ALMOST HOLOMORPHIC EXTENSIONS OF ULTRADIFFERENTIABLE FUNCTIONS

By

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Abstract. For a smooth function f on $\mathbb{R}^n$, we construct an extension F to $\mathbb{C}^n$ with $\bar{\partial}F$ vanishing to a high order on $\mathbb{R}^n$ and give precise estimates of how the degree of smothness is reflected in the degree of vanishing. This analysis is used to define the $\bar{\partial}$ operator on $(n, n-1)$ forms with singularities on $\mathbb{R}^n$.

1 Introduction

Let f be a function on the real line or the unit circle. One can analyze its regularity properties in terms of the existence of extensions F of f to the complex plane such that $\bar{\partial}F$ tends to zero with a certain speed as we approach the line (or the circle). We shall refer to F as an almost holomorphic extension of f. A simple and useful result of this type is that f is of class C^∞ on $\mathbb{R}$ if and only if there exists an extension F such that $|\bar{\partial}F(x+iy)| \leq C_N|y|^N$ for any $N > 0$. This idea is of importance in connection with problems of quasianalyticity, approximation theory and operator calculus; we refer the reader to [7] and [8], where as far as we know the method was first discussed systematically, as well as to the survey [9], where many results and applications are described. Even earlier results for C^∞-functions were also used in [10], [13] and [12].

The purpose of this article is to study similar constructions in several variables. In the basic cases, i.e., in the case of rotationally symmetric growth conditions, this has already been done (see, e.g., [6], [5] and [11]), but here we shall pay particular attention to growth conditions that may be different in different directions. This complication is already present in one variable. If we measure the regularity of a function in terms of the decay of its Fourier transform

$$\hat{f}(t) = \frac{1}{2\pi i}\int f(x)e^{-ixt}dx,$$

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 89 (2003)

it is not hard to see that decay of $\hat{f}$ on the positive and negative half axes corresponds to existence of almost holomorphic extensions to the lower and upper half planes, respectively. The problem, however, becomes much more delicate in several variables. This is perhaps most clearly seen in connection with the edge of the wedge theorem. This theorem says that if f has a holomorphic extension for $y = \operatorname{Im} z$ belonging to certain cones with vertices at the origin in $i\mathbb{R}^n$, then f also has an (in general different) extension to the convex hull of the union of these cones. Similar phenomena occur even if we consider only almost holomorphic extensions, and one can therefore expect to get a one-to-one correspondence between rates of decay of $\hat{f}$ and the size of $\bar{\partial}F$ only if the growth scales satisfy certain convexity conditions. Assuming such convexity conditions, in Section 4 we obtain quite precise results in that direction. The main work behind these results is done in Section 3, where we introduce a Legendre type transform that relates the decay of $\hat{f}$ and $\bar{\partial}F$. We also use the corresponding results in one dimension, which we review and formulate as precisely as we have been able to in Section 2.

A related, and more difficult, question is to find the optimal almost holomorphic extensions — if they exist. By this, we mean that, given an extension of f that satisfies

$$|\bar{\partial}F(x+iy)| \leq e^{-g(y)}$$

for a certain function g, find the largest function $\tilde{g} \geq g$ for which there exists an extension whose $\bar{\partial}$ is controlled by $e^{-\tilde{g}}$. The edge of the wedge theorem treats the case when g equals infinity in certain cones, and $\tilde{g}$ is then the function that equals infinity in the convex hull of these cones. In general, we have not been able to determine $\tilde{g}$, and we do not even know if an optimal choice exists; but we give a partial result in Theorem 4.3.

Almost holomorphic extensions are also related to Beurling's generalized theory of distributions, the so-called ultradistributions, cf. [3] and [4]. Beurling's idea was to construct more general distributions by using a smaller class of test functions. He showed that one obtains such ultradistributions as limits of holomorphic functions in the upper or lower halfplanes and determined the optimal growth conditions on the holomorphic functions which ensure that the limits exist in the ultradistribution sense. Although the original discussion is phrased differently, one can define the limit ultradistribution of a holomorphic function h defined in the complement of $\mathbb{R}$ by

$$h^*.f = -\int \bar{\partial}F \wedge hdz$$

if f is a compactly supported test function and F an almost holomorphic extension. Alternately, we can think of the ultradistribution h^* as $\bar{\partial}(hdz)$. The difference between these two viewpoints of an ultradistribution — boundary value of a holomorphic function versus $\bar{\partial}$ of an $(n, n-1)$-form — is more decisive in several variables. A secondary purpose of this paper is to generalize two key results of Beurling [4] to the second interpretation. In particular, we determine which growth conditions

$$\int |\omega| e^{-g(y)} < \infty$$

on an $(n, n-1)$-form outside of $\mathbb{R}^n$ allow a definition of $\bar{\partial}\omega$ across $\mathbb{R}^n$ which is both local and continuous with respect to this weighted L^1-norm (cf. Theorems 4.8 and 4.9).

We thank the referee for many helpful comments on the manuscript.

2 Functions of one variable

In this section, we review the one-variable theory; the results here are due principally to Dynkin, [7] and [8], and Beurling [3]. We consider functions f defined on the entire real line and ask for conditions in terms of the regularity of f which imply that f has an extension F to $\mathbb{C}$ with $\bar{\partial}F(x+iy)$ tending rapidly to zero as y tends to zero. For an integrable function f on $\mathbb{R}$, we define its Fourier transform as

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} f(x) dx.$$

The regularity of f will be measured in terms of the decay of its Fourier transform. Let $h \in C(\mathbb{R})$ be nonnegative, concave and increasing on $\mathbb{R}^+$, and concave and decreasing on $\mathbb{R}^-$. We also assume that $\lim_{t \pm\infty} h(t) = \infty$. We let H denote the class of such weight functions. We are concerned with functions f on $\mathbb{R}$ such that $\hat{f}$ is majorized in some way by e^{-h}. Let

$$\mathcal{A}_h = \{f \in C(\mathbb{R}) : \int |\hat{f}| e^h < \infty\}.$$

From the concavity of h, it follows that h is subadditive, and this in turn implies that $\mathcal{A}_h$ is closed under multiplication, since

$$\begin{aligned} \int |\widehat{fg}| e^h dt &\leq \int \int |\hat{f}(t-s)||\hat{g}(s)| e^{h(t)} dt ds \\ &= \int \int |\hat{f}(t)||\hat{g}(s)| e^{h(t+s)} dt ds \leq \int |\hat{f}(t)| e^{h(t)} dt \int |\hat{g}(s)| e^{h(s)} ds. \end{aligned}$$

Along with $\mathcal{A}_h$, we also consider the class of functions whose Fourier transforms are uniformly bounded by e^{-h},

$$\widetilde{\mathcal{A}}_h = \{f \in C(\mathbb{R}) : \sup |\hat{f}| e^h < \infty\}.$$

The size of $\bar{\partial}F$, where F is an extension of f to $\mathbb{C}$, is measured with another class of weight functions. Let g be a nonnegative function on $\mathbb{R} \setminus \{0\}$ which is convex, decreasing on $\mathbb{R}^+$, increasing on $\mathbb{R}^-$, and satisfies $\lim_{y \pm 0} g(y) = \infty$. Let G denote the class of such functions. Define

$$\mathcal{B}_g = \{f \in C(\mathbb{R}) : \exists F \in C(\mathbb{C}) \text{ such that } F = f \text{ on } \mathbb{R} \\ \text{and } \sup |\bar{\partial}F(x+iy)|e^{g(-y)} < \infty\}.$$

The two classes of weight functions H and G are related via a variant of the Legendre transform introduced by Beurling [3].

Definition 1. Let h be nonnegative and increasing on $\mathbb{R}^+$ and decreasing on $\mathbb{R}^-$. Then

$$h^\sharp(y) = \sup_{\{t:\ yt>0\}} (h(t) - ty).$$

Let g be nonnegative and decreasing on $\mathbb{R}^+$ and increasing on $\mathbb{R}^-$. Then

$$g^\flat(t) = \inf_{\{y:\ yt>0\}} (g(y) + ty).$$

Thus the values of $h^\sharp(y)$ for $y > 0$ only depend on the values of $h(t)$ for $t > 0$. Note also that $h^\sharp(y) \geq h(1/y) - 1$, so that $\lim_{y \to \pm 0} h^\sharp(y) = \infty$ if $\lim_{t \to \pm\infty} h(t) = \infty$. Similarly, one can verify that $\lim_{t \to \pm\infty} g^\flat(t) = \infty$ if $\lim_{y \to \pm 0} g(y) = \infty$. Note also that $h^\sharp$ is always convex on each semi-axis, and $g^\flat$ is always concave on each semi-axis.

If y is fixed and $g = h^\sharp$, the function $t \mapsto g(y) + ty$ is the smallest affine function with slope y that majorizes h. Taking the infimum over all y, we therefore get the smallest concave majorant of h. We have therefore proved half of the next proposition; the remaining part can be obtained in a similar way.

Proposition 2.1. *On each semi-axis $h^{\sharp\flat}$ is the smallest concave majorant of h, and $g^{\flat\sharp}$ is the largest convex minorant of g. In particular, if $h \in H$, then $h = h^{\sharp\flat}$ and if $g \in G$, then $g = g^{\flat\sharp}$.*

The next result on almost holomorphic extensions is a somewhat more precise version of a theorem of Dynkin [8].

Theorem 2.2. *Let $h \in H$ and $f \in C(\mathbb{R})$. Then f can be extended to a function F in $C(\mathbb{C})$ such that*

$$\sup_x \int_{-\infty}^{\infty} |\partial F(x+iy)/\partial\bar{z}| e^{g(-y)} dy \leq \frac{1}{2} \int_{-\infty}^{\infty} |\hat{f}(t)| e^{h(t)} dt,$$

and, if h is of class C^2,

$$\sup_{\mathbf{C}} |\partial F(x+iy)/\partial \bar{z}| e^{g(-y)} \leq \frac{1}{2} \sup_{\mathbf{R}} \frac{|\hat{f}|e^h}{-h''},$$

where $g = h^\sharp$.

Proof. We construct the extension for $\operatorname{Im} z \leq 0$, the case of $\operatorname{Im} z \geq 0$ being similar. We also assume h to be smooth for $t \neq 0$ and strictly concave; the general case then follows if we approximate h from below by such functions. Then $\phi = h'$ is a decreasing map from $\mathbb{R}^+$ to (a subset of) $\mathbb{R}^+$. Let ψ be the inverse of ϕ. Since

$$g(y) = \sup_{t>0} (h(t) - ty)$$

for $y > 0$, and since the supremum is attained at a point where the derivative vanishes, we have

$$g(\phi(t)) = h(t) - t\phi(t).$$

For $y > 0$, let

$$F(x-iy) = \int_{\phi(t)>y} e^{it(x-iy)} \hat{f}(t)dt; \tag{2.1}$$

the integration is thus performed from $-\infty$ to the positive number $\psi(y) = \phi^{-1}(y)$. Since $g \geq 0$, the integrand satisfies, if $t > 0$,

$$\begin{aligned} \left|e^{it(x-iy)}\hat{f}(t)\right| &= e^{ty-h(t)}|\hat{f}(t)|e^{h(t)} \\ &\leq e^{t\phi(t)-h(t)}|\hat{f}(t)|e^{h(t)} = e^{-g(\phi)}|\hat{f}|e^h \leq |\hat{f}|e^h. \end{aligned}$$

The same conclusion clearly holds also for negative values of t, so by Lebesgue's theorem and the Fourier inversion formula, F is indeed a continuous extension of f. Next, since the integrand in (2.1) is holomorphic in $z = x - iy$, we have

$$|\partial F(x-iy)/\partial \bar{z}| \leq (1/2)|\psi'(y)|e^{ty-h(t)}|\hat{f}(t)|e^{h(t)}\Big|_{t=\psi(y)},$$

and hence

$$|\partial F(x-iy)/\partial \bar{z}|e^{g(y)} \leq (1/2)|\psi'(y)||\hat{f}(t)|e^{h(t)}\Big|_{t=\psi(y)}$$

(recall that $h(t) - ty = g(y)$ if $t = \psi(y)$). Since

$$\psi'(y) = \frac{1}{\phi'(t)} = \frac{1}{h''(t)}$$

if $t = \psi(y)$, it follows that

$$\sup_{y\leq 0} |\partial F(x+iy)/\partial \bar{z}|e^{g(-y)} \leq (1/2) \sup |\hat{f}|e^h/|h''|.$$

For the L^1-norms, we get

$$\int_0^\infty |\partial F(x-iy)/\partial\bar{z}|e^{g(y)}dy \leq \frac{1}{2}\int_0^\infty |\hat{f}|e^h\Big|_{t=\psi(y)}|\psi'|dy \leq \int_0^\infty |\hat{f}|e^h dt.$$

This proves the theorem. □

Theorem 2.2 has a simple partial converse stating that if a function has an extension with $\bar{\partial}F$ small near the x-axis, then $\hat{f}$ must decrease rapidly as t tends to infinity.

Theorem 2.3. *Let $f \in C^1(\mathbb{R})$ and suppose that there is a C^1-extension F to $\mathbb{C}$ such that*

$$\int_{|y|<1} |F|dxdy + \int_{|y|<1} |\bar{\partial}F(x+iy)|e^{g(-y)}dxdy \leq C.$$

Then

$$|\hat{f}|e^h \leq (1+o(1))\frac{C}{2\pi} \tag{2.2}$$

when $|t| \to \infty$, where $h = g^\flat$.

Proof. We prove (2.2) for $t > 0$. Let $\chi(y)$ be a cutoff function such that $\chi(y) = 1$ for $-1/2 < y < 0$ and $\chi(y) = 0$ for $y < -1$. Then

$$\hat{f}(t) = \frac{1}{2\pi}\int f(x)e^{-itx}dx = \frac{1}{2\pi}\int_{y<0} (\chi\bar{\partial}F + F\bar{\partial}\chi)e^{-itz}\wedge dz,$$

by Stokes' theorem. Hence

$$\begin{aligned} |\hat{f}(t)| &\leq \frac{C}{2\pi}e^{-t/2} + \frac{1}{2\pi}\int_{-1<y<0} |\bar{\partial}F|e^{g(-y)}e^{-\left(g(-y)+(-y)t\right)} \\ &\leq \frac{C}{2\pi}e^{-t/2} + e^{-h(t)}\frac{1}{2\pi}\int_{-1<y<0} |\bar{\partial}F|e^{g(-y)}dxdy. \end{aligned}$$

This proves the theorem, since $h(t) = o(|t|)$. □

Theorem 2.3 is not a complete converse to Theorem 2.2 since we require integrability of $\bar{\partial}F$ in the x-direction as well. However, if F satisfies only

$$\sup_x \int_{|y|<1} |\bar{\partial}F|e^{g(-y)}dy < \infty,$$

we may replace f by $f_\lambda(x) = e^{-\lambda x^2}f(x)$ for $\lambda > 0$. Then each f_λ satisfies the hypothesis of Theorem 2.3; thus we get pointwise control of $\hat{f} * e_\lambda$, where

$$e_\lambda = \frac{1}{\lambda\sqrt{\pi}}e^{-t^2/\lambda}$$

is an approximate identity as $\lambda \to 0$.

Observe also that no condition of convexity of g is needed in Theorem 2.3. However,

$$h = g^\flat = (g^\flat)^{\sharp\flat} = (g^{\flat\sharp})^\flat$$

since $g^\flat$ is concave. Therefore, the estimate obtained in Theorem 2.3 is the same as what would be obtained if g is replaced by its largest convex minorant $g^{\flat\sharp}$, so we could just as well have assumed from the start that g was convex (on each semi-axis).

The situation is different in Theorem 2.2; there the assumption that h be concave is indispensable. In fact, if the statement of Theorem 2.2 holds for some (nonconcave) h, then we get from Theorem 2.3 that, in general terms,

$$|\hat{f}|e^h \leq C \quad \text{implies} \quad |\hat{f}|e^{h^{\sharp\flat}} \leq C',$$

which can only hold if h differs from its smallest concave majorant $h^{\sharp\flat}$ by an additive constant. In other words, if the statement of Theorem 2.2 holds, then h is uniformly close to a concave function.

In the applications later on, we will have use for an extension F of f with the desired decay of $|\bar{\partial}F|$ which, in addition, has compact support if f has. This is not automatic from our construction, but it does hold under a small additional assumption on g. To prove this we first need a technical result.

Proposition 2.4. *Assume that $g \in G$ satisfies the extra condition that g is C^2 outside the origin and that*

$$g''/g \to \infty \quad \text{as } y \to 0. \tag{2.3}$$

Let I and J be intervals such that $I \subset\subset J \subset \mathbb{R}$, and suppose that $f \in C(J)$ has a continuous extension F to $J \times [0, \epsilon]$ such that

$$|\bar{\partial}F|e^{g(y)} \leq 1.$$

If furthermore $f = 0$ in some neighbourhood of I, then

$$|F(x+iy)|e^{g(y)} \leq C, \quad x \in I,\ y \in [0, \epsilon].$$

In fact, we even have that $|F|e^g \to 0$ when $y \to 0$.

The proof of Proposition 2.4 is based on the following maximum principle for almost holomorphic functions.

Proposition 2.5. *Let F be a C^1-function in a smoothly bounded domain Ω in $\mathbb{C}$, and let $\varphi \in C(\overline{\Omega})$ be subharmonic in Ω. Then*

$$\sup_{\Omega} |F|e^{\varphi} \leq c(\sup_{\partial\Omega} |F|e^{\varphi} + \sup_{\Omega} |\bar{\partial}F|e^{\varphi}), \tag{2.4}$$

where c depends only on the domain Ω. Moreover, if K is a compact subset of $\partial\Omega$ such that F vanishes on a neighbourhood of K in $\partial\Omega$, then $|F(z)|e^{\varphi(z)}$ tends to 0 when z tends to K.

Proof. The proof (cf. [2]) is based on a calculation of $\Delta|F|^2e^{2\varphi}$. Let

$$\vartheta = e^{-2\varphi}\frac{\partial}{\partial z}e^{2\varphi}$$

be the negative of the formal adjoint of $\frac{\partial}{\partial \bar{z}}$ in $L^2(e^{2\varphi})$, so that

$$\int_{\Omega} \frac{\partial u}{\partial \bar{z}}\bar{v}e^{2\varphi} = -\int_{\Omega} u\overline{\vartheta v}e^{2\varphi}$$

if u or v vanishes on $\partial\Omega$. A calculation, shows that

$$\begin{aligned}\Delta|F|^2e^{2\varphi} &= 2\mathrm{Re}\left(\vartheta\frac{\partial F}{\partial \bar{z}}\right)\bar{F}e^{2\varphi} + \left|\frac{\partial F}{\partial \bar{z}}\right|^2 e^{2\varphi} + |\vartheta F|^2e^{2\varphi} + 2(\Delta\varphi)|F|^2e^{2\varphi}\\ &\geq 2\mathrm{Re}\left(\vartheta\frac{\partial F}{\partial \bar{z}}\right)\bar{F}e^{2\varphi},\end{aligned}$$

since φ is subharmonic. Let $G \leq 0$ be the Green function for Ω with pole at $a \in \Omega$, and let Pg denote the Poisson integral of a function g on $\partial\Omega$. Then

$$\begin{aligned}|F(a)|^2e^{2\varphi(a)} &= P\big[|F|^2e^{2\varphi}\big](a) + \int G\Delta|F|^2e^{2\varphi}\\ &\leq \sup_{\partial\Omega}|F|^2e^{2\varphi} + 2\mathrm{Re}\int G\vartheta\Big(\frac{\partial F}{\partial \bar{z}}\Big)\bar{F}e^{2\varphi}\\ &= \sup_{\partial\Omega}|F|^2e^{2\varphi} - 2\mathrm{Re}\int\Big(\frac{\partial G}{\partial z}\frac{\partial F}{\partial \bar{z}}\bar{F} + G\Big|\frac{\partial F}{\partial \bar{z}}\Big|^2\Big)e^{2\varphi}\\ &\leq \sup_{\partial\Omega}|F|^2e^{2\varphi} + 2\sup_{\Omega}|F|e^{\varphi}\int_{\Omega}\Big|\frac{\partial G}{\partial z}\Big|\sup_{\Omega}\Big|\frac{\partial F}{\partial \bar{z}}\Big|e^{\varphi}\\ &\quad + 2\sup_{\Omega}\Big|\frac{\partial F}{\partial \bar{z}}\Big|^2e^{2\varphi}\int_{\Omega} -G.\end{aligned}$$

Since $\int_{\Omega} G$ and $\int_{\Omega} |\partial G/\partial z|$ are uniformly bounded for $a \in \Omega$, we may take the supremum over all $a \in \Omega$. Letting

$$B = \sup_{\Omega}|F|e^{\varphi} \quad \text{and} \quad C = \sup_{\Omega}\Big|\frac{\partial F}{\partial \bar{z}}\Big|e^{\varphi},$$

we then get

$$B^2 \lesssim \sup_{\partial\Omega} |F|^2 e^{2\varphi} + BC + C^2,$$

from which we conclude that

$$B^2 \lesssim \sup_{\partial\Omega} |F|^2 e^{2\varphi} + C^2.$$

This concludes the proof of (2.4). To see the last statement, we use the inequality

$$|F(a)|^2 e^{2\varphi(a)} \leq P\big[|F|^2 e^{2\varphi}\big](a) + 2\sup_{\Omega} |F| e^{\varphi} \int_{\Omega} \left|\frac{\partial G}{\partial z}\right| \sup_{\Omega} \left|\frac{\partial F}{\partial \bar{z}}\right| e^{\varphi}$$
$$+ 2\sup_{\Omega} \left|\frac{\partial F}{\partial \bar{z}}\right|^2 e^{2\varphi} \int_{\Omega} -G.$$

Since we now know that $|F|^2 e^{2\varphi}$ and $|\bar{\partial} F|^2 e^{2\varphi}$ are uniformly bounded, we get

$$|F(a)|^2 e^{2\varphi(a)} \leq P\big[|F|^2 e^{2\varphi}\big](a) + A \int_{\Omega} \left|\frac{\partial G}{\partial z}\right| + |G|.$$

Since $\int_\Omega G$ and $\int_\Omega |\partial G / \partial z|$ tend to 0 as a approaches the boundary of Ω, the theorem follows. □

With the aid of Proposition 2.5, we can now give the

Proof of Proposition 2.4. Let χ be a cutoff function, $0 \leq \chi \in C_c^\infty(\mathbb{R})$, such that $\chi = 1$ on I and with support in the neighbourhood of I where f vanishes. We may also assume that χ is convex where $\chi < 1/2$. Let $\psi(x + iy) = \chi(x) g(y)$. Since $\Delta\psi = \chi'' g + \chi g''$, it follows from the assumption (2.3) that ψ is subharmonic for $0 < y \leq \epsilon$ if ϵ is sufficiently small. Let now R be a rectangle

$$R = \big((I - \delta) \cup (I + \delta)\big) \times (0, \epsilon),$$

where $\delta > 0$ is chosen so that $f = 0$ on $(I - \delta) \cup (I + \delta)$, and $\chi = 0$ on $\mathbb{R} \setminus \big((I-\delta) \cup (I+\delta)\big)$, and let Ω be a smoothly bounded domain obtained by rounding off the corners slightly. We apply Proposition 2.5 with $\varphi = \varphi_\lambda = \psi(\cdot + i\lambda)$ for small λ. Since $\varphi_\lambda \leq \psi \leq g$, it follows that

$$|\bar{\partial} F| e^{\varphi_\lambda} \leq 1$$

in Ω. Moreover,

$$\sup_{\partial\Omega} |f| e^{\varphi_\lambda} \leq C,$$

where C is a constant depending on ϵ and on $\sup_{\partial\Omega} |F|$. By Proposition 2.5, $|F| e^{\varphi_\lambda} \leq C$; and letting $\lambda \to 0$, we get that $|F| e^g \leq C$ in $I \times (0, \epsilon)$. □

It follows easily that we can control the support of the function F that extends f.

Theorem 2.6. *Assume that $g \in G$ satisfies the extra condition that g is C^2 outside the origin and that* (2.3) *holds. Let U and V be open sets, $U \subset\subset V \subset \mathbb{R}$. Suppose that f has compact support in U and that f has an extension F to $\mathbb{C}$ such that*

$$|\bar{\partial} F| e^g \leq 1.$$

Then there is an extension $\tilde{F}$ of f with support in $V \times (-\epsilon, \epsilon)$ such that

$$|\bar{\partial} \tilde{F}| e^g \leq C.$$

Clearly the condition (2.3) is fulfilled in all reasonable cases.

Proof. Let $\tilde{F} = \chi F$, where $\chi = \chi(x)$ is a cutoff function with compact support in V which equals 1 on U. Then

$$\bar{\partial} \tilde{F} = F \bar{\partial} \chi + \chi \bar{\partial} F,$$

so by Proposition 2.4 we have that $|\bar{\partial} \tilde{F}| e^g \leq C$ as claimed. □

Beurling's idea was to use the functions with rapidly decreasing Fourier transforms as test functions for a generalized theory of distributions. It then becomes necessary to determine which growth conditions can be satisfied by functions with compact support. The answer is contained in the following theorem, which is essentially taken from [3]; a proof is given in [1].

Theorem 2.7. *Let $h \in H$. There exists a nontrivial function with compact support satisfying $|\hat{f}| e^h \leq C$ if and only if*

$$\int_{-\infty}^{\infty} \frac{h(t) dt}{1 + t^2} < \infty. \tag{2.5}$$

If this condition is satisfied, then there are nonnegative compactly supported functions χ_n with $|\hat{\chi}_n| e^h \leq C_n$ such that $\chi_n \to \delta_0$ in the sense of distributions.

One can also express the integrability condition (2.5) in terms of $g = h^\sharp$; see [3] for a proof.

Proposition 2.8. *Let $h \in H$ and put $g = h^\sharp$. Then*

$$\int_1^{\infty} \frac{h(t) dt}{1 + t^2} < \infty$$

if and only if

$$\int_{-1}^{1} \log g(y) dy < \infty. \tag{2.6}$$

Beurling expressed the generalized distributions, ultradistributions, dual to the class $\mathcal{A}_h$ as differences of boundary values of functions holomorphic in the upper and lower half planes. Equivalently, one may define the ultradistributions as $\bar{\partial}$ of holomorphic forms defined outside the real line. This point of view fits better with the generalization to higher dimensions that we have in mind.

Let Ω be a domain in $\mathbb{C}$ which intersects $\mathbb{R}$, and let ω be a holomorphic $(1,0)$-form in $\Omega \setminus \mathbb{R}$ satisfying an estimate of the form

$$\int_\Omega |\omega| e^{-g} < \infty. \tag{2.7}$$

We want to define $\bar{\partial}\omega$ as an ultradistribution on $\Omega \cap \mathbb{R}$. This definition should have the property that $\bar{\partial}\omega = 0$ if and only if ω extends holomorphically to Ω; moreover, we require the $\bar{\partial}$ operator to be continuous in the sense that if

$$\omega_n e^{-g} \to 0$$

in $L^1_{\rm loc}(\Omega)$, then $\bar{\partial}\omega_n.f \to 0$ for each test function in the class.

We shall see that this is possible precisely when the weight g satisfies (2.6). First we prove that $\bar{\partial}\omega$ has no definition satisfying the above requirements if (2.6) is not fulfilled. This is a consequence of the next theorem, which is given (with a different proof) in [4]

Theorem 2.9. *Suppose that g does not satisfy* (2.6) *(and that* (2.3) *holds). Then for any holomorphic 1-form ω in $\Omega \setminus \mathbb{R}$ such that $|\omega|e^{-g} \in L^1_{loc}(\Omega)$ and any $K \subset\subset \Omega$, we can find polynomials $p_n(z)$ such that*

$$\int_K |\omega - p_n dz| e^{-g} \to 0.$$

Proof. For a given compact set $K \subset \Omega$ we have to show that ω is in the closure in $L^1(K, e^{-g})$ of the (restrictions to K of) polynomials. By the Hahn–Banach theorem, it is therefore enough to prove that if α is a $(0,1)$-form on K such that $|\alpha|e^g$ is bounded and

$$\int_K \alpha \wedge p dz = 0 \tag{2.8}$$

for all polynomials p, then $\int \alpha \wedge \omega = 0$. But (2.8) implies that $\alpha = \bar{\partial}F$, where F is continuous in $\mathbb{C}$ and identically zero in the unbounded component of $\mathbb{C} \setminus K$. Hence the restriction f of F to the real axis has compact support. Since $|\bar{\partial}F|e^g$ is bounded, it follows from Theorem 2.3 that $|\hat{f}|e^h$ is bounded if $h = g^\flat$. So f is actually a smooth function, which by Proposition 2.8 and Theorem 2.7 belongs to a quasianalytic class. Therefore, f must vanish identically. It now follows from Proposition 2.4 that $|F(x+iy)|e^{g(y)} \to 0$ uniformly as $y \to 0$. Hence Stokes' theorem gives

$$\lim_{\epsilon \to 0} \int_{|\mathrm{Im}\, z| > \epsilon} \alpha \wedge \omega = \int_{|\mathrm{Im}\, z| = \epsilon} F\omega \to 0,$$

so $\int \alpha \wedge \omega = 0$. □

Theorem 2.9 shows that forms satisfying $\bar\partial\omega = 0$ are dense in $L^1_{\mathrm{loc}}(e^g)$ if g does not satisfy (2.6), so $\bar\partial\omega$ has no local definition which is continuous in the $L^1(e^g)$-topology in that case. (One can, however, always give a sense to $\bar\partial\omega$ as a *hyperfunction*; see [11] for this.)

We next assume that g satisfies condition (2.6) and let ω be a holomorphic $(1,0)$-form in Ω satisfying (2.7). Let f be a test function with compact support which has an extension F such that $|\bar\partial F|e^g \le 1$. We then define

$$\bar\partial\omega.f = \int \omega \wedge \bar\partial F,$$

where the extension F is chosen with support in Ω, which is possible by Theorem 2.6. Proposition 2.4 implies that the definition is independent of the choice of F, and it is also clear from the definition that $\bar\partial\omega$ is continuous in $L^1_{\mathrm{loc}}(e^g)$. The next proposition is a variant of the "Corollary to Theorems I and II" in [4].

Proposition 2.10. *Assume g satisfies (2.6) (and (2.3)). Let ω be a holomorphic form in $\Omega \setminus \mathbb{R}$ satisfying (2.7), and assume that $\bar\partial\omega$ is defined as above. Then $\bar\partial\omega = 0$ if and only if ω extends holomorphically across $\mathbb{R}$.*

Proof. We only have to check that if $\bar\partial\omega = 0$, then ω extends holomorphically. Let F be a function with compact support in Ω, and suppose $|\bar\partial F|e^g \le C$. Consider

$$I = \frac{1}{\pi}\int \frac{\omega \wedge \bar\partial F}{z-a},$$

where $a \in \Omega \setminus \mathbb{R}$. If we decompose F as $F = F_1 + F_2$, where F_2 is supported in a small neighbourhood of a and F_1 vanishes near a, we see that

$$I = \frac{1}{\pi}\int \frac{\bar\partial(F_2\omega)}{z-a} = F(a)\omega(a).$$

We can then fix $x_0 \in \Omega \cap \mathbb{R}$ and choose F so that $F \equiv 1$ near x_0. Then I depends holomorphically on a if a is near x_0, and we thus see that ω extends holomorphically across x_0. □

3 Multivariable Legendre type transforms

We are now going to consider multidimensional analogues to the results in the previous section. Our first task is to find appropriate generalizations of the transforms $h \to h^\sharp$ and $g \mapsto g^\flat$. As before, h always denotes a nonnegative function defined in $\mathbb{R}^n$ such that $h(0) = 0$, and g denotes a function in $\mathbb{R}^n \setminus \{0\}$ such that $g(y) \to \infty$ when $y \to 0$. If k is any function in $\mathbb{R}^n \setminus \{0\}$ and $a \in S^{n-1}$, i.e., $a \in \mathbb{R}^n$ and $|a| = 1$, then set

$$k^a(\eta) = k(\eta a), \quad \eta > 0,$$

to be the restriction of k to the ray from 0 determined by a. Let $k^{a\sharp}$ and $k^{a\flat}$ denote the functions $(k^a)^\sharp$ and $(k^a)^\flat$, respectively, i.e.,

$$k^{a\sharp}(\eta) = \sup_{s>0}(k^a(s) - s\eta) \quad \text{and} \quad k^{a\flat}(s) = \inf_{\eta>0}(k^a(\eta) + \eta s),$$

and extend the definition to all of $\mathbb{R}$ by setting $h^{a\sharp}(\eta) = \infty$ for $\eta < 0$ and $g^{a\flat}(s) = -\infty$ for $s < 0$.

Remark 1. This just means that $g^{a\flat}(s) = -Lg^a(-s)$ if $g^a(\eta)$ is defined as ∞ for $\eta < 0$ and L is the usual Legendre transform; similarly, $h^{a\sharp}(\eta) = -Lh^a(-\eta)$ if $h^a = -\infty$ for negative η. □

Definition 2. If h and g are nonnegative functions in $\mathbb{R}^n \setminus \{0\}$, then

$$g^\flat(t) = \sup_{a\in S^{n-1}} g^{a\flat}(a \cdot t) \quad \text{and} \quad h^\sharp(y) = \inf_{a\in S^{n-1}} h^{a\sharp}(a \cdot y).$$

One readily verifies that this definition coincides with the previously given one in the case when $n = 1$. Note also that $g^\flat$ is always lower semicontinuous and $h^\sharp$ is always upper semicontinuous.

We say that a function k in $\mathbb{R}^n \setminus \{0\}$ is convex (concave) on rays if k^a is convex (concave) for all $a \in S^{n-1}$. We also say that k has convex sub- (super-) level sets if all sets of the form $K_A = \{y : g(y) \leq A\}$ ($V_A = \{y : g(y) \geq A\}$) are convex. We then have

Proposition 3.1. *Suppose that g is nonnegative in $\mathbb{R}^n \setminus \{0\}$ and convex and decreasing on rays. Then $g^{\flat\sharp}$ is the smallest majorant of g which has convex superlevel sets. Similarly, if h is increasing and concave on rays, then $h^{\sharp\flat}$ is the largest minorant of h which has convex sublevel sets.*

Proof. We prove the first statement, the proof of the second one being similar. Since $h^{a\sharp}(s)$ is decreasing, it follows that $\{s \in \mathbb{R};\ h^{a\sharp}(s) \geq A\}$ is an interval and hence $\{y \in \mathbb{R}^n : h^{a\sharp}(a \cdot y) \geq A\}$ is a halfspace. Therefore,

$$\{y \in \mathbb{R}^n : h^{\sharp}(y) \geq A\} = \{y \in \mathbb{R}^n : \inf_a h^{a\sharp}(a{\cdot}y) \geq A\} = \bigcap_a \{y \in \mathbb{R}^n : h^{a\sharp}(a{\cdot}y) \geq A\}$$

is convex. Taking $h = g^b$, we find that $g^{b\sharp}$ has convex superlevel sets as claimed.

We next derive a convenient formula for $g^{b\sharp}$, assuming that g is convex on rays. Let $h = g^b$. If $a \cdot y > 0$, then

$$h^{a\sharp}(a \cdot y) = \sup_{\eta>0} \big(h(\eta a) - \eta a \cdot y\big)$$

$$= \sup_{\eta>0} \sup_{b\in S^{n-1}} \big(g^{bb}(\eta b \cdot a) - \eta a \cdot y\big) = \sup_{b\in S^{n-1}} \sup_{\eta>0} \big(g^{bb}(\eta b \cdot a) - \eta a \cdot y\big)$$

$$= \sup_{b\in S^{n-1}} \sup_{\eta>0} \Big(g^{bb}(\eta) - \eta \frac{a \cdot y}{b \cdot a}\Big) = \sup_{b\in S^{n-1}} g^{bb\sharp}\Big(\frac{a \cdot y}{b \cdot a}\Big).$$

However, since g^b is assumed to be convex, $g^{bb\sharp} = g^b$. Therefore,

$$h^{a\sharp}(a \cdot y) = \sup_{b\in S^{n-1}} g\Big(b\frac{a \cdot y}{b \cdot a}\Big);$$

and hence

$$g^{b\sharp}(y) = \inf_a \sup_b g\Big(b\frac{a \cdot y}{b \cdot a}\Big) = \inf_a \sup_{\xi\cdot a = y\cdot a} g(\xi).$$

Let us abbreviate the last expression

$$\inf_a \sup_{\xi\cdot a = y\cdot a} g(\xi) = \tilde{g}(y).$$

We have thus seen that if g is convex on rays, then $g^{b\sharp} = \tilde{g}$; and since for any a we may choose $\xi = y$, it is clear that for any function γ, $\tilde{\gamma} \geq \gamma$. Now let γ be a function with convex superlevel sets. Then for a given point y, let $A = \gamma(y)$ and let a be the normal vector of a supporting hyperplane to $\{x : \gamma(x) \geq A\}$ through y. Then

$$\sup_{\xi\cdot a = y\cdot a} \gamma(\xi) = \gamma(y),$$

so

$$\tilde{\gamma}(y) = \gamma(y).$$

Altogether we see that if g is convex on rays, γ has convex superlevel sets, and $g \leq \gamma$, then

$$g^{b\sharp} = \tilde{g} \leq \tilde{\gamma} = \gamma,$$

so the proof is complete. □

In our later applications, it will be desirable to deal with functions g for which $g^\flat$ is concave on rays. This is not automatically fulfilled, since even though each function $t \mapsto g^{a\flat}(a \cdot t)$ has this property, in general the concavity will be destroyed when we take the supremum over all $a \in S^{n-1}$. We shall see, however, that if g has convex superlevel sets then $g^\flat$ will be concave on rays. To this end, we introduce yet another transform, g°, which always has this property, and prove that $g^\flat = g^\circ$ if g has convex level sets.

Definition 3. For $a \in S^{n-1}$ and $\eta > 0$, let

$$g_a(\eta) = \sup_{y \cdot a = \eta} g(y),$$

and define

$$g^\circ(sa) = g_a^\flat(s), \quad a \in S^{n-1}, \ s > 0.$$

Since any function $g^\flat$ is concave, it follows that $g^\circ(t)$ is concave on rays.

Proposition 3.2. *Assume g is upper semicontinuous and convex on rays. Then $g^\circ \geq g^\flat$, and equality holds if g has convex superlevel sets. In particular, $g^\flat$ is concave on rays if g has convex level sets.*

In the proof of Proposition 3.2, we use the following lemma, which is a variant of von Neumann's minmax theorem.

Lemma 3.3. *Let $G(\eta, \xi) \geq 0$ be upper semicontinuous on $\mathbb{R}^+ \times \mathbb{R}^N$. Assume that $\eta \mapsto G(\eta, \xi)$ is convex for each fixed ξ and that, for fixed $\eta > 0$, any set $\{\xi;\ G(\eta, \xi) \geq A\}$ is connected. Assume also that $G(\eta, \xi)$ tends to ∞ when η tends to either 0 or ∞; that for each $\eta > 0$, $\sup_t G(\eta, t) > G(\eta) =: \limsup_{|t| \to \infty} G(\eta, t)$; and that $G(\eta)$ is continuous. Then*

$$\inf_\eta \sup_\xi G(\eta, \xi) = \sup_\xi \inf_\eta G(\eta, \xi).$$

Proof. Extend the domain of definition of $G(\eta, \xi)$ to $[0, \infty] \times \widehat{\mathbb{R}^N}$, where $\widehat{\mathbb{R}^N}$ is the one-point compactification of $\mathbb{R}^N$, by putting it equal to $G(\eta)$ when ξ is the point at infinity and to infinity when η equals 0 or ∞. Since for any η,

$$\sup_\xi G(\eta, \xi) \geq \sup_\xi \inf_\eta G(\eta, \xi),$$

it follows that

$$\inf_\eta \sup_\xi G(\eta, \xi) \geq \sup_\xi \inf_\eta G(\eta, \xi).$$

To obtain the converse inequality, let

$$A = \inf_{\eta} \sup_{\xi} G(\eta, \xi)$$

and let $E = \{G \geq A\}$. Let

$$E^{\xi} = \{\eta \in [0, \infty] : (\eta, \xi) \in E\} \quad \text{and} \quad E_{\eta} = \{\xi \in \widehat{\mathbb{R}^N} : (\eta, \xi) \in E\}.$$

Then E satisfies the hypothesis in the geometric-topological Lemma 3.4 below. Thus E^{ξ_0} is the full halfline $[0, \infty]$ for some $\xi_0 \in \widehat{\mathbb{R}^N}$, and hence $\inf_{\eta} G(\eta, \xi_0) \geq A$, which proves the lemma provided that ξ_0 is not ∞. However, $\sup_t G(\eta_0, t) = A$ for some $\eta_0 \neq 0, \infty$, and from the last assumption on G it follows that E_{η_0} is compact in $\mathbb{R}^N$, and hence $\xi_0 \in \mathbb{R}^N$. □

Lemma 3.4. *Suppose that E is a closed set in* $[0, \infty] \times \widehat{\mathbb{R}^N}$ *such that each* E_{η} *is connected, compact and nonempty in* $\widehat{\mathbb{R}^N}$ *and that* $E_0 = E_{\infty} = \widehat{\mathbb{R}^N}$. *Moreover, assume that each* E^t *consists of at most two components. Then there exists t such that* E^t *is the whole interval* $[0, \infty]$.

Proof. Consider a fixed E_{η_0}. Each point (η_0, t) in this set belongs either to the left component or the right component of E^t. (If it belongs to both, we are done of course.) We call such a point a left point or a right point, respectively. We claim that there is an η_0 such that E_{η_0} contains both left points and right points. In fact, if all points of E_{η} are left points for $\eta < \eta_1$ then E_{η_1} contains left points as well by compactness. Hence if η_0 is the supremum of all η such that E_{η} only has left points, and η_1 is the infimum of all η such that E_{η} only has right points, then $\eta_0 = \eta_1$, since otherwise E_{η} would be empty for $\eta_0 < \eta < \eta_1$.

Now consider E_{η_0} with left points as well as right points. It is easy to see that both the set of left points and the set of right points are closed. Since E_{η} is connected, they must have a common point. This concludes the proof. □

Proof of Proposition 3.2. We have to prove that $g^{\circ} \geq g^{\flat}$ and that equality holds if g has convex level sets. We verify the statement at a fixed point, which with no loss of generality we may take to be $t = (s, 0, \ldots, 0)$ with $s > 0$. Then

$$g^{\circ}(t) = \inf_{\eta > 0} \sup_{\xi'} \big(g(\eta, \xi') + \eta s\big) = \inf_{\eta > 0} \sup_{\xi'} \big(g(\eta, \eta\xi') + \eta s\big).$$

On the other hand, letting $a = (a_1, a')$, we have

$$\begin{aligned}
g^\flat(t) &= \sup_{a\in S^{n-1};\ a_1>0} g^{a\flat}(a_1 s) = \sup_a \inf_{\eta>0} \big(g(\eta a) + \eta a_1 s\big) \\
&= \sup_a \inf_{\eta>0} \big(g(\eta a/a_1) + \eta s\big) = \sup_a \inf_{\eta>0} \big(g(\eta, \eta a'/a_1) + \eta s\big) \\
&= \sup_{\xi'\in\mathbf{R}^{n-1}} \inf_{\eta>0} \big(g(\eta, \eta\xi') + \eta s\big).
\end{aligned}$$

Since inf sup $\geq$ sup inf, it follows that $g^\circ \geq g^\flat$. Now assume that g has convex level sets, and let

$$G(\eta, \xi') = g(\eta, \eta\xi') + \eta s.$$

The set of ξ' such that $G(\eta, \xi') \geq A$ is the intersection of a superlevel set of g with a hyperplane, and is therefore convex and hence connected. Since g is convex on rays, $\eta \mapsto G(\eta, \xi')$ is convex. Lemma 3.3 therefore implies that $g^\circ = g^\flat$. □

It is clear from the proof that $g^\circ = g^\flat$ as soon as g has the property that all the sets $K_A = \{g \geq A\}$ have connected intersections with all hyperplanes outside the origin.

The definitions of the transforms $g \mapsto g^\flat$ and $g \mapsto g^\circ$ perhaps become clearer if we consider a more restricted class of functions. Let g be a function in $C^\infty(\mathbb{R}^n \setminus \{0\})$ which has the property that the gradient map

$$y \mapsto \nabla g(y)$$

is bijective from $\mathbb{R}^n \setminus \{0\}$ to itself. We then define $g^*(t)$ as the critical value of

$$y \mapsto g(y) + y \cdot t,$$

i.e., as the value of $g(y) + y \cdot t$ at the unique point y where $\nabla g(y) + t = 0$. We then have

Proposition 3.5. *Assume that $g \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and that $y \mapsto \nabla g(y)$ is bijective in $\mathbb{R}^n \setminus \{0\}$.*

(i) If g is strictly convex on rays, then $g^* = g^\flat$.

(ii) If g has convex level sets, then $g^* = g^\circ$.

This proposition has the amusing corollary that g^* has convex level sets if g is strictly convex on rays (since any function $h = g^\flat$ has convex level sets) and that g^* is concave on rays if g has convex level sets.

Lemma 3.6. *Let $G(\eta, \xi)$ be a smooth function in $\mathbb{R}^n_\eta \times \mathbb{R}^m_\xi$. Assume that either*

(i) *for each η, the set*

$$M_\eta = \{\xi : \; G(\eta,\xi) = \sup_\xi G(\eta,\xi)\}$$

only consists of one single point, or

(ii) $n = 1$ *and for each η, M_η is connected.*

Then, if the min-max $\inf_\eta \sup_\xi G(\eta,\xi)$ *is attained at some point, it is also attained at some critical point.*

Proof. Assume that $\inf_\eta \sup_\xi G(\eta,\xi) = G(0,0)$ and furthermore that (i) holds. We must prove that $(\partial G/\partial\eta)(0,0) = 0$. If this is not the case, then (possibly after a linear change of variables) we may assume that $(\partial G/\partial\eta_1)(0,\xi) > 0$ for all ξ in a compact neighbourhood E of 0. Then $G(-\epsilon, 0, \dots, \xi) < G(0,\xi)$ for all sufficiently small ϵ and $\xi \in E$. By assumption, there is $\xi(\epsilon)$ such that $G(-\epsilon, 0, \dots, 0, \xi(\epsilon)) \geq G(0,0)$. Since $\xi(\epsilon)$ lies outside of E, when $\epsilon \to 0$, we get a point $\xi_0 \neq 0$ with

$$G(0,\xi_0) \geq G(0,0) = \sup_\xi G(0,\xi).$$

This contradicts our assumption that M_0 consists of only one point.

Next, assume that (ii) holds, so that M_0 is connected. Then if $(\partial G/\partial\eta)(0,\xi) \neq 0$ for all $\xi \in M_0$, it must be either strictly negative or strictly positive on M_0. The proof is then concluded as in the case (i). □

Remark 2. Probably the lemma holds if we assume that any M_η has a neighbourhood basis $\{U\}$ such that $\Pi_k(U)$ are trivial for $k \leq n-1$. That this assumption is necessary can be seen from the example

$$G(\eta,\xi) = f(|\xi|) + \xi \cdot \eta,$$

where $f(|\xi|) = |\xi|^2 e^{-|\xi|^2}$. Then $\inf_\eta \sup_\xi G(\eta,\xi)$ is attained when $\eta = 0$ and $|\xi| \neq 0$, but $\partial G/\partial\eta \equiv \xi$, so no such point is critical. In this example, M_0 is a sphere S^{n-1} and thus $\Pi_{n-1}(M_0)$ is not trivial. □

We can now prove Proposition 3.5. Let $t \neq 0$ be a point in $\mathbb{R}^n$, which without loss of generality we can take to be $t = (t_1, 0, \dots, 0)$. By definition,

$$g^\circ(t) = \inf_\eta \sup_{y'} (g(\eta, y') + \eta t_1).$$

If g has convex superlevel sets the $\sup_{y'}$ is attained in a connected set; so Lemma 3.6 implies that $g^\circ(t)$ is a critical value of $g(y) + y \cdot t$. Therefore, $g^\circ = g^*$; so we have proved the second part of Proposition 3.5. The first part is proved in a similar way from the definition of $g^\flat$.

4 Almost holomorphic extensions from $\mathbb{R}^n$

We are now ready to give the analog of Theorem 2.2 in higher dimensions. For $z \in \mathbb{C}^n$, let $z = x + iy$.

Theorem 4.1. *Let h be a weight function that is concave on rays, and let $g = h^\sharp$. Suppose that f is a tempered distribution on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} |\hat{f}(t)| e^{h(t)} dt < \infty.$$

Then f can be extended to a function F satisfying

$$\sup_x \int_{|y|<1} |\bar{\partial} F(x+iy)| e^{g(-y)} \leq 1/2 \int |\hat{f}(t)| e^{h(t)} dt, \tag{4.1}$$

and, if h is of class C^2,

$$\sup_{z\in\Omega} |\bar{\partial} F(z)| e^{g(-y)} \leq 1/2 \sup \sum |\hat{f}(t)| e^{h(t)} \frac{\sum h^{jk} t_j t_k}{|t|^2}. \tag{4.2}$$

Proof. Write

$$f(x) = \int \hat{f}(t) e^{ix\cdot t} dt = \int_{|a|=1} f_a(a\cdot x) dS(a),$$

where

$$f_a(s) = \int_0^\infty \hat{f}(ra) e^{irs} r^{n-1} dr.$$

Then $\hat{f}_a(r)$ is $\hat{f}(ra) r^{n-1}$ for $r > 0$ and 0 for $r < 0$.

By Theorem 2.2, we can extend each function f_a to a function $F_a(\zeta)$ in $\mathbb{C}$ such that (with $\zeta = \xi + i\eta$)

$$\sup_\xi \int \left| \frac{\partial F_a}{\partial \bar{\zeta}} (\xi + i\eta) \right| e^{h^{a\sharp}(-\eta)} d\eta \leq 1/2 \int_0^\infty |\hat{f}_a| e^{h^a}.$$

If

$$F(z) = \int_{|a|=1} F_a(a\cdot z) dS(a),$$

then

$$|\bar{\partial} F| \leq \int \left| \frac{\partial F_a}{\partial \bar{\zeta}} (a\cdot x + ia\cdot y) \right| dS(a).$$

Since $g = \inf_a h^{a\sharp}$, we have for any $x \in \mathbb{R}^n$ that

$$\begin{aligned}\int_{|y|<1} |\bar{\partial} F(x+iy)| e^{g(-y)} dy &\leq \int_{|a|=1} dS(a) \int_{|y|<1} \left| \frac{\partial F_a}{\partial \bar{\zeta}} (a\cdot x + ia\cdot y) \right| e^{h^{a\sharp}(-y\cdot a)} \\ &\leq \int_{|a|=1} dS(a) \int |\hat{f}_a| e^{h^a} = \int |\hat{f}| e^{h(t)} dt.\end{aligned}$$

This proves (4.1), and (4.2) follows in the same way. $\square$

Just as in one variable, Theorem 4.1 has a partial converse.

Theorem 4.2. *Let g be a weight function and let $h = g^\sharp$. Assume that $f \in C(\mathbb{R}^n)$ can be continuously extended to a function F such that*

$$\sup_y \int_{\mathbb{R}^n} |y \cdot \bar\partial F(x+iy)| e^{g(-y)} + F(x+iy)dx < \infty.$$

Then

$$|\hat f(t)| e^{h(t)} \leq C.$$

Proof. Again, this is reduced to the one-dimensional case. Let $\chi(y_1)$ be a cut-off function which equals 1 for $-1/2 < y_1 < 0$ and vanishes for $y_1 < -1$. Then

$$\begin{aligned}\hat f(t) = \int f(x) e^{-it\cdot x} dx &= \int_{y_1<0} \frac{\partial(\chi F)}{\partial \bar z_1}(z_1, x') e^{-it\cdot x + t_1 y_1} dx dy_1 \\ &= \int \chi \frac{\partial F}{\partial \bar z_1}(z_1, x') e^{-it\cdot x + t_1 y_1} dx dy_1 + \int \frac{\partial \chi}{\partial \bar z_1} F e^{-it\cdot x + t_1 y_1} dx dy_1.\end{aligned}$$

Hence

$$|\hat f(t)| \leq \sup_{y_1} e^{-g(-y_1)+t_1 y_1} + Ce^{-t/2} \leq Ce^{-g^{ab}(a\cdot t)}$$

if $a = (1, 0, \ldots, 0)$. Similarly,

$$|\hat f(t)| \leq Ce^{-g^{ab}(a\cdot t)}$$

for any $a \in S^{n-1}$; taking supremum over all a, we get

$$|\hat f(t)| \leq Ce^{-h(t)}.$$

□

Note that in Theorem 4.2 we need only a good bound on the "normal component" $y \cdot \bar\partial F$ of $\bar\partial F$ for some extension F in order to get a good decrease of the Fourier transform. On the other hand, Theorem 4.1 says that if $\hat f$ is rapidly decreasing, one gets the existence of a (possibly different) extension F where all of $\bar\partial F$ is small.

If g is both convex on rays and has convex superlevel sets, then by Proposition 3.1, the weight functions g and h in Theorem 4.2 are dual in the sense that $g = h^\sharp$ and $h = g^\flat$. Moreover, in this case it follows from Proposition 3.2 that h is concave on rays, so Theorem 4.1 applies; and then Theorems 4.1 and 4.2 are almost converses of each other.

Now assume that f has an extension such that $|\bar\partial F| \leq e^{-g(-y)}$. It then follows that $\hat f$ is controlled by e^{-h}, where $h = g^\flat$. However, in case we do not assume that

g has convex superlevel sets, $h = g^b$ will not in general be concave on rays, so we cannot conclude that f has an extension F with $|\bar{\partial}F| \leq e^{-g^{b\sharp}(-y)}$. Such a statement would be an optimal smooth form of the edge of the wedge theorem, but we have been unable to decide whether it holds. We give a weaker substitute for such an extension which is sufficient to obtain a smooth form of the edge of the wedge theorem.

Consider two truncated cones C_1 and C_2 in $\mathbb{R}^n_y$, and let g be ∞ in $C_1 \cup C_2$ and 0 in the complement. Since $g^{b\sharp} \geq g$ and $g^{b\sharp}$ has convex superlevel sets, it follows that $g^{b\sharp} = \infty$ in the convex hull C of $C_1 \cup C_2$; and the fact that f has an extension with $\bar{\partial}F = 0$ in $\mathbb{R}^n_x \times C$ follows from the edge of the wedge theorem.

Theorem 4.3. *Assume that $f \in C(\mathbb{R}^n)$ has extensions F_1 and F_2 satisfying*

$$\int |\bar{\partial}F_j(x+iy)|e^{g_j(-y)} + F_j(x+iy)dx \leq 1,$$

where each g_j is convex on rays and has convex superlevel sets. Then there is a function g, convex on rays and with convex superlevel sets, such that $g \geq \max g_j$, and such that f has an extension F with

$$|\bar{\partial}F| \leq e^{-\frac{1}{2}g(-2y)}.$$

Proof. By Theorem 4.2, $|\hat{f}(t)| \leq e^{-h_j(t)}$ if $h_j = g_j^b$. Hence

$$|\hat{f}(t)| \leq e^{-(h_1(t)+h_2(t))/2}.$$

Since $h = (h_1 + h_2)/2$ is concave on rays, Theorem 4.1 implies that f has an extension with $|\bar{\partial}F| \leq e^{-h^\sharp(-y)}$. One easily verifies that

$$h^\sharp(y) = \frac{1}{2}g(2y) \quad \text{if } g = (2h)^\sharp = (g_1^b + g_2^b)^\sharp.$$

Finally, it is clear that $g \geq g_j^{b\sharp} = g_j$ and that g has convex superlevel sets. □

Next we turn to the generalization of Proposition 2.4.

Theorem 4.4. *Let $g(y)$ be a weight function which is convex on rays and satisfies the regularity assumption*

$$\lim_{y \to 0} \sum \frac{\partial^2 g}{\partial y_j \partial y_k} y_j y_k / |y|^2 g(y) = \infty. \tag{4.3}$$

Let U be a domain in $\mathbb{R}^n$ and let F be a C^1-function in a neighbourhood V of U in $\mathbb{C}^n$ such that $f = 0$ in U. Assume that

$$|y \cdot \bar{\partial}F|/|y| \leq e^{-g(y)}.$$

Then $|F(x+iy)|e^g$ tends to 0 as y tends to 0 for x in any compact set $K \subset\subset U$.

Proof. For $x_0 \in U$ and $a \in S^{n-1}$, let

$$F_{x_0,a}(\zeta) = F(x_0 + \zeta a)$$

for ζ close to 0 in $\mathbb{C}$. Then $F_{x_0,a}(\zeta) = 0$ when ζ is real, and

$$|\bar{\partial}_\zeta F_{x_0,a}(\zeta)| \leq e^{-g^a(\operatorname{Im}\zeta)}.$$

By Proposition 2.4, $|F_{x_0,a}(\zeta)|e^{g^a(\operatorname{Im}\zeta)}$ tends to 0 as $\operatorname{Im}\zeta$ tends to 0, and the result follows. □

We now discuss when there are functions with compact support satisfying our assumption on rapid decay of the Fourier transform. Our first result is essentially due to Beurling [3]; again it follows by a reduction to the one variable case. For a proof, see [4].

Theorem 4.5. *Let h be a nonnegative subadditive function on $\mathbb{R}^n$. Then the following conditions are equivalent.*

(i) *There is a nontrivial function with compact support such that*

$$\int |\hat{f}(t)|e^{h(t)}dt < \infty.$$

(ii) *For all $a \in S^{n-1}$,*

$$\int_1^\infty \frac{h^a(s)ds}{s^2} < \infty.$$

(iii) *There is a nontrivial function with compact support such that*

$$|\hat{f}(t)|e^{h(t)} \leq C.$$

(iv) *There is a sequence of functions χ_k satisfying*

$$\int |\hat{\chi}_k(t)|e^{h(t)}dt < \infty$$

such that χ_k tends to a Dirac measure at the origin in the sense of distributions.

This theorem applies in particular in our setting because of the following result.

Proposition 4.6. *Suppose h is nonnegative, concave on rays and has convex sublevel sets. Then h is subadditive.*

Proof. Since h has convex sublevel sets, we have for $0 < \lambda < 1$

$$h(t + u) = h(\lambda t/\lambda + (1 - \lambda)u/(1 - \lambda)) \leq \max(h(t/\lambda), h(u/(1 - \lambda))).$$

Since h is concave on rays, it follows that $h(st) \leq sh(t)$ for any $s > 1$, so

$$h(t+u) \leq \max((1/\lambda)h(t), 1/(1-\lambda)h(u)).$$

Taking $\lambda = h(t)/(h(t)+h(u))$, we find that

$$h(t+u) \leq h(t) + h(u).$$

The proof is complete. □

Later on, we shall have use for the translation of Theorem 4.5 to conditions in terms of extensions with small $\bar{\partial}$.

Theorem 4.7. *Let g be a weight function which is convex on rays and has convex superlevel sets. Then there is a nontrivial function f in $C^\infty(\mathbb{R}^n)$ with compact support which can be extended to a function F in $\mathbb{C}^n$ satisfying*

$$|\bar{\partial}F(x+iy)| \leq e^{-g(-y)} \tag{4.4}$$

if and only if

$$\int_0^1 \log g_a(s)ds < \infty \tag{4.5}$$

for all $a \in S^{n-1}$. In this case, there is also a sequence of functions satisfying (4.4) *whose restrictions to $\mathbb{R}^n$ tend to a Dirac measure at the origin in the sense of distributions.*

Proof. Suppose that f has compact support and can be extended to a function F which satisfies (4.4). By Theorem 4.4, we may assume that F has compact support. By Theorem 4.2, $|\hat{f}|e^h \leq C$, so by Theorem 4.5,

$$\int_1^\infty \frac{h^a(s)ds}{s^2} < \infty$$

for all $a \in S^{n-1}$. By Proposition 3.2, $h^a = g_a^\flat$; the integrability of $\log g_a$ then follows from Proposition 2.8. By Theorem 4.1, this argument is reversible. □

We henceforth refer to condition (4.5) as the nonquasianalyticity condition. If it is fulfilled, then just as in Section 2, we can find functions with arbitrarily small support satisfying condition (4.5); and this class of functions will be dense in D.

Our final objective in this section is the generalization of Theorems 2.9 and 2.10. We consider $\bar{\partial}$-closed forms of bidegree $(n, n-1)$ defined in $\Omega \setminus \mathbb{R}^n$, where Ω is a domain in $\mathbb{C}^n$. If such a form ω has polynomial growth near $\mathbb{R}^n$, it defines a current across $\mathbb{R}^n$; and $\bar\partial\omega$ is then a distribution on $\Omega \cap \mathbb{R}^n$. We shall now see that $\bar\partial\omega$ can be defined in a similar way, provided a more liberal growth condition $|\omega| \le e^{g(-y)}$, where g satisfies the nonquasianalyticity condition (4.5). First we state the generalization of Theorem 2.9, which says that a local and continuous definition of $\bar\partial\omega$ is not possible if the growth condition is not nonquasianalytic.

Theorem 4.8. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain intersecting $\mathbb{R}^n$, and let ω be a $\bar\partial$-closed $(n, n-1)$-form in $\Omega \setminus \mathbb{R}^n$. Let g be a weight function which is convex on rays, has convex superlevel sets, and does not satisfy* (4.5). *Suppose that*

$$\int |\omega| e^{g(-y)} < \infty$$

(and that g satisfies the technical condition (4.3)*). Let $K \subset\subset \Omega$. Then there are smooth forms v_k in Ω of bidegree $(n, n-2)$ such that*

$$\int_K |\omega - \bar\partial v_k| e^{-g(-y)} \to 0.$$

Proof. Let α be a $(0,1)$-form with support in K such that

$$\int \alpha \wedge \bar\partial v = 0 \tag{4.6}$$

for all smooth $(n, n-2)$-forms in Ω and

$$|\alpha| e^{g(-y)} \le C.$$

We need to prove that $\int \alpha \wedge \omega = 0$. However, (4.6) implies that $\bar\partial\alpha = 0$; therefore, we can solve $\bar\partial F = \alpha$ with a compactly supported F. Then F is a continuous function, and Theorem 4.7 implies that $F \equiv 0$ in $\mathbb{R}^n$. By Theorem 4.4, $|F| \le Ce^{g(-y)}$ on K. Hence

$$\int \alpha \wedge \omega = \lim_{\epsilon \to 0} \int_{|y| > \epsilon} \bar\partial F \wedge \omega = \lim_{\epsilon \to 0} \int_{|y| = \epsilon} F \wedge \omega = 0. \qquad \square$$

Let us now consider weight functions g which satisfy the non quasianalytic growth condition (4.5). We shall see that if ω is a $\bar\partial$-closed $(n, n-1)$-form in $\Omega \setminus \mathbb{R}^n$ such that

$$\int_{\Omega \setminus \mathbb{R}^n} |\omega| e^{-g(-y)} < \infty, \tag{4.7}$$

then we can define $\bar\partial\omega$ across $\mathbb{R}^n$ in such a way that $\bar\partial\omega = 0$ if and only if ω is exact. To simplify, we shall no longer pay attention to the exact growth of ω, but instead consider at the same time all forms ω that satisfy (4.7) for *some* g such that (4.5) holds. We may then as well suppose that $g(y)$ is radial, since if g satisfies (4.5) then

$$\tilde{g}(y) = \sup_{|\xi| \geq |y|} g(\xi)$$

will also satisfy (4.5). (To see this, notice first that (4.5) holds for all a if it holds for a set of a whose convex hull contains 0.)

Now introduce the space

$$\mathcal{B}_g = \{f \in C_c^\infty(\mathbb{R}^n) : \text{there exists } F \in C_c^\infty(\mathbb{C}^n) \text{ with } F = f \text{ on } R^n \\ \text{and } |\bar{\partial} F| e^{g(-y)} \leq C\}.$$

For ω satisfying (4.7), define the action of $\bar{\partial}\omega$ on a function f in $\mathcal{B}_g$ by

$$\bar{\partial}\omega . f = -\int \bar{\partial} F \wedge \omega.$$

Note that if $g_1 \geq g$ and $F \in C_c^\infty(\mathbb{C}^n)$ with $|\bar{\partial} F| \leq e^{-g(-y)}$, we can approximate F by $F_k = \chi_k * F$, where χ_k is an approximate of unity in $\mathcal{B}_{g_1}$ (cf. Theorem 4.7). Then $F_k \in \mathcal{B}_{g_1}$ and

$$\int \bar{\partial} F_k \wedge \omega \to \int \bar{\partial} F \wedge \omega,$$

since

$$\phi(x) = \int \bar{\partial} F(\cdot + x) \wedge \omega$$

is continuous by Lebesgue's theorem. Hence $\bar{\partial}\omega$ is uniquely defined by its action on any $\mathcal{B}_{g_1}$; and, by Theorem 4.4, $\bar{\partial}\omega . f$ does not depend on the extension F chosen.

Theorem 4.9. *Let Ω be a convex domain in $\mathbb{C}^n$ which intersects $\mathbb{R}^n$ ($n > 1$). Let ω be a $\bar{\partial}$-closed $(n, n-1)$-form in $\Omega \setminus \mathbb{R}^n$ such that*

$$\int |\omega| e^{-g(-y)} < \infty$$

for some weight function g satisfying the nonquasianalyticity condition (4.5) *(and condition* (4.3)*). Then the following are equivalent.*

(i) $\bar{\partial}\omega = 0$ *across* $\mathbb{R}^n$.

(ii) *For any $\Omega' \subset\subset \Omega$, there is an $(n, n-2)$-form u in $\Omega' \setminus \mathbb{R}^n$ such that $\bar{\partial} u = \omega$ and*

$$\int |u| e^{-g'(-y)} < \infty$$

for some weight function g' that satisfies the nonquasianalyticity condition (4.5).

(iii) *For any $\Omega' \subset\subset \Omega$, there is an $(n, n-2)$-form u in $\Omega' \setminus \mathbb{R}^n$ such that $\bar{\partial} u = \omega$.*

Proof. It is clear that (ii) implies (i), since

$$\int \bar{\partial} F \wedge \omega = \lim_{\epsilon \to 0} \int_{\operatorname{Im} z = \epsilon} \bar{\partial} F \wedge u = 0$$

if $|\bar{\partial} F| e^{-g}$ is bounded for some weight function g which tends sufficiently rapidly to infinity.

To prove that (i) implies (ii) we shall use Cauchy–Fantappie kernels. Let $S(\zeta, z) = (S_1, \ldots, S_n)$ be defined in $\Omega_z \times \Omega_\zeta$ and satisfy

$$\operatorname{Re} \langle S, \zeta - z \rangle \geq \delta |\zeta - z|^2,$$

and let

$$s = \sum_1^n S_j d(\zeta_j - z_j).$$

Let K be the Cauchy–Fantappie kernel

$$K = \frac{1}{(2\pi i)^n} \frac{s \wedge (\bar{\partial}_{\zeta, z} s)^{n-1}}{\langle S, \zeta - z \rangle^n},$$

and let K_q be the component of K of bidegree (n, q) in z and $(0, n-1-q)$ in ζ. Then K satisfies

$$\bar{\partial}_\zeta K_q + \bar{\partial}_z K_{q-1} = 0$$

for $z \neq \zeta$; and if u is a q-form with compact support in Ω, *Koppelman's formula*

$$u(z) = \int \bar{\partial} u(\zeta) \wedge K_q(z, \zeta) + \bar{\partial}_z \int u(\zeta) \wedge K_{q-1}(z, \zeta)$$

holds. We shall define S so that S is holomorphic in ζ for $|\operatorname{Im} \zeta| < |\operatorname{Im} z|/2$. In particular, this means that if z is outside of $\mathbb{R}^n$, all the coefficients of K are holomorphic functions of ζ near $\mathbb{R}^n$ and hence belong to $\mathcal{B}_g$ for any choice of g. Assume for the moment that S is chosen in this way, and assume that $\bar{\partial}\omega = 0$ across $\mathbb{R}^n$. Let F be a function with compact support in Ω, such that

$$|\bar{\partial} F| e^{\tilde{g}(-y)} \leq C,$$

where $\tilde{g} \geq g$ and satisfies the nonquasianalyticity condition. We claim that, for z outside of $\mathbb{R}^n$,

$$\begin{aligned} F\omega(z) &= \int \bar{\partial} F \wedge \omega \wedge K_{n-1}(z, \zeta) + \bar{\partial}_z \int F\omega(\zeta) \wedge K_{n-2}(z, \zeta) \\ &=: \omega_1 + \bar{\partial} u. \end{aligned}$$

Indeed, this follows from Koppelman's formula if the support of F does not intersect $\mathbb{R}^n$, since ω is closed there. On the other hand, if the support of F does not contain z, the formula follows since by assumption $\bar{\partial}\omega = 0$ across $\mathbb{R}^n$, so

$$\int \bar{\partial}(FK_{n-1}) \wedge \omega = 0.$$

Our claim then follows in general from a decomposition of F, just as in the proof of Proposition 2.10.

Now choose F equal to 1 on Ω' and have compact support in Ω. Observe that the kernel K_{n-2} vanishes for $|\mathrm{Im}\,\zeta| < |\mathrm{Im}\,z|/2$, since S is holomorphic with respect to ζ there. Taking $g'(y)$ equal to $g(y/2)$ plus a term of logarithmic growth to compensate for the growth of the kernel near $\mathbb{R}^n$, we see that u satisfies the desired estimate

On the other hand, ω_1 is a bounded form if $\tilde{g}$ is chosen properly and satisfies $\bar{\partial}\omega_1 = 0$ across $\mathbb{R}^n$. Since ω_1 is bounded, this just means that ω_1 is closed in the sense of currents; and it is then well-known that we can solve $\bar{\partial}v = \omega_1$ with v bounded.

Thus all that remains is the construction of S. For this, let

$$S^0 = (\zeta - \bar{z}),$$

and let

$$S^1 = (\bar{\zeta} - \bar{z}).$$

Then if $z = x + iy$ and $\zeta = \xi + i\eta$,

$$\mathrm{Re}\,\langle S^0, \zeta - z\rangle = (\xi - x)^2 + y^2 - \eta^2 \geq \delta|\zeta - z|^2$$

if $|\mathrm{Im}\,z| > 2|\mathrm{Im}\,\zeta|$, and S^1 satisfies the same inequality everywhere. Let

$$S = \chi S^0 + (1 - \chi)S^1,$$

with $\chi = \chi(|\mathrm{Im}\,z|/|\mathrm{Im}\,\zeta|)$ a suitable cut-off function. This completes the proof that (i) implies (ii).

Finally, we show that (iii) implies (ii), for which we again use the Cauchy–Fantappié kernel. Choose our cut-off function F to be equal to 1 in a neighbourhood of Ω' and to have compact support in Ω. This time we choose the function χ in the definition of S so that $S = S_0$ if z lies in Ω', ζ lies in the support of $\bar{\partial}F$ and η is small enough. One verifies that this can be done in such a way that $\mathrm{Re}\,\langle S, \zeta - z\rangle > \delta|z - \zeta|^2$. Now suppose that $\omega = \bar{\partial}u$ in $\Omega \setminus \mathbb{R}^n$. We claim that if z is outside $\mathbb{R}^n$, then

$$Fu(z) = \int K_{n-2}(z,\zeta) \wedge \bar{\partial}(Fu)(\zeta) + \bar{\partial}_z \int K_{n-3}(z,\zeta) \wedge Fu(\zeta).$$

(The last term should be interpreted as 0 if $n = 2$.) Indeed, this follows from Koppelman's formula applied to $F_1 F u$, where F_1 vanishes near $\mathbb{R}^n$ and equals 1 for $|\mathrm{Im}\,\zeta| > |\mathrm{Im}\,z|/2$, if we observe that K_{n-2} vanishes where F_1 is not equal to 1. Taking $\bar\partial$ of this equation and restricting to Ω', we find that $\omega = \bar\partial u'$, where u' is given by

$$u'(z) = \int K_{n-2}(z,\zeta) \wedge \bar\partial F \wedge u + \int K_{n-2}(z,\zeta) \wedge F\omega.$$

By construction, the first term involves only the values of u at a fixed distance to $\mathbb{R}^n$ and so is bounded for z in Ω'. The second term is handled just as in the previous proof. □

Corollary 4.10. *Let $g(y) = g(|y|)$ and let E be the space of exact forms in $\Omega \setminus \mathbb{R}^n$. Then E is closed in $L^1_{loc}(e^{-g})$ if and only if*

$$\int_0 \log g(t)dt < \infty.$$

Proof. From Theorem 4.8, it follows that E is dense, and hence not closed, in the space of all forms in $L^1_{loc}(e^{-g})$ that are closed in $\Omega \setminus \mathbb{R}^n$ if $\log g$ is not integrable. On the other hand, if $\log g$ is integrable, Theorem 4.9 says that ω lies in E if and only if $\bar\partial\omega = 0$ across $\mathbb{R}^n$; and this is clearly a closed condition. □

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(Received August 27, 2001 and in revised form December 2, 2001)