Number Theory Learning Seminar
Perfectoid Spaces

2014 - 2015

Abstract

These are informal notes[1] of all the lectures released throughout the 2014-2015 Number Theory Learning Seminar at Stanford. For more details on the aim of the course, please refer to the syllabus, available at `http://math.stanford.edu/~conrad/Perfseminar/refs/Syllabus1415.pdf`. Throughout the text, the references are cited as in the syllabus.

Part I - Adic Spaces

1 Lecture 1: Motivation for adic spaces

1.1 Introduction

Today we give some motivation for the introduction of the category of adic spaces, but before doing so we expand on a question asked during the lecture. Although one can functorially associate a rigid analytic variety to a scheme locally of finite type over a field k, why is a systematic theory of rigid-analytic varieties needed, as opposed to ones tied directly to algebraic schemes? The point is that we cannot meaningfully work "locally" in analytic settings (eg, use analytic methods to solve algebraic problems via a GAGA technique) without an intrinsic analytic theory.

For another example one can also think to the case when we deform, say, a proper smooth scheme X_0 over $\mathbf{F}_p$ to a proper formal scheme $\mathfrak{X}$ over $W(\mathbf{F}_p) = \mathbf{Z}_p$ (this arises in Serre-Tate deformation theory, where one lifts abelian varieties over $\mathbf{F}_p$ via deforming their p-divisible group, though one a priori only obtains an abelian formal scheme over $\mathbf{Z}_p$, which may well not be algebraizable). This latter may have *non-algebraic* proper rigid-analytic "generic fiber"! So a general theory is needed to handle such cases.

This first lecture note is organized as follows. We give motivation for the introduction of adic spaces, via some examples and the clear analogy with scheme theory. We shall discuss an example more in detail, expanding on it in the appendix.

1.2 Overview on adic spaces

To start off, consider A an algebra of finite type over an algebraically closed field k, and let X^0 be $\mathrm{MaxSpec}(A)$, and $X = \mathrm{Spec}(A)$. Regard them simply as topological spaces with

[1]Notes prepared by me, Alessandro Maria Masullo, meaning that any typo or mistake contained in them is entirely mine. Revised versions of the notes will appear on the website promptly. Please email me with any comment: `amasullo@stanford.edu`.

their Zariski topology. For field extensions K/k, as for instance $\mathbf{C}/\overline{\mathbf{Q}}$, we have a k-algebra homomorphism

$$A \to A \otimes_k K$$

but there is no natural map $\mathrm{MaxSpec}(A_K) \to \mathrm{MaxSpec}(A)$ (e.g, for $\mathfrak{m} = (X - \pi) \in \mathrm{MaxSpec}(\mathbf{C}[X])$ we have $\mathfrak{m} \cap A = (0) \notin \mathrm{MaxSpec}(A)$ for $A = \overline{\mathbf{Q}}[X]$). This makes the use of X^0 inconvienient for globalizing or dealing with geometric arguments incorporating ground field extensions.

Again, consider instead rings A which do not contain a field (e.g. $A = \mathbf{Z}[t]$). Then there is simply no associated geometric object for them in classical algebraic geometry. Such issues are fixed by $X = \mathrm{Spec}(A)$, although if A is a k-algebra of finite type then X^0 already captures a lot of X, from a topological viewpoint. Let us recall the following:

Definition 1.2.1 A *constructible subset* C of a Noetherian topological space is a finite union of locally closed subsets (or, equivalently, a finite Boolean expression in open subsets).

Let X and X^0 be as above. Given a constructible set C in X, the assignment

$$C \mapsto C^0 := X^0 \cap C$$

yelds an inclusion-preserving bijection

$$\{\text{constructible sets in } X\} \leftrightarrow \{\text{constructible sets in } X^0\}$$

in both directions, and such $C \subset X$ is open, resp. closed, if and only if $C^0 \subset X^0$ is open, resp. closed in the Zariski topology. This works the same for any scheme X locally of finite type over a field, with X^0 its subspace of closed points. A sheaf theory on a topological space is developed on a basis for the topology, and is well posed once one has enough inclusion relations and, correspondengly, transition maps.

The above dscussion yelds the following::

Proposition 1.2.2 *Let X be a scheme locally of finite type over a field, and X^0 its subspace of closed points. We have an equivalence of categories of sheaves of sets*

$$\mathbf{Shv}(X) \simeq \mathbf{Shv}(X^0)$$

defined by $\mathscr{F} \mapsto \mathscr{F}|_{X^0}$, where the categories of sheaves on X and X^0, respectively, are to be considered with respect to the Zariski topology.

Now recall X is of finite type over a field k, and consider the stalk functor at $x \in X$

$$\mathbf{Shv}(X) \to \mathbf{Set}$$

given by sending a Zariski sheaf $\mathscr{F}$ on X to its stalk at x, $\mathscr{F}_x$. Due to the equivalence between the Zariski site on X and that on X^0, we expect to be able to describe the stalk functor purely in terms of X^0 and its sheaf theory. As we have

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}^0(U^0)$$

with clear meaning of notation, this is in turn equivalent to characterizing the set

$$\{U^0 \subset X^0 \mid U \ni x\}$$

purely in terms of X^0, that is, without "mentioning" x or X.

Definition 1.2.3 Let S be a set, Σ a collection of non-empty subsets of S. We say $F \subset \Sigma$ is a *prime filter* if the following properties are satisfied:
(1) given $U, U' \in F$, then $U \cap U' \in F$ (so in particular $U \cap U' \neq \varnothing$).
(2) given $U \in F$, and $U' \supseteq U$, with $U \in \Sigma$, then also $U' \in F$.
(3) given $U_1, \ldots, U_n \in \Sigma$ such that $\cup U_i \in F$, then some U_i is in F.

As soon as Σ is non-empty, a Zorn's Lemma argument ensures prime filters on Σ exist. It T is a topological space, and F is a prime filter of open sets, then for any $\mathscr{F}$ sheaf of sets on T, the *F-stalk of $\mathscr{F}$* is defined to be

$$\varinjlim_{U \in F} \mathscr{F}(U).$$

One has the following::

Theorem 1.2.4 *Let X be a scheme locally of finite type over a field k, and X^0 be its subspace of closed points. Then the correspondence*

$$X \to \{\text{prime filters of non-empty open subsets of } X^0\}$$

given by $x \mapsto \{\text{open } U^0 \subset X^0 \mid x \in U\}$ is bijective.

Proof. We call X' the set of prime filters of non-empty open subsets of X^0, and prove that X and X' are in bijective correspondence. Given that injectivity of the map $X \to X'$ is clear, we discuss surjectivity, reducing to the case X is affine (left to the reader as an exercise). Let F be a prime filter on X^0. We set

$$\mathscr{O}_{X',F} := \varinjlim_{U \in F} \mathscr{O}_{X_0}(U)$$

and

$$\mathfrak{m}_F := \varinjlim_{U \in F} \{f \in \mathscr{O}_{X^0}(U) \mid \{y \in U \mid f(y) \neq 0\} \notin F\}.$$

Call $A := \mathscr{O}_{X',F}$, so $\mathfrak{m}_F \subset A$ is an ideal. Moreover, if $f \in A \setminus \mathfrak{m}_F$ is defined on some $U \in F$, then

$$\{x \in U \mid f(x) \neq 0\} \in F$$

implying that f is invertible in A. Then A is a local ring with maximal ideal $\mathfrak{m}_F$. We have a natural composition map

$$B := \Gamma(X, \mathscr{O}_X) \to A \to A/\mathfrak{m}_F,$$

whose kernel is a prime ideal $P \subset B$. Let $b \in B$. Then we have

$$X_b \cap X^0 \in F \quad \text{if and only if} \quad b \notin P,$$

where $X_b = \{x \in X \mid b(x) \neq 0\}$. On the other hand, the X_b's form a basis for the Zariski topology on X. By property (3) in Definition 1.2.3, since we can write every element of F as a union of basic open sets X_b, we get maps $B_b \to A$, yelding an isomorphism $\varinjlim_{b \notin P} B_b \simeq A$. This implies that $A = B_P$, so P is a point in X, with the property that $F = \{U^0 \mid P \in U\}$, as desired. $\square$

Remark 1.2.5 In [vdP] and [vdPS], van der Put and Schneider applied the same idea to $\mathrm{MaxSpec}(A)$ for affinoid algebras A over non-archimedean fields k, using quasi-compact admissible open subspaces.

Let us discuss in detail a toy example.

Example 1.2.6 We consider $\mathbf{Q}$ with the usual archimedean topology. We write $(a,b)_q$ for the real interval (a,b) with rational endpoints, and $(a,b)_q^0$ for its intersection with $\mathbf{Q}$. We call this latter an *open rational interval*, and likewise we say $[a,b]_q^0$ is a closed rational interval. We say a union U of such rational open intervals is *admissible* if every closed rational interval $B \subset U$ is covered by finitely many open rational intervals $I_1, \ldots, I_n$ among those defining U.

Let $U \subset \mathbf{Q}$ be an admissible open subset, and $U_i \subset \mathbf{Q}$ admissible open subsets. We say $\{U_i\}$ is an *admissible cover* of U if

$$U = \bigcup_i U_i$$

and every closed rational interval $B \subset U$ is covered by finitely many of the U_i's. We have the following (proved in the Appendix):

Proposition 1.2.7 *The assignment $U \mapsto U^0 := U \cap \mathbf{Q}$ yelds an inclusion-preserving bijection*

$$\{\text{open sets in } \mathbf{R}\} \leftrightarrow \{\text{admissible open sets in } \mathbf{Q}\}$$

sending covers to admissible covers.

Remark 1.2.8 Note that *not* every set-theoretic cover of an admissible open by admissible opens is admissible (much as in rigid-analytic geometry), namely if $U = (0,1)_q^0$ and $U_i = (0,x_i)_q^0 \cup (x_i,1)_q^0$ for an increasing sequence of rationals x_i with $x_i \to 1/\sqrt{2}$ then the U_i's are admissible opens in $\mathbf{Q}$ that cover U as a set yet violates the admissibility requirement using any rational interval $[a,b]_q^0$ with $0 < a < 1/\sqrt{2} < b < 1$ since the irrational $1/\sqrt{2}$ is approximated arbitrarily well by rationals.

With respect to this mild Grothendieck topology on $\mathbf{Q}$, we have an equivalence between the respective sheaf theories over $\mathbf{Q}$ and over $\mathbf{R}$. Basically, the real line is the correct topological space underlying the category of sheaves on $\mathbf{Q}$ with respect to the "topology" we have just introduced, thus playing the role Spec played for MaxSpec.

We remark that the equivalence $\mathbf{Shv}(\mathbf{R}) \simeq \mathbf{Shv}(\mathbf{Q})$ also means the sheaf condition for a sheaf of set $\mathscr{F}$ on $\mathbf{Q}$ and on $\mathbf{R}$ respectively must be checked with respect to the appropriate notion of covering, meaning that we may well have a nonzero abelian sheaf $\mathscr{F}$ on $\mathbf{Q}$ with the property that for all $q \in \mathbf{Q}$, the stalk of $\mathscr{F}$ at q is zero, although $\mathscr{F}$ is not. Consider, for example, the skyscraper sheaf $\mathscr{F}$ at, say $\sqrt{2} \in \mathbf{R}$, with $\mathscr{F}_{\sqrt{2}} = G$, a nonzero abelian group. Then

$$\mathscr{F}(U) = \begin{cases} G & \text{if "}\sqrt{2} \in U\text{"} \\ 0 & \text{otherwise} \end{cases}$$

where the quotation marks simply mean the condition $\sqrt{2} \in U$ is expressed without mentioning $\sqrt{2}$. We have $\mathscr{F}_q = 0$ for all $q \in \mathbf{Q}$, although $\mathscr{F}$ is not the zero sheaf.

Remark 1.2.9 Roughly, Example 1.2.6 indicates the sense in which the totally disconnected "canonical topology" on an affinoid space will be seen inside an adic space associated to it.

Definition 1.2.10 A topological space T is *sober* if every irreducible closed subset has a unique generic point.

Examples of sober topological spaces are locally Hausdorff spaces and schemes (or any classical algebraic variety with positive dimension). The easiest non-sober space is an infinite set with the cofinite topology, as the space itself is irreducible and it has no generic point.

The following theorem, for which the reader may refer to [MM, IX] (§ 1 (definition of "locale"), § 2 Cor. 4, § 5 Prop. 2), essentially says that if we have a sober topological space T, we can reconstruct it from its sheaf theory, that is, from knowledge of $\mathbf{Shv}(T)$. First, if X and Y are topological spaces, recall that for any morphism

$$f : X \to Y$$

we have an induced functor

$$f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$$

and in fact an adjoint pair (f_*, f^{-1}), where f^{-1} is the inverse image functor

$$f^{-1} : \mathbf{Shv}(Y) \to \mathbf{Shv}(X),$$

and f^{-1} is exact (in the sense that it commutes with fiber products and equalizers, or equivalently with all finite limits). A *morphism* $\mathbf{Shv}(X) \to \mathbf{Shv}(Y)$ is a pair of functors (h', h), with

$$h : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$$

and

$$h' : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$$

for which h' is left adjoint to h, and h' is exact.

Theorem 1.2.11 *If X and Y are sober topological spaces, then the natural map*

$$\mathrm{Hom}_{\mathbf{Top}}(X, Y) \to \mathrm{Mor}(\mathbf{Shv}(X), \mathbf{Shv}(Y))/\sim$$

is a bijection, where $\sim$ denotes natural equivalence for adjoint pairs.

As an example, one may consider $X = \{*\}$ to be a single point, yelding

$$\mathrm{Mor}(\mathbf{Set}, \mathbf{Shv}(Y)) = |Y|,$$

the set underlying Y (via stalks and skyscraper sheaves of sets).

Towards adic spaces

In light of our previous discussion, we mention that Huber shows in [H1] that for an affinoid algebra A over a complete nonarchimedean field k there is a naturally associated quasi-compact sober space $\mathrm{Spa}(A)$ containing $\mathrm{Sp}(A)$ as a subset so that the inclusion induces an equivalence of categories

$$\mathbf{Shv}(\mathrm{Spa}(A)) \simeq \mathbf{Sh}(\mathrm{Sp}(A)),$$

where we regard $\mathrm{Sp}(A)$ with the usual topology as in Tate's theory.

Remark 1.2.12 Huber also shows in [H1] (as we'll discuss later) that $U \mapsto U \cap \mathrm{Sp}(A)$ is an inclusion-preserving bijection between the sets of quasi-compact opens in $\mathrm{Spa}(A)$ and quasi-compact admissible opens in $\mathrm{Sp}(A)$, with finite covers corresponding to finite admissible covers. Turning back to Example 1.2.6 and the Appendix, it is unclear if *all* open subsets of $\mathrm{Spa}(A)$ all come from admissible open subsets of $\mathrm{Sp}(A)$, although the two notions match perfectly in the quasi-compact setting.

We give a closer look to van der Put's paper [vdP]. Here van der Put introduced the notion of a *generalized point* of $\mathrm{Sp}(A)$ via prime filters of quasi-compact admissible open subsets. Consequently, the maximal such generalized points are called *closed*, and van der Put proved in [vdP] that they are in natural bijection with the set

$$M(A) := \{|\cdot| : A \to \mathbf{R} \mid |\cdot| \in P_k\},$$

where P_k is simply a notation to mean the set of bounded multiplicative seminorms on A, extending the given absolute value on k. Recall that boundedness, for such seminorms, means that for all elements a in A

$$|a| \leq C \cdot \|a\|$$

where $\|\cdot\|$ is a (in fact, any, as they are all equivalent) Banach k-algebra norm on A and $C > 0$ is a constant which may depend on the Banach norm. $M(A)$ is the Berkovich spectrum of A.

Likewise $\mathbf{Q}$ in Example 1.2.6, $\mathrm{Sp}(A)$ has not "enough points", and the defects which this fact yelds can already be seen, for example, extending the nonarchimedean base field (eg. from $\mathbf{Q}_p$ all the way to $\mathbf{C}_p$) as follows. Let

$$f : A \to B$$

be a map of k-affinoid algebras, and call

$$f_K : K\widehat{\otimes}_k A \to K\widehat{\otimes}_k B$$

We have the following:

Proposition 1.2.13 *The map $f : A \to B$ as above is flat (in the commutative-algebraic sense) if and only if the morphism $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ is flat (that is, is flat on stalks of the respective structure sheaves).*

Proof. We call $X = \mathrm{Sp}(B)$ and $Y = \mathrm{Sp}(A)$. Let $x \in X$ and $y = \mathrm{Sp}(f)(x)$ in Y correspond, respectively, to a maximal ideal $\mathfrak{m}'$ of B and $\mathfrak{m}$ of A. The local ring $\mathscr{O}_{X,x}$ of X at x is generally not $B_{\mathfrak{m}'}$. However, the natural local map of local noetherian rings

$$B \to \mathscr{O}_{X,x}$$

factors through $B_{\mathfrak{m}'}$ and induces an isomorphism of completions ${B_{\mathfrak{m}'}}^\wedge \simeq \mathscr{O}_{X,x}^\wedge$ (see [BGR, 7.3.2/3]. Likewise, $A_{\mathfrak{m}}^\wedge \simeq \mathscr{O}_{Y,y}^\wedge$, and a local map between local noetherian rings is flat if and only if the induced map between completions is flat (see [Mat, 22.4]). Since $f : A \to B$ is flat if and only if $A_{\mathfrak{m}} \to B_{\mathfrak{m}'}$ is flat for all $\mathfrak{m}'$, this in turn is equivalent to the flatness of the completion of the induced stalk map

$$\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$$

for all x and $y = \mathrm{Sp}(f)(x)$. But that in turn is equivalent to the flatness of the induced map on stalks (without completion), or in other words the flatness of $\mathrm{Sp}(f)$. $\square$

Now the question is, given that f is flat, whether or not f_K is flat as well. The answer is yes, but to prove it it's used Raynaud's theory on formal models for rigid spaces. The metric completion inherent in such scalar extension is the main source of difficulties, but at a more geometric level one has the annoyance that, roughly speaking, that in the following diagram the dotted arrows do not exist:

$$\begin{tikzcd} \mathrm{Sp}(A_K) \arrow[d, dashed] & \mathrm{Sp}(B_K) \arrow[l, "\mathrm{Sp}(f_K)"'] \arrow[d, dashed] \\ \mathrm{Sp}(A) & \mathrm{Sp}(B) \arrow[l, "\mathrm{Sp}(f)"] \end{tikzcd}$$

A glance at formal models

Let k be a nonarchimedean field, with valuation ring $\mathcal{O}$. Roughly, a formal model for a rigid space X over k is a quasi-compact formal scheme $\mathfrak{X}$ over $\mathcal{O}$, which is locally isomorphic to

$$\mathrm{Spf}\left(\mathcal{O}\{t_1, \ldots, t_n\}/(f_1, \ldots, f_m)\right).$$

If $0 < |\pi| < 1$, then $\mathcal{O}\{t_1, \ldots, t_n\}[\frac{1}{\pi}]$ is a Tate algebra, so one can associate to such $\mathfrak{X}$ a quasi-compact quasi-separated rigid space over k by gluing affinoids

$$\mathrm{Sp}\left(k\langle t_1, \ldots, t_n\rangle/(f_1, \ldots, f_m)\right),$$

yelding a "generic fiber" $\mathfrak{X}^{\mathrm{rig}}$ of the formal scheme $\mathfrak{X}$ over $\mathcal{O}$.

More in detail, if X is a quasi-compact quasi-separated rigid space over k, then Raynaud proved X has the form $\mathfrak{X}^{\mathrm{rig}} := \mathfrak{X} \otimes k$ for some formal scheme $\mathfrak{X}$ as above flat over $\mathcal{O}$, and in the affinoid case, explicitly, if we let

$$A := k\langle t_1, \ldots, t_n\rangle/I$$

for some ideal I of $k\langle t_1, \ldots, t_n\rangle$, then for $X = \mathrm{Sp}(A)$ we can choose

$$\mathfrak{X} = \mathrm{Spf}\left(\mathcal{O}\{t_1, \ldots, t_n\}/(\mathcal{O}\{t_1, \ldots, t_n\} \cap I)\right).$$

(If $|k^\times| \subset \mathbf{R}^\times_{>0}$ is not discrete then some work is needed to show $\mathcal{O}\{t_1, \ldots, t_n\} \cap I$ is finitely generated).

Suppose $\mathfrak{X}$ and $\mathfrak{X}'$ are two formal models for the same quasi-compact, quasi-separated rigid space X over k. There can well exist two such. In fact, if $\mathfrak{X}$ is a formal model for X, then "blowing up" $\mathfrak{X}$ along an open ideal ("supported on the special fiber") does not change the generic fiber X. However, they are related in the sense that there always exists another formal model $\mathfrak{X}''$ of X, flat over $\mathcal{O}$, with unique maps $\mathfrak{X}'' \to \mathfrak{X}$ and $\mathfrak{X}'' \to \mathfrak{X}'$ respecting the generic fiber identifications with X. The reader may refer to [Bosch].

Let us state the following Theorem of van der Put and Schneider, in a preliminary form, for now.

Theorem 1.2.14 *Let X be a quasi-compact, quasi-separated rigid space over k. Then*

$$\varprojlim_{\mathfrak{X}^{\mathrm{rig}} \simeq X} |\mathfrak{X}|$$

is homeomorphic to the adic space attatched to X, which is to be defined later.

In their paper [vdPS], such a space goes under the name of "space of the prime filters of quasi-compact open subsets". As a concluding remark, we underline that the Berkovich spectrum of an affinoid k-algebra A, which we called $M(A)$ (the space of rank 1 valuations on A subject to some conditions) will turn out to be exactly the maximal Hausdorff quotient of $\mathrm{Spa}(A)$. We will Beware however that although $\mathrm{Spa}(A)$ will be defined as a certain set of (possibly higher-rank!) valuations on A, the inclusion $M(A) \hookrightarrow \mathrm{Spa}(A)$ will *not* be continuous (since "rational domains" in the target will be open yet have preimage that is compact Hausdorff and essentially never open away from trivial situations)! Nonetheless, the image of this generally discontinuous section is useful: the "rank 1" points (arising from $M(A)$ inside $\mathrm{Spa}(A)$) will turn out to be sufficiently abundant in $\mathrm{Spa}(A)$ that for certain problems it is sufficient to work locally near such points (even though $\mathrm{Spa}(A)$ generally has many points corresponding to "higher-rank" valuations, in a sense to be made precise later).

1.3 Appendix: an example

First of all, we recall the setting of Example 1.2.6.

Throughout, for an open subset $V \subset \mathbf{R}^n$, V^0 will denote $V \cap \mathbf{Q}^n$. In particular, for real numbers a_i and b_i, $\prod_{i=1}^n [a_i, b_i]$ will denote the closed box in $\mathbf{R}^n$ and $\prod_{i=1}^n [a_i, b_i]^0$ will denote $\prod_{i=1}^n [a_i, b_i] \cap \mathbf{Q}^n$. Similar conventions will hold for open boxes. In the case a_i, b_i $(i = 1, \ldots, n)$ are rational numbers, we shall write $\prod_{i=1}^n [a_i, b_i]_q$ for the closed box in $\mathbf{R}^n$ with *rational* endpoints, and $\prod_{i=1}^n [a_i, b_i]_q^0$ for its intersection with $\mathbf{Q}^n$. We shall call this latter *closed rational box.*

Definition 1.3.1 A subset $U \subset \mathbf{Q}^n$ is called *admissible open* if it can be written in the form $U = \cup_j I_j$ with I_j open rational boxes such that for any closed rational box $B \subset U$, there exist finitely many j, say $j_1, \ldots, j_m$, such that $B \subset I_{j_1} \cup \cdots \cup I_{j_m}$.

Any admissible open subset of $\mathbf{Q}^n$ is open by definition, and it's natural to ask whether the converse holds, that is, if all open subsets $V \subset \mathbf{R}^n$ give rise to admissible open subsets V^0 in $\mathbf{Q}^n$. The answer is yes for $n = 1$, no for $n > 1$. The argument for the case $n = 1$, and counterexample for the case $n > 1$, are due to Zev Rosengarten after the lecture.

Proposition 1.3.2 *Every open set $V \subset \mathbf{Q}$ is admissible open.*

Proof. Let $V \subset \mathbf{Q}$ be open, and let $U \subset \mathbf{R}$ be the maximal open subset of $\mathbf{R}$ such that $U \cap \mathbf{Q} = V$. Such U exists, as one can take the union of all open subsets W of $\mathbf{R}$ such that $W \cap \mathbf{Q} = V$. Write $U = \cup_j I_j$, with the I_j open intervals with rational endpoints. We claim that the representation $V = \cup_j I_j^0$ exhibits V as an admissible open set. To see this, let us consider a closed box $[a, b]_q$, with $[a, b]_q^0 \subset V$. Then $[a, b]_q \subset U$. Indeed,

$$(U \cup (a, b)) \cap \mathbf{Q} = V$$

so $(a, b)_q \subset U$ by maximality of U, and $a, b \in U$ because $a, b \in V$. By compactness of $[a, b]_q$, we have $[a, b]_q \subset \cup_{k=1}^N I_{j_k}$ for some $j_1, \ldots, j_N$. Then we have

$$[a, b]_q^0 \subset \cup_{k=1}^N I_{j_k}^0,$$

which gives the required finite subcover, and completes the proof. $\square$

The case $n > 1$: a counterexample

Now, for $n > 1$ we will give an example of an open set $V \subset \mathbf{R}^n$, such that $V^0 \subset \mathbf{Q}^n$ is not an admissible open set. First, we give the example for the case $n = 2$. In the last instance, the fact that $\partial[a, b] \subset V$ in the case $n = 1$ (refer to the proof of the above Proposition 1.3.2) is the key fact that fails in dimension greater that 1, as we will clarify in the end. Let V be the region

$$(0, \sqrt{2}) + \{y < |x|\} \subset \mathbf{R}^2.$$

We claim that V^0 is not admissible. Indeed, consider the closed rational box

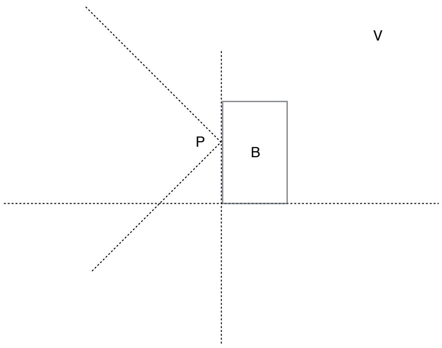

Figure 1: $V = P + \{y < |x|\} \subset \mathbf{R}^2$, with $P = (0, \sqrt{2})$. $B = [0, 1] \times [0, 2]$.

$$B := ([0, 1] \times [0, 2])^0 \subset V^0.$$

There cannot be a finite collection of open rational boxes $I_1, \ldots, I_N \subset V^0$, covering B, as for any such collection, we can argue the following way. There exists some rational number $r > \sqrt{2}$, such that the point $(0, r)$ is in B but $(0, r)$ is not contained in $I_1 \cup \cdots \cup I_N$. Indeed, if the left edge of I_j has nonnegative x-coordinate, or if the top edge has y-coordinate smaller than $\sqrt{2}$, then $(0, r) \notin I_j$. The only other possibility for $I_j \subset U$ is that the bottom edge has y-coordinate greater than $\sqrt{2}$. Let m be the minimum of the y-coordinates of the bottom edges of all such boxes in our finite collection, and choose $r \in (\sqrt{2}, m) \cap \mathbf{Q}$, such that the point $(0, r)$ of $\mathbf{Q}^2$ is contained in B. We have

$$(0, r) \notin I_1 \cup \cdots \cup I_N,$$

as desired. This shows that V^0 is not admissible. The case $n > 2$ is constructed from the case $n = 2$ taking $U := (V \times \mathbf{R}^{n-2})^0$, and we automatically have that $U \subset \mathbf{Q}^n$ is not admissible.

Amusingly, an analogous counterexample in the rigid analytic case also seems to require $n \geq 2$: see [Con, Ex. 2.2.12].

Final remarks

In fact, all that is needed to produce such non admissible open $U \subset \mathbf{Q}^n$ lies in the following:

Proposition 1.3.3 *Let C in $\mathbf{R}^n$ be a closed subset satisfying the following properties.*

(1) *$C \cap \mathbf{Q}^n$ is dense in C.*

(2) *There exists a closed rational box $B \subset \mathbf{R}^n$ such that*

(a) $B \cap C \neq \varnothing$.

(b) $B \cap C \cap \mathbf{Q}^n = \varnothing$.

Then $U := (\mathbf{R}^n \setminus C)^0$ *is non-admissible.*

We remark that such couple (C, B) cannot exist in the case $n = 1$.

Proof. Suppose U is admissible. We have $B^0 \subset U$ by $(2.b)$, and $B^0 \subset \cup_{i=1}^m I_i$ by assumption, for open rational boxes $I_i \subset U$.
Let $x \in B \cap C$, which exists by $(2.a)$. There exists a sequence of points with rational coordinates $\{q_n\}_{n \geq 0}$ contained in B, and converging to x in $\mathbf{R}^n$. By Bolzano-Weierstrass, up to renumbering indexes and extracting a subsequence, we can assume $\{q_n\}$ is contained in the closure in $\mathbf{R}^n$ of one of the I_i's. Call such closure B', which, therefore, contains x.

We claim $x \in B' \cap C$ is an isolated point in $B' \cap C$.

Step 1 First, we show x cannot lie in the interior of B'. If not so, there exists an open subset $W \subset B'$ in $\mathbf{R}^n$, containing x. Then $W \cap C$ is nonempty, and hence is open in C. Since $C \cap \mathbf{Q}^n$ is dense in C by property (1), $W \cap C \cap \mathbf{Q}^n$ must be nonempty, thus implying that $B' \cap \mathbf{Q}^n$ is not contained in U. A contradiction.

Step 2 x is, thus, contained in the boundary of B'. If it is not an isolated point in $B' \cap C$, there exists another point $x' \in B' \cap C$ with the property that the whole segment $L_{xx'}$ with endpoints x and x' is still contained in $B' \cap C$. By density of $\mathbf{Q}$ in $\mathbf{R}$, and since x', likewise x, must be contained in the boundary of B', $C \cap B' \cap \mathbf{Q}^n$ would be nonempty, which, again, contradicts the fact that $B' \cap \mathbf{Q}^n \subset U$.

We can, therefore, shrink B' in such a way that $B' \cap C = \{x\}$. Notice that we can substitute B with B', and we are reduced to the case $B \cap C = \{x\}$, and C is convex, as all that matters is C in a neighbourhood of x. The same argument proposed above for $n = 2$ now establishes that B^0 cannot be covered by finitely many open rational boxes in U. □

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