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On k −Quasi Class Q^* Operators

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Abstract Let T be a bounded linear operator on a complex Hilbert space H. In this paper we introduce a new class of operators: $k-$ quasi class Q^* operators. An operator T is said to be $k-$ quasi class Q^* if it satisfies $\left\|T^*T^k\right\|^2 \leq \frac{1}{2}\left(\left\|T^{k+2}x\right\|^2 + \left\|T^k x\right\|^2\right)$, for all $x \in H$, where k is a natural number. We prove the basic properties of this class of operators.

Keywords: *k −quasi class Q^*, quasi class Q^*, k −quasi − ∗ − paranormal operators, quasi− ∗ − paranormal operators*

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1. Introduction

Throughout this paper, let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $L(H)$ denote the C^* algebra of all bounded operators on H. For $T \in L(H)$, we denote by $\ker T$ the null space, by $T(H)$ the range of T and by $\sigma(\mathrm{T})$ the spectrum of T. The null operator and the identity on H will be denoted by 0 and I, respectively. If $T$is an operator, then $T^*$is its adjoint, and$\|T\| = \|T^*\|$. For an operator $T \in L(H)$, as usual $|T| = (T^*T)^{\frac{1}{2}}$.

We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non-negative integers by $\mathbb{N}$ and the complex conjugate of a complex number λ by $\bar{\lambda}$. The closure of a set M will be denoted by $\bar{M}$. An operator $T \in L(H)$ is a positive operator, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$. We write $r(T)$ for the spectral radius. It is well known that $r(T) \leq \|T\|$. The operator T is called normaloid if$r(T) = \|T\|$. The operator T is an isometry if $\|Tx\| = \|x\|$, for all $x \in H$. The operator T is called unitary operator if $T^*T = TT^* = I$.

An operator $T \in L(H)$, is said to be paranormal [4], if $\|Tx\|^2 \leq \|T^2x\|$for any unit vector x in H. Further, T is said to be $*-$paranormal [1,9], if $\|T^*x\|^2 \leq \|T^2x\|$ for any unit vector x in H. An operator $T \in L(H)$, is said to be quasi−paranormal operator if $\|T^2x\|^2 \leq \|T^3x\|\|Tx\|$, for all $x \in L(H)$.

Mecheri [7] introduced a new class of operators called k −quasi paranormal operators. An operator $T$is called k −quasi −paranormal if $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^k x\|$,for all $x \in H$, where k is a natural number. An operator T is called quasi − ∗ − paranormal [8,11], if $\|T^*Tx\|^2 \leq \|T^3x\|\|Tx\|$, for all $x \in H$.

An operator T is called $k-$quasi− ∗ −paranormal if $\|T^*T^k x\|^2 \leq \|T^{k+2}x\|\|T^k x\|$ for all $x \in H$, where k is a natural number, [6].

Shen, Zuo and Yang [13] introduced a new class of operator quasi− ∗ −class A. An operator $T \in L(H)$ is said to be a quasi− ∗ −class A, if $T^*|T^2|T \geq T^*|T^*|^2T$.

Mecheri [12] introduced k −quasi− ∗ −class Aoperator. An operator $T \in L(H)$ is said to be a $k-$ quasi − ∗ −class A, if $T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k$.

Duggal, Kubrusly, Levan [3] introduced a new class of operators, the class Q. An operator $T \in L(H)$ belongs to class Q if $T^{*2}T^2 - 2T^*T + I \geq 0$, or equivalent $\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$, for all $x \in H$.

Senthilkumar, Prasad [10] introduced a new class of operators, the classQ^*. An operator $T \in L(H)$ belongs to class Q^* if $T^{*2}T^2 - 2TT^* + I \geq 0$, or equivalent $\|T^*x\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ for all $x \in H$.

Senthilkumar, Naik and Kiruthika [2] introduced a new class of operators, the quasi class Q^*. An operator $T \in L(H)$ is said to belong to the quasi class Q^* if $T^{*3}T^3 - 2(T^*T)^2 + T^*T \geq 0$, or equivalent $\|T^*Tx\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2)$ for all $x \in H$.

Now we introduce the class of $k-$ quasi class Q^* operators defined as follows:

Definition 1.1. An operator $T \in L(H)$ is said to be of the k −quasi class Q^* if

$$\left\|T^*T^k x\right\|^2 \leq \frac{1}{2}\left(\left\|T^{k+2}x\right\|^2 + \left\|T^k x\right\|^2\right),$$

for all $x \in H$, where k is a natural number.

Remark 1.2. For $k = 1$, a 1 −quasi class Q^* operators is a quasi class Q^* operators.

2. Main Results

Proposition 2.1. An operator $T \in L(H)$ is of the k −quasi class Q^*, if and only if

$$T^{*k}\left(T^{*2}T^2-2TT^*+I\right)T^k\geq 0,$$

where k is a natural number.

Proof: Since T is operator of the k −quasi class Q^*, then

$$2\left\|T^*T^kx\right\|^2\leq\left\|T^{k+2}x\right\|^2+\left\|T^kx\right\|^2,$$

for all $x\in H$, where k is a natural number.

$$\left\langle T^{k+2}x,T^{k+2}x\right\rangle-2\left\langle T^*T^kx,T^*T^kx\right\rangle+\left\langle T^kx,T^kx\right\rangle\geq 0$$

$$\Rightarrow\left\langle T^{*(k+2)}T^{k+2}x,x\right\rangle-2\left\langle\left(T^*T^k\right)^*T^*T^kx,x\right\rangle$$

$$+\left\langle T^{*k}T^kx,x\right\rangle\geq 0$$

$$\Rightarrow\left\langle T^{*(k+2)}T^{k+2}x,x\right\rangle-2\left\langle T^{*k}TT^*T^kx,x\right\rangle$$

$$+\left\langle T^{*k}T^kx,x\right\rangle\geq 0$$

$$\Rightarrow\left\langle T^{*k}\left(T^{*2}T^2-2TT^*+I\right)T^kx,x\right\rangle\geq 0,$$

for all $x\in H$, where k is a natural number.

The last relation is equivalent to

$$T^{*k}\left(T^{*2}T^2-2TT^*+I\right)T^k\geq 0.$$

From the definition of the class Q^* operators, quasi class Q^* operators and the proposition 2.1 we see that every operator of the class Q^* and every operator of the quasi class Q^* is also an operator of the k −quasi class Q^*. Thus, we have the following implication:

$$class\ Q^*\subseteq quasi\ class\ Q^*\subseteq k-quasi\ class\ Q^*$$
$$\subseteq(k+1)-quasi\ class\ Q^*.$$

Corollary 2.2. A weighted shift operator T with decreasing weighted sequence (α_n) is an operator of the k −quasi class Q^* if and only if

$$\alpha_{n+k}^2\alpha_{n+k+1}^2-2\alpha_{n+k-1}^2+1\geq 0,$$

for all n.

Proof: Since T is a weighted shift operator, its adjoint T^* is also a wighted shift operator, then:

$$T(e_n)=\alpha_ne_{n+1},$$
$$T^*e_n=\overline{\alpha_{n-1}}e_{n-1},$$
$$TT^*e_n=\alpha_{n-1}^2e_n,$$
$$\left(T^{*2}T^2\right)e_n=\alpha_n^2\alpha_{n+1}^2e_n.$$

Since, T is an operator of the k −quasi class Q^*, then after some calculations we have:

$$T^{*k}\left(T^{*2}T^2-2TT^*+I\right)T^k\geq 0$$

$$\Rightarrow\alpha_n^2\alpha_{n+1}^2\ldots\alpha_{n+k-1}^2\left(\alpha_{n+k}^2\alpha_{n+k+1}^2-2\alpha_{n+k-1}^2+1\right)\geq 0$$

$$\Rightarrow\alpha_{n+k}^2\alpha_{n+k+1}^2-2\alpha_{n+k-1}^2+1\geq 0.$$

Now we will give an example of 2 −quasi class Q^* operator which is not 1 −quasi class Q^* operator.

Example 2.3. Consider the operator T in l_2 defined by $T(x)=(0,\alpha_1x_1,\alpha_2x_2,\alpha_3x_3,\ldots)$, where

$$\alpha_1=2,\alpha_n=1,n\geq 2.$$

Then T is an operator of the 2 −quasi class Q^*, but this operator is not 1 −quasi class Q^*.

Given:

$$T(x)=(0,\alpha_1x_1,\alpha_2x_2,\alpha_3x_3,\ldots),$$
$$T^*(x)=(\alpha_1x_2,\alpha_2x_3,\alpha_3x_4,\ldots).$$

Now from the proposition 2.1 and corollary 2.2, for 2 −quasi class Q^* operator we have:

$$\left\langle T^{*2}\left(T^{*2}T^2-2TT^*+I\right)T^2x,x\right\rangle$$
$$=\alpha_1^2\alpha_2^2\left(\alpha_3^2\alpha_4^2-2\alpha_2^2+1\right)\left\|x_1\right\|^2$$
$$+\alpha_2^2\alpha_3^2\left(\alpha_4^2\alpha_5^2-2\alpha_3^2+1\right)\left\|x_2\right\|^2+\ldots=0.$$

But for 1 −quasi class Q^* operator we have:

$$\left\langle T^*\left(T^{*2}T^2-2TT^*+I\right)Tx,x\right\rangle$$
$$=\alpha_1^2\left(\alpha_2^2\alpha_3^2-2\alpha_1^2+1\right)\left\|x_1\right\|^2$$
$$+\alpha_2^2\left(\alpha_3^2\alpha_4^2-2\alpha_2^2+1\right)\left\|x_2\right\|^2+\ldots<0.$$

In the following we prove that if T is an operator of the k −quasi class Q^* and if the range of T^k is dense, then T is an operator of the class Q^*.

Proposition 2.4. Let $T\in L(H)$ be an operator of the k − quasi class Q^*. If T^k has dense range, then T is an operator of the class Q^*.

Proof: Since T^k has dense range, then $\overline{T^k(H)}=H$. Let be $y\in H$. Then there exist a sequence $\{x_n\}_{n=1}^{\infty}$ in H such that $T^kx_n\to y,\ n\to\infty$. Since T is an operator of the k −quasi class Q^*, then:

$$\left\langle T^{*k}\left(T^{*2}T^2-2TT^*+I\right)T^kx_n,x_n\right\rangle\geq 0$$

$$\Rightarrow\left\langle\left(T^{*2}T^2-2TT^*+I\right)T^kx_n,T^kx_n\right\rangle\geq 0,\forall n\in\mathbb{N}.$$

By the continuity of the inner product, we have

$$\left\langle\left(T^{*2}T^2-2TT^*+I\right)y,y\right\rangle\geq 0,y\in H.$$

So,

$$T^{*2}T^2-2TT^*+I\geq 0.$$

Therefore T is an operator of the class Q^*.

In the following we give the relations between k −quasi class Q^* and k −quasi − ∗ −paranormal operators.

Hoxha and Braha [[6], Proposition 2.1] prove that an operator $T\in L(H)$ is of the k −quasi − ∗ −paranormal if and only if $T^{*k}\left(T^{*2}T^2-2\lambda TT^*+\lambda^2\right)T^k\geq 0$, for all $\lambda\in\mathbb{R}$.

From this we have that every k− quasi − ∗ −paranormal is operator of the k −quasi class Q^*. Also, every quasi − ∗ −paranormal is operator of the quasi class Q^*.

Proposition 2.5. Let $T \in L(H)$. If $\lambda^{-\frac{1}{2}}T$ is an operator of the $k-$ quasi class Q^*, then T is a $k-$ quasi $-*-$paranormal operator for all $\lambda > 0$.

Proof: Let $\lambda^{-\frac{1}{2}}T$ be an operator of $k-$ quasi class Q^*, for all $\lambda > 0$, then:

$$\left(\lambda^{-\frac{1}{2}}T\right)^{*k}\left(\left(\lambda^{-\frac{1}{2}}T\right)^{*2}\left(\lambda^{-\frac{1}{2}}T\right)^{2}-2\left(\lambda^{-\frac{1}{2}}T\right)\left(\lambda^{-\frac{1}{2}}T\right)^{*}+I\right)$$
$$\bullet\left(\lambda^{-\frac{1}{2}}T\right)^{k}\geq 0$$
$$\Rightarrow \lambda^{-\frac{k}{2}}T^{*k}\left(\lambda^{-2}T^{*2}T^{2}-2\lambda^{-1}TT^{*}+I\right)\lambda^{-\frac{k}{2}}T^{k}\geq 0$$
$$\Rightarrow \frac{1}{\lambda^{k+2}}T^{*k}\left(T^{*2}T^{2}-2\lambda TT^{*}+\lambda^{2}\right)T^{k}\geq 0$$
$$\Rightarrow T^{*k}\left(T^{*2}T^{2}-2\lambda TT^{*}+\lambda^{2}\right)T^{k}\geq 0, \forall \lambda > 0.$$

By this it is proved that the T is $k-$ quasi $-*-$paranormal operator.

Remark 2.6. If $\lambda^{-\frac{1}{2}}T$ is an operator of the quasi class Q^*, then T is a quasi$-*-$paranormal operator for all $\lambda > 0$.

Proposition 2.7. If $T \in L(H)$ is an operator of the $k-$quasi class Q^* and T^2 is an isometry, then T is $k-$quasi$-*-$paranormal operator.

Proof: Let T be an operator of the $k-$quasi class Q^*, then:

$$2\left\|T^{*}T^{k}x\right\|^{2}\leq\left\|T^{k+2}x\right\|^{2}+\left\|T^{k}x\right\|^{2}$$
$$=\left(\left\|T^{k+2}x\right\|-\left\|T^{k}x\right\|\right)^{2}+2\left\|T^{k+2}x\right\|\left\|T^{k}x\right\|.$$

Since operator T^2 is an isometry, then $\|T^2x\| = \|x\|$, for all $x \in H$.

Then,

$$\left\|T^{2}x\right\|=\|x\|$$
$$\Rightarrow\left\|T^{4}x\right\|=\left\|T^{2}x\right\|$$
$$\Rightarrow\ldots\Rightarrow\left\|T^{k+2}x\right\|=\left\|T^{k}x\right\|,$$

so we have,

$$\left\|T^{*}T^{k}x\right\|^{2}\leq\left\|T^{k+2}x\right\|\left\|T^{k}x\right\|, \text{ for all } x\in H.$$

So, T is $k-$ quasi$-*-$paranormal operator.

In the following we give the relation between $k-$quasi class Q^* and $k-$quasi$-*-$class A operators.

Proposition 2.8. If $T \in L(H)$ belongs to the $k-$quasi$-*-$class A, for k a natural number, then T is an operator of $k-$quasi class Q^*.

Proof: Since T belongs to $k-$ quasi $-*-$ class A operators, we have:

$$T^{*k}\left|T^{2}\right|T^{k}\geq T^{*k}\left|T^{*}\right|^{2}T^{k},$$

where k is a natural number.

Let $x \in H$. Then,

$$2\left\|T^{*}T^{k}x\right\|^{2}=2\left\langle T^{*}T^{k}x,T^{*}T^{k}x\right\rangle$$
$$=2\left\langle T^{*k}TT^{*}T^{k}x,x\right\rangle$$
$$=2\left\langle T^{*k}\left|T^{*}\right|^{2}T^{k}x,x\right\rangle$$
$$\leq 2\left\langle T^{*k}\left|T^{2}\right|T^{k}x,x\right\rangle$$
$$=2\left\langle\left|T^{2}\right|T^{k}x,T^{k}x\right\rangle$$
$$\leq 2\left\|\left|T^{2}\right|T^{k}x\right\|.\left\|T^{k}x\right\|$$
$$=2\left\|T^{k+2}x\right\|.\left\|T^{k}x\right\|$$
$$\leq\left\|T^{k+2}x\right\|^{2}+\left\|T^{k}x\right\|^{2}.$$

Therefore,

$$2\left\|T^{*}T^{k}x\right\|^{2}\leq\left\|T^{k+2}x\right\|^{2}+\left\|T^{k}x\right\|^{2}.$$

Hence, T is an operator of the $k-$quasi class Q^*.

Remark 2.9. If $T \in L(H)$ belongs to the quasi$-*-$class A, then T is an operator of quasi class Q^*.

In following we give an example of operator T which is operator of the quasi class Q^*, but not quasi$-*-$class A.

Example 2.10. Let $K = \bigoplus_{n=1}^{+\infty} H_n$, where $H_n \cong H$. Given positive operators A, $B \in L(H)$, define the operator $T_{A,B}$ on K as follows:

$$T_{A,B}=\begin{pmatrix}0&0&0&0&0&\cdots\\A&0&0&0&0&\cdots\\0&B&0&0&0&\cdots\\0&0&B&0&0&\cdots\\0&0&0&B&0&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

The operator $T_{A,B}$ is quasi$-*-$class A if and only if $AB^2A \geq A^4$.

Let A and B be operator as

$$A=\begin{pmatrix}1&1\\1&2\end{pmatrix}^{\frac{1}{2}} \text{ and } B=\begin{pmatrix}1&2\\2&8\end{pmatrix}^{\frac{1}{4}}.$$

Then,

$$A\left(B^{2}-A^{2}\right)A=\begin{pmatrix}-0.3359\ldots&-0.2265\ldots\\-0.2265\ldots&0.8244\ldots\end{pmatrix}\ngeq 0.$$

Hence T is not quasi$-*-$class A.

Then a computation shows that the operator $T_{A,B}$ is quasi class Q^* if and only if

$$T^{*}\left(T^{*2}T^{2}-2TT^{*}+I\right)T=A\left(B^{4}-2A^{2}+I\right)A\geq 0.$$

So,

$$A\left(B^{4}-2A^{2}+I\right)A=\begin{pmatrix}1&1\\1&2\end{pmatrix}^{\frac{1}{2}}\begin{pmatrix}0&0\\0&5\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix}^{\frac{1}{2}}\geq 0.$$

Therefore T is operator of the quasi class Q^*.

Proposition 2.11. Let $T \in L(H)$. If $\|T^*\| \leq \frac{1}{\sqrt{2}}$, then T is operator of the k −quasi class Q^*.

Proof: From $\|T^*\| \leq \frac{1}{\sqrt{2}}$, we have $\|T^*x\|^2 \leq \frac{1}{2}$, $\forall x \in H$. Then,

$$\left\langle T^*x, T^*x\right\rangle - \frac{1}{2}\left\langle x, x\right\rangle \leq 0, \forall x \in H$$
$$\Rightarrow \left\langle \left(I - 2TT^*\right)x, x\right\rangle \geq 0, \forall x \in H$$
$$\Rightarrow I - 2TT^* \geq 0$$
$$\Rightarrow T^{*2}T^2 - 2TT^* + I \geq 0$$
$$\Rightarrow T^{*k}\left(T^{*2}T^2 - 2TT^* + I\right)T^k \geq 0,$$

so T is operator of the k −quasi class Q^*.

Proposition 2.12. If T is an operator of the k −quasi class Q^* and if T commutes with an isometric operator S, then TS is an operator of the k −quasi class Q^*.

Proof: Let $A = TS$. Then

$$A^{*k}\left(A^{*2}A^2 - 2AA^* + I\right)A^k$$
$$= (TS)^{*k}\left((TS)^{*2}(TS)^2 - 2(TS)(TS)^* + I\right)(TS)^k$$
$$= (TS)^*(TS)^*\ldots(TS)^*((TS)^*(TS)^*(TS)(TS)$$
$$-2(TS)(TS)^* + I)(TS)(TS)\ldots(TS)$$
$$= S^*T^*S^*T^*\ldots S^*T^*\left(S^*T^*S^*T^*TSTS - 2TSS^*T^* + I\right)$$
$$\bullet TSTS\ldots TS$$
$$= S^{*k}T^{*k}(T^{*2}T^2 - 2TT^* + I)T^kS^k \geq 0,$$

Hence TS is an operator of the k −quasi class Q^*.

Proposition 2.13. Let T be an operator of the k −quasi class $Q^*$and if T is unitarily equivalent to operator S, then S is an operator of the k −quasi class Q^*.

Proof: Since T is unitarily equivalent to operator S, there is an unitary operator U such that $S = U^*TU$.

Since T is an operator of the k −quasi class Q^*, then

$$T^{*k}\left(T^{*2}T^2 - 2TT^* + I\right)T^k \geq 0.$$

Hence,

$$S^{*k}\left(S^{*2}S^2 - 2SS^* + I\right)S^k$$
$$= \left(U^*TU\right)^{*k}\left(\left(U^*TU\right)^{*2}\left(U^*TU\right)^2 - 2\left(U^*TU\right)\left(U^*TU\right)^* + I\right)$$
$$\bullet\left(U^*TU\right)^k$$
$$= U^*T^*UU^*T^*U\ldots U^*T^*U$$
$$\left(U^*T^*UU^*T^*UU^*TUU^*TU - 2U^*TUU^*T^*U + I\right)$$
$$\bullet U^*TUU^*TU\ldots U^*TU$$
$$= U^*T^{*k}\left(T^{*2}T^2 - 2TT^* + I\right)T^kU \geq 0,$$

so, S is an operator of the k −quasi class Q^*.

Proposition 2.14. Let M be a closed T invariant subset of H. Then, the restriction $T_{|M}$ of a $k-$ quasi class Q^* operator T to M is a k −quasi class Q^* operator.

Proof: Let be $u \in M$. Then

$$\left\|\left(T_{|M}\right)^*\left(T_{|M}\right)^k u\right\|^2 = \left\|T^*T^ku\right\|^2$$
$$\leq \frac{1}{2}\left(\left\|T^{k+2}u\right\|^2 + \left\|T^ku\right\|^2\right)$$
$$= \frac{1}{2}\left(\left\|\left(T_{|M}\right)^{k+2}u\right\|^2 + \left\|\left(T_{|M}\right)^k u\right\|^2\right).$$

This implies that $T_{|M}$ is an operator of k −quasi class Q^*.

Proposition 2.15. Let $T \in L(H)$ be a $k-$ quasi class Q^*operator, the range of T^k not to be dense, and

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} on\ H = \overline{T^k(H)} \oplus \ker T^{*k}.$$

Then, A is an operator of the class Q^* on $\overline{T^k(H)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof: Suppose that T is an operator of k −quasi class Q^*. Since T^k does not have dense range, we can represent T as the upper triangular matrix:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} on\ H = \overline{T^k(H)} \oplus \ker T^{*k}.$$

Since T is an operator of k −quasi class Q^*, we have

$$T^{*k}\left(T^{*2}T^2 - 2TT^* + I\right)T^k \geq 0.$$

Therefore

$$\left\langle\left(T^{*2}T^2 - 2TT^* + I\right)x, x\right\rangle$$
$$= \left\langle\left(A^{*2}A^2 - 2AA^* + I\right)x, x\right\rangle \geq 0,$$

for all $x \in \overline{T^k(H)}$.

Hence

$$A^{*2}A^2 - 2AA^* + I \geq 0.$$

This shows that A is an operator of the class Q^* on $\overline{T^k(H)}$.

Let P be the orthogonal projection of H onto $\overline{T^k(H)}$.

For any

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \overline{T^k(H)} \oplus \ker T^{*k},$$

We have

$$\left\langle C^kx_2, x_2\right\rangle = \left\langle T^k(I-P)x, (I-P)x\right\rangle$$
$$= \left\langle (I-P)x, T^{*k}(I-P)x\right\rangle = 0.$$

Thus $T^{*k} = 0$.

Since,

$$\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta,$$

where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$ by [[5], Corollary 7].

Since, $\sigma(A) \cap \sigma(C)$ have no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$ and $C^k = 0$.

3. Conclusion

In this paper we introduce a new class of operators: $k-$ quasi class Q^* operators. It is proved that the following impication is true

$$classQ^* \subseteq quasi\ classQ^* \subseteq k - quasi\ classQ^* \subseteq (k+1) - quasi\ classQ^*.$$

With example it is shown that, exist a $2-$quasi class Q^* operator which is not $1-$quasi class Q^* (Example 2.3). Further, it is proved that if T is an operator of the $k-$quasi class Q^* and if the range of T^k is dense, then T is an operator of the class Q^* (Proposition 2.4).

It is shown the relation between $k-$quasi class Q^* and $k-$quasi $-*-$paranormal operators (Proposition 2.5, Remark 2.6 and Proposition 2.7). Also it is shown the relation between $k-$quasi class Q^* and $k-$quasi $-*-$class A operators (Proposition 2.8, Remark 2.9 and Example 2.10).

Finally is proved that every operator which satisfy the condition $\|T^*\| \leq \frac{1}{\sqrt{2}}$ is operator of the $k-$quasi class Q^* (Proposition 2.11).

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