-style-type: none"> <li>A. Physical, mechanical, chemical, and electrical properties of ferrous metals</li> <li>B. Physical, mechanical, chemical, and electrical properties of nonferrous metals</li> <li>C. Physical, mechanical, chemical, and electrical properties of engineered materials (e.g., polymers, concrete, composites)</li> <li>D. Corrosion mechanisms and control</li> </ul>   |             |
| <b>12. Fluid Mechanics and Dynamics of Liquids</b>  | <b>8–12</b> |
| <ul style="list-style-type: none"> <li>A. Fluid properties (e.g., Newtonian, non-Newtonian)</li> <li>B. Dimensionless numbers (e.g., Reynolds number, Froude number)</li> <li>C. Laminar and turbulent flow</li> <li>D. Fluid statics</li> <li>E. Energy, impulse, and momentum equations (e.g., Bernoulli equation)</li> <li>F. Pipe flow and friction losses (e.g., pipes, valves, fittings, Darcy-Weisbach equation, Hazen-Williams equation)</li> <li>G. Open-channel flow (e.g., Manning equation, drag)</li> <li>H. Fluid transport systems (e.g., series and parallel operations)</li> <li>I. Flow measurement</li> <li>J. Turbomachinery (e.g., pumps, turbines)</li> </ul> |             |
| <b>13. Fluid Mechanics and Dynamics of Gases</b>  | <b>4–6</b>  |
| <ul style="list-style-type: none"> <li>A. Fluid properties (e.g., ideal and non-ideal gases)</li> <li>B. Dimensionless numbers (e.g., Reynolds number, Mach number)</li> <li>C. Laminar and turbulent flow</li> <li>D. Fluid statics</li> <li>E. Energy, impulse, and momentum equations</li> <li>F. Duct and pipe flow and friction losses</li> <li>G. Fluid transport systems (e.g., series and parallel operations)</li> <li>H. Flow measurement</li> <li>I. Turbomachinery (e.g., fans, compressors, turbines)</li> </ul>   |             |
| <b>14. Electricity, Power, and Magnetism</b>  | <b>7–11</b> |
| <ul style="list-style-type: none"> <li>A. Electrical fundamentals (e.g., charge, current, voltage, resistance, power, energy)</li> <li>B. Current and voltage laws (Kirchhoff, Ohm)</li> <li>C. DC circuits</li> <li>D. Equivalent circuits (series, parallel, Norton's theorem, Thevenin's theorem)</li> <li>E. Capacitance and inductance</li> <li>F. AC circuits (e.g., real and imaginary components, complex numbers, power factor, reactance and impedance)</li> <li>G. Measuring devices (e.g., voltmeter, ammeter, wattmeter)</li> </ul>  |             |

**15. Heat, Mass, and Energy Transfer**

**9–14**

- A. Energy, heat, and work
- B. Thermodynamic laws (e.g., 1st law, 2nd law)
- C. Thermodynamic equilibrium
- D. Thermodynamic properties (e.g., entropy, enthalpy, heat capacity)
- E. Thermodynamic processes (e.g., isothermal, adiabatic, reversible, irreversible)
- F. Mixtures of nonreactive gases
- G. Heat transfer (e.g., conduction, convection, and radiation)
- H. Mass and energy balances
- I. Property and phase diagrams (e.g., T-s, P-h)
- J. Phase equilibrium and phase change
- K. Combustion and combustion products (e.g., CO, CO<sub>2</sub>, NO<sub>x</sub>, ash, particulates)
- L. Psychrometrics (e.g., relative humidity, wet-bulb)