

High-Frequency Perturbational Analysis
of the Surface Point-Source
Response of a Layered Fluid¹

by

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0. *Abstract*

We linearize the relation between the density and velocity profiles of a layered fluid subject to a specified surface point traction, and study a precritical projection of its surface motion. We decompose the resulting linear map into a high-frequency leading term and a lower-order (smoother) remainder, and show that the spectral analysis of the leading term may be described in terms of ray geometry. The leading term is generally well-conditioned, but may become poorly conditioned in low-velocity zones. These results imply stability estimates for the linearized acoustic reflection inversion problem as well as for the nonlinear problem and yield insight into the behaviour of numerical algorithms for the determination of the density and velocity of a layered fluid from its surface response.

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1. Introduction

The transient response of a (linearly) acoustic fluid to a highly localized source of mechanical energy reflects the mechanical constitution of the fluid. In this paper, we consider the surface motion of a layered acoustic half-space resulting from a traction applied at a point on the (otherwise stress-free) surface. We linearize the relation between the density and velocity profiles and the surface motion, and study the high-frequency asymptotics of this linearized relation. We show that a strictly precritical projection of the high-frequency response is related to the density and velocity perturbations by a linear map whose spectral structure (sensitivity analysis) may be understood in terms of ray geometry. In particular, this map is generally well-conditioned, but may become ill-conditioned for perturbations in low-velocity zones.

The results are so formulated that they imply stability results for the linearized acoustic reflection inversion problem and for the nonlinear inverse problem as well. We sketch these extensions, together with implications for the design of numerical algorithms; detailed discussions will appear elsewhere.

We state the main results of this analysis in Section 2, after description of the boundary-value problem of a layered fluid with prescribed surface pressure. In section 3 we discuss the relation of our results to previous work on the layered acoustic inverse problem, and point out features such as the role of regularity of the reference profiles in ensuring the validity of the results, and the importance of proper truncation in the definition of the Radon transform. Section 4 is devoted to the relation between the (suitably) truncated Radon transform of the point-source field and the solutions of the plane-wave equations: namely, these have the same high-frequency asymptotics. The plane-wave equations are one (-space)-dimensional scalar hyperbolic boundary value problems. Their high-frequency asymptotics are derived in Section 5. The leading order (in frequency) behaviour of a precritical projection of the point-source response, defined *via* the truncated Radon transform, is related in Section 6 to the spectral analysis of a 2-by-2 matrix multiplication operator. The spectrum of this operator is then estimated in terms of simple ray-theoretic constructions. Section 7 sketches the implications of the high-frequency analysis for the

broad band linearized problem, the nonlinear inverse problem, and the design of numerical algorithms. Section 8 restates our main conclusions.

We include two appendices. Appendix A gives the computation of the perturbed plane-wave equations satisfied by the truncated Radon transform. Appendix B describes the high-frequency asymptotics of the perturbational point-source response. We use this expansion in section 4 to justify our choice of truncation, but it may also be of interest in its own right.

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2. Notations, Statement of Main Results

We write $(\mathbf{x}, z) = (x_1, x_2, z)$ for the Cartesian coordinates of a point in three-dimensional Euclidean space.

We shall write ρ for the material density function and λ for the incompressibility function of a layered fluid: thus, both ρ and λ depend on z alone.

Formally, the displacement response to a transient pressure load $f(\mathbf{x}, t)$ across $\{z=0\}$ is the 3-vector solution \mathbf{u} to the linear elastic boundary value problem in the limit of vanishing shear modulus (see Achenbach (1973), p. 78):

$$\rho \ddot{\mathbf{u}} = \nabla \lambda \nabla \cdot \mathbf{u}, \quad z > 0 \quad (2.1a)$$

$$\lambda(0) \nabla \cdot \mathbf{u}(\mathbf{x}, 0, t) = f(\mathbf{x}, t) \quad (2.1b)$$

$$\mathbf{u} \equiv 0, \quad t \ll 0 \quad (2.1c)$$

As \mathbf{u} is uniquely determined by (2.1a), (2.1b'), (2.1c), we can view the surface trace of the normal velocity \dot{u}_3 ("seismogram") as the *response* of the medium to the load f : thus we write

$$F(\rho, \lambda) = \dot{u}_3|_{z=0}$$

F may be viewed as a scattering operator: it associates to the mechanical parameter distribution a remote measurement of the acoustic field, generated by a known source.

Because the equations of motion (2.1a) are invariant under translation in \mathbf{x} and t , the response to a load f is the (\mathbf{x}, t) -convolution of f with the response to the point-source impulsive load

$$\lambda(0) \nabla \cdot \mathbf{u}(\mathbf{x}, 0, t) = \delta(\mathbf{x}) \delta(t) \quad (2.1bi)$$

As the properties of the convolution operator are relatively well understood, we shall concentrate exclusively on (2.1bi). We call the resulting surface normal velocity $F(\rho, \lambda) = \dot{u}_3(z=0)$ the *point-source impulse-response*.

Instead of studying F directly, we shall study its formal linearization DF , defined by means of the perturbational boundary value problem:

$$\begin{aligned}
\rho &\leftarrow \rho + \delta\rho \\
\lambda &\leftarrow \lambda + \delta\lambda \\
\mathbf{u} &\leftarrow \mathbf{u} + \delta\mathbf{u}
\end{aligned}$$

$$\begin{aligned}
\rho \partial_t^2 \delta\mathbf{u} - \nabla \lambda_0 \nabla \cdot \delta\mathbf{u} &= -(\delta\rho \partial_t^2 \mathbf{u} - \nabla \delta\lambda \nabla \cdot \mathbf{u}) \\
\lambda(0) \nabla \cdot \delta\mathbf{u}(\mathbf{x}, 0, t) &\equiv 0 \\
\delta\mathbf{u} &\equiv 0, \quad t \ll 0.
\end{aligned}$$

(We shall assume throughout that $\delta\rho(0) = \delta\lambda(0) = 0$.) Then set

$$DF(\rho, \lambda) \cdot (\delta\rho, \delta\lambda) := \partial_t \delta u_3(\cdot, 0, \cdot)$$

Under suitable conditions, DF is actually the derivative of F , and estimates for DF determine the local behaviour of F . See Section 7 for more discussion on this point.

We are particularly interested in the extent to which the response $F(\rho, \lambda)$ is characteristic of, or sensitive to, the medium (ρ, λ) . That is, when do small changes in ρ, λ result in small changes in $F(\rho, \lambda)$ and vice versa? For linear problems, the simplest and most important measure of sensitivity is the *condition number* (see e.g. Golub and Van Loan, 1984, pp. 25 ff.): unfortunately, it will turn out that DF itself is very poorly conditioned (infinite condition number). Fortunately, a certain “precritical” projection of DF is a well-conditioned functional of $\delta\rho, \delta\lambda$.

This *precritical projection* is naturally defined in terms of the *truncated Radon Transform* of the vertical component of displacement:

$$\tilde{U}(p, z, \tau) := \int d\mathbf{x} u_3(\mathbf{x}, z, \tau + p \cdot \mathbf{x}) \eta(|\mathbf{x}|, z, \tau + p \cdot \mathbf{x}, p)$$

(We will henceforth use p to denote the magnitude of the vector slope \mathbf{p} .) Here the cutoff function $\eta(r, z, t, p)$ is suitably smooth in r, z, t , and p , and at the surface satisfies

$$\begin{aligned}
\eta(r, 0, t, p) &= 1, \quad 0 \leq r \leq d_1(p) \\
\eta(r, 0, t, p) &= 0, \quad d_2(p) \leq r
\end{aligned}$$

for suitable *cutoff radii* d_1, d_2 . It is important that the cutoff radii $d_1 < d_2$ be allowed to depend explicitly on the incidence vector (or vector slowness, or ray parameter) p , and that η be allowed to depend on the time t and on the depth z as well. A suitable dependence is discussed below.

The precritical projection is defined by restricting the truncated Radon Transform to a certain subset of (τ, z, p) -space.

We call a pair (z, p) *precritical* if

$$c(z)p < 1$$

where the *sound speed* c is defined by $c = \rho^{-1/2} \lambda^{1/2}$, and Δ -precritical for $\Delta > 0$ if

$$c(z)p \leq \sqrt{1 - \Delta^2}.$$

(The introduction of $\Delta > 0$ is necessary to make our results stable against small perturbations in ρ, c .)

We associate to p the vertical travel time function

$$S(z, p) := \int_0^z d\zeta \frac{\sqrt{1 - c^2(\zeta)p^2}}{c(\zeta)}$$

which is well-defined when (z, p) is precritical.

We call a pair (τ, p) $(\Delta-)$ precritical if $\tau \leq 2S(z, p)$ and (z, p) is $(\Delta-)$ precritical.

For each p , we define the Δ -precritical depth by

$$Z_\Delta(p) = \inf\{z : c(z)p = \sqrt{1 - \Delta^2}\}$$

and correspondingly the Δ -precritical time by

$$S_\Delta(p) = S(Z_\Delta(p), p)$$

Notice that these functions are defined for $0 \leq p \leq \sqrt{1 - \Delta^2}$ when $z \geq 0$ and $c(0) = 1$, which normalization we shall adopt. We shall also have use of the inverse function of $Z_\Delta(p)$, which by definition is monotone nonincreasing:

$$P_\Delta(z) = \sup\{p : c(z')p \leq \sqrt{1 - \Delta^2} \text{ for } 0 \leq z' \leq z\}$$

Thus a point (z, p) or (τ, p) is Δ -precritical if

$$0 \leq p \leq P_\Delta(z)$$

or

$$0 \leq z \leq Z_\Delta(p)$$

or

$$0 \leq \tau \leq 2S_{\Delta}(p)$$

We illustrate the functions $P_{\Delta}(z)$ (figure 1b) and $S_{\Delta}(p)$ (figure 1c) for a typical (coarsely blocked) seismic velocity profile (figure 1a).

We are now in position to define the Δ -precritical projection of the linearized response as the restriction of the truncated Radon transform of DF to the set of Δ -precritical (τ, p) , thus:

$$\tilde{L}(\delta p, \delta \lambda)(\tau, p) = \int d\mathbf{x} \, \eta(|\mathbf{x}|, 0, \tau + \mathbf{p} \cdot \mathbf{x}, p) DF(\rho, \lambda) \cdot (\delta \rho, \delta \lambda)(\mathbf{x}, \tau + \mathbf{p} \cdot \mathbf{x})$$

$$\text{for } (\tau, p) \in R_{\Delta} := \{(\tau, p) : 0 \leq \tau \leq 2S_{\Delta}(p)\}$$

$$\tilde{L}(\delta p, \delta \lambda)(\tau, p) = 0, \quad (\tau, p) \notin R_{\Delta}$$

Our first main result is:

If the function η is chosen properly, we may decompose \tilde{L} as

$$\tilde{L} = L + M, \tag{2.2}$$

where L is defined by the solution of the plane-wave equations

$$\begin{aligned} (\rho \partial_{\tau}^2 - \partial_z \Lambda \partial_z) U &= 0 \\ \partial_z U(\tau, 0, p) &= \delta(\tau) \\ U &\equiv 0, \quad \tau < 0 \end{aligned} \tag{2.3}$$

$$\begin{aligned} (\rho \partial_{\tau}^2 - \partial_z \Lambda \partial_z) \delta U &= -(\delta \rho \partial_{\tau}^2 - \partial_z \delta \Lambda \partial_z) U \\ \partial_z \delta U(\tau, 0, p) &\equiv 0 \\ \delta U &\equiv 0, \quad \tau < 0 \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} c(z) &= \lambda(z)^{\frac{1}{2}} \rho(z)^{-\frac{1}{2}} \\ \Lambda(z, p) &:= \lambda(z) (1 - c^2(z) p^2)^{-\frac{1}{2}} \\ \delta \Lambda(z, p) &= \delta \lambda(z) (1 - c^2(z) p^2)^{-\frac{1}{2}} + \frac{1}{2} \lambda(z) (1 - c^2(z) p^2)^{-\frac{3}{2}} (\delta \lambda(z) \rho(z) - \lambda(z) \delta \rho(z)) p^2 \rho(z)^{-2} \end{aligned}$$

and

$$L(\delta\rho, \delta\lambda)(\tau, p) = \begin{cases} \partial_\tau \delta U(\tau, 0, p), & (\tau, p) \in R_\Delta \\ 0 & \text{else} \end{cases}$$

and M is smoothing: precisely, if we assume for $\delta\rho, \delta\lambda$ the forms

$$\begin{aligned} \delta\rho(z) &= \chi_\rho(z) e^{i\zeta z} \\ \delta\lambda(z) &= \chi_\lambda(z) e^{i\zeta z} \end{aligned}$$

where χ_ρ and χ_λ are smooth envelopes, then

$$M(\delta\rho, \delta\lambda) = O(\zeta^{-N})$$

for any $N = 1, 2, 3, \dots$

Thus the high-frequency asymptotics of L and \tilde{L} are exactly the same.

Remark. We shall take advantage of the linearity of the perturbational problem to write $\delta\rho$ and $\delta\lambda$ as if they were complex, with the tacit understanding that only the real parts have significance.

To define the cutoff radii in such a way that the decomposition (2.2) is valid, introduce the Δ -precritical exit radius

$$X_\Delta(p) = \int_0^{Z_\Delta(p)} dz \frac{pc(z)}{\sqrt{1 - c^2(z)p^2}}$$

That is, a ray (of geometric optics) with ray parameter p leaves the slab $\{0 \leq z \leq Z_\Delta(p)\}$ at a point $(x, Z_\Delta(p))$ with $|x| = X_\Delta(p)$. (For this and other material on ray optics in layered media, see Aki and Richards, pp. 643 ff.). Since we have assumed that the velocity distribution $c(z)$ is smooth, there is a positive gap $D_\Delta(p)$ so that the ray cannot re-enter the slab at a point with x-coordinate less than

$$X_\Delta(p) + D_\Delta(p)$$

(An estimate for $D_\Delta(p)$ will be given in Section 4.) Further, a ray with ray parameter p cannot return to the surface $\{z = 0\}$ at a point with x-coordinate less than

$$2X_{\Delta}(p) + D_{\Delta}(p)$$

This situation is illustrated in Figure 2.

Now define the cutoff radii by

$$\begin{aligned} d_1(z, p) &= 2X_{\Delta}(p) + .1D_{\Delta}(p) \\ d_2(z, p) &= 2X_{\Delta}(p) + .9D_{\Delta}(p) \end{aligned} \tag{a}$$

This choice of d_1, d_2 gives a correct truncation of the Radon transform at the surface. The truncation below the surface is more delicate; details are given in section 4.

Our second major result is:

The operator L decomposes as

$$L = E + K \tag{2.5}$$

where K is a small perturbation in the sense that we have for the condition numbers

$$\kappa(L) \leq C \kappa(E)$$

where the constant C depends on the properties of the reference medium (ρ, λ) .

Therefore the stability analysis of L reduces to that of E . The operator E is best described by introducing the quantities ("reflectivities")

$$\begin{aligned} r &= \partial_z \log p; \quad l = \partial_z \log \lambda \\ \delta r &= \partial_z (\delta \rho / \rho); \quad \delta l = \partial_z (\delta \lambda / \lambda) \end{aligned}$$

as the primary descriptors of the medium and its perturbation. Then E has the form

$$E(\delta r, \delta l)(\tau, p) = (A \delta r + B \delta l)(S^{-1}(\frac{\tau}{2}, p)) \tag{2.6}$$

where $A(z, p)$ and $B(z, p)$ are algebraic expressions in p , $r(z)$, $l(z)$, and $c(z)$, which we compute explicitly. Thus E is a multiplication operator followed by a change of variables.

The explicit formulas for the coefficient A and B in (2.6) lead to our third set of results, concerning the spectral bounds σ_{\min} and σ_{\max} and the condition number κ of the operator E :

- (a) *these quantities are local: that is, when E is restricted to perturbations $(\delta r, \delta l)$ vanishing outside of an interval $[z_1, z_2]$, they depend only on properties of ρ, λ in $[z_1, z_2]$*
- (b) *for perturbations vanishing outside $[z_1, z_2]$, σ_{\min} and σ_{\max} are functions of*

$$\sup_{z_1 \leq z \leq z_2} c(z), \quad \inf_{z_1 \leq z \leq z_2} c(z),$$

$$\sup_{z_1 \leq z \leq z_2} |c'(z)|, \quad \text{and} \quad \inf_{z_1 \leq z \leq z_2} (c(z)P_{\Delta}(z) = \gamma(z))$$

whereas κ is a function only of the third and fourth of these quantities.

- (c) *Suppose that the first three of the above quantities are fixed. Then:*

- (i) *as $\gamma \rightarrow 1$*

$$\sigma_{\min} \rightarrow 2, \quad \sigma_{\max} \rightarrow \infty$$

$$\text{so} \quad \kappa = \frac{\sigma_{\max}}{\sigma_{\min}} \rightarrow \infty$$

- (ii) *as $\gamma \rightarrow 0$*

$$\sigma_{\min}, \sigma_{\max} \rightarrow 0 \text{ in such a way that } \kappa \rightarrow \infty$$

- (iii) *For $\sup |c'(z)|$ small, as a function of γ , κ is convex upward and the minimum value $\kappa_{\text{opt}} \simeq 2.8$ at $\gamma_{\text{opt}} = .85$. As $\sup |c'(z)|$ increases, κ_{opt} increases whereas γ_{opt} decreases.*

The quantity $\gamma = cP_{\Delta}$ is identically $= \sqrt{1 - \Delta^2}$ so long as c increases monotonically. In intervals $[z_1, z_2]$ in which $c(z)$ is less than $\sup \{c(z): 0 \leq z \leq z_1\}$, $\gamma < \sqrt{1 - \Delta^2}$. We call such regions low-velocity zones. A typical plot of γ versus z appears as Figure 1(d). Clearly it is possible that $\gamma \ll \sqrt{1 - \Delta^2}$ in zones of anomalously low velocity.

By combining (a)-(c), we arrive at our major conclusion concerning the spectral analysis of DF .

If we choose Δ so that $\sqrt{1 - \Delta^2} \sim .9$, the Δ -precritical projection of the linearized high-frequency response is well-conditioned except possibly in low-velocity zones.

The instability in low-velocity zones is severe: that is, the data is substantially insensitive to high-frequency perturbations of ρ and λ localized in regions of low velocity. Otherwise put, the

resolution possible in low-velocity zones is much less than that for the overlying high-velocity structure.

The regions of interest in exploration seismology and other applications are often precisely the low-velocity regions. As the lack of resolution in low-velocity zones is a feature of the *problem*, which no amount or type of data processing can ameliorate, we are led to conclude that other information is necessary to stabilize the layered acoustic inverse problem.

Fortunately, such information is often available. For instance, the density and compressional velocity of sedimentary rocks are highly correlated. Accordingly, we are led to consider the restriction of the layered acoustic problem by the pointwise constraint

$$\rho(z) = G(c(z)) \quad (2.7)$$

and its linearization.

Since the constraint (2.7) is local, the problem for highly localized velocity perturbations becomes automatically well-conditioned, as the point version is scalar. In the limit of small aperture, we recover the stability of the plane-wave response for the velocity perturbation problem. The low-velocity zone problem persists, in that the precritical response is less sensitive to perturbations localized in low-velocity zones. If the velocity perturbation is weighted according to the local volume of plane-wave components, however — as is quite natural — the relation becomes entirely well-conditioned. Details are discussed in Section 6.

Remark. The analysis given in the sequel is valid under the assumptions:

- (i) $\log \rho, \log \lambda$ infinitely differentiable for $0 \leq z < \infty$
- (ii) for suitable real k_*, k^* ,

$$k_* \leq \log \rho(z), \log \lambda(z) < k^* \\ \text{for all } z, 0 \leq z < \infty.$$

These assumptions may be weakened somewhat. Indeed, rigorous linkage of the linear and non-linear problems, and the study of convergence of numerical methods, requires that the properties of the linearized map depend only on finitely many derivatives of ρ, λ . The derivation of such

refined results is technically involved, and will be given elsewhere. We will occasionally point out the dependence of various constructions on a finite number of derivatives of ρ, λ however, when it is convenient and instructive to do so.

3. Discussion, Relation to Previous Work

The main application of our results is to the study of the *reflection inversion problem* for a layered acoustic medium: in the notation of Section 2, given surface trace data g , solve the functional equation

$$F(\rho, \lambda) = g$$

for the acoustic parameters ρ, λ , or some related best-fit problem.

This inverse problem has a long history as a model for the interpretation of reflection seismograms; see Claerbout (1976), Cohen and Bleistein (1979), Clayton and Stolt (1981), for a small sample. It has also served as a model in ocean acoustics (Frisk 1980), in the theory of ultrasonic biomedical tomography (Greenleaf, 1983) and (with less justification) in the theory of non-destructive materials evaluation, for which it is flawed because of neglect of shear effects (as is also the case with seismology).

In the geophysical literature, most work has concerned the *velocity* model: that is, the velocity $c = \lambda^{\frac{1}{2}} \rho^{-\frac{1}{2}}$ is allowed to vary with depth, but the density is required to remain constant (and known): see e.g. Lahlou et al. (1983), Tarantola (1984). A number of authors have considered the "complete" inverse problem of separate recovery of ρ and λ : a partial list is Raz (1981a,b), Coen (1981a, b), (1982), Clayton and Stolt (1981), Carrion, Kuo and Stoffa (1984), Howard (1983), Deift and Stickler (1981), Bregman, Chapman and Bailey (1985), Yagle and Levy (1985), Eiges and Raz (1985).

The bulk of this work has depended on the use of some transform of the acoustic field in the horizontal variables to produce a suite of one-dimensional inverse problems, about which a great deal is known. In our previous papers on this subject (Coen and Symes, 1981 and Santosa and Symes, 1985) we have studied Coen's (1981a) suggestion that two plane-wave components (slant stacks) of the surface data be used to construct the density and velocity profiles of a layered fluid. Coen's results imply that a layered fluid is uniquely determined by its point source response, and suggests an algorithm for constructing $\rho(z)$ and $\lambda(z)$. This algorithm has been implemented numerically (Howard, 1983; Santosa and Symes, 1985) and appears to be excessively noise-

sensitive.

Another approach is exemplified by the work of Clayton and Stolt (1981), who study the linearized problem about some (piecewise) smooth reference profiles. Clayton and Stolt in fact allow the reference density and sound velocity to depend on all of the spatial coordinates, and determine the high-frequency asymptotics of the response. In the layered case, the high-frequency asymptotics of the reference field ("WKBJ" Green's operators") may be expressed in closed form (i.e. via quadratures), and rather explicit expressions relate perturbations in ρ and λ to perturbations in the surface trace. Clayton and Stolt note that this method is restricted to precritical reflected waves, and that "... Density is distinguishable from bulk modulus only if a sufficient range of precritical incidence angles is available" (Clayton and Stolt, 1981, p. 1559 (Abstract); see also p. 1563). Because of their straightforward use of the Fourier transform, they are forced to assume that the data are given on the entire surface $\{z=0\}$ (i.e. "infinite aperture") and for all time (see p. 1560).

In this paper we combine a horizontal transform, which preserves locality in time and depth, and high frequency asymptotics to give a spectral description of the linearized problem at smooth ρ , λ . We use a truncated version of the Radon transform, which involves only finite data aperture (and manifestly convergent integrals), yet yields the same high-frequency asymptotics as the formal untruncated (infinite aperture) transform. We also use the full range of precritical data available for each depth, and give quantitative estimates of the *condition* of the resulting precritically projected linearized problem. We find that the sole source of ill-conditioning (i.e. data insensitivity) in this problem is the possible presence of low-velocity zones: that is, the condition of the problem is local in depth, and becomes irretrievably poor in low-velocity zones.

It is interesting to compare these results with those obtained *via* Coen's (1981a) approach and similar approaches, which rely (essentially) on a very small part of the data set, e.g. two plane wave components. As noted above, these methods tend to be excessively noise-sensitive. The question naturally arises: can the estimation of density and velocity profiles be stabilized by use of a larger (hence redundant) subset of the point-source data?

To explain the source of instability in Coen's method, we introduce the notions of *aperture*, that is, the solid angle subtended by the normal vectors of the plane wave components in the restricted data set (two components in Coen's algorithm), and *slowness aperture*, which is depth-dependent and consists of the solid angle subtended by the rays corresponding to those plane-wave components, at each depth. In the typical seismic layered-earth model, the velocity tends to increase with depth, so that each ray is (on the average) convex upward, and eventually turns, except for the ray at precisely normal incidence. In Coen's (1981) algorithm the slab of depth Z is probed with plane-wave components which are required to be precritical in the entire slab: that is, the corresponding family of rays is not allowed to turn in the depth interval $[0, Z]$. If, as is typical, the wave velocity increases by a factor of four over the slab, then the aperture (which equals the slowness aperture at the surface) is at most .25 for any pair of precritical plane wave components, one of which is assumed to be normally incident. See Figure 3.

Now in one of the steps in Coen's algorithm, a quantity extracted from the data is divided by the *square* of the slowness aperture, i.e. multiplied by perhaps a factor of 16 (near the surface). As will be explained below, this quantity has imbedded in it some unavoidable high-frequency error (phase shift) caused by the travel-time/depth conversion. This error is magnified by the division just mentioned, and passes through the rest of the computation. Although the error is confined initially near the surface if the non-normal ray is chosen to be near turning at $z = Z$ (so that the slowness aperture is relatively large at $z = Z$), the computation of the density and incompressibility profiles is progressive in depth, so that both are contaminated throughout the slab.

For a detailed analysis of error propagation in Coen's algorithm, see Coen and Symes (1981), Santosa and Symes (1985).

We partly remedy this problem by using more of the data. Specifically, we allow the aperture, used to determine the profiles to depth z , to vary with z , so that *the slowness aperture remains as large as possible* throughout the slab. Thus a large aperture is used at shallow depths, where the rays can make a large initial angle with the vertical without turning. As the depth

increases, we narrow the aperture, so that the plane-wave components used to determine the profiles are uniformly precritical throughout the slab except in low-velocity zones. As we shall show, this device restores the maximum possible degree of stability to the inverse problem, at the price of increasing its computational complexity. See Figure 4.

The bulk of previous work on the layered acoustic inverse problem has depended either on the special properties of perturbation about a constant background ("Born approximation") or on the reduction to a suite of one-dimensional scalar inverse problems, whose solution is relatively well-understood. The limitations of perturbation about constant background are obvious and well-documented. On the other hand, the second mode of analysis is unavailable for the problem of determining a layered elastic medium (i.e. density and Lamé parameters) from either

- A single (e.g. normal) component of surface motion
- or
- the response in a fluid layer overlying an elastic half-space.

These problems have great practical interest. Unfortunately, transforms in the horizontal variables do not decouple this ("P-SV") problem into scalar inverse problems. The present analysis was developed precisely to overcome this difficulty. Indeed, Paul Sacks has recently used the techniques developed here to establish uniqueness and stability results for the determination of the elastic profiles from the normal component of surface motion. See Sacks (1985), also Sacks and Symes (1985).

In this paper we restrict our attention to the high-frequency asymptotics of the perturbational relation about smooth reference profiles. We consider only smooth reference profiles for several reasons. Our analysis extends without much modification to reference profiles with a few discontinuities, although the ray-optical bookkeeping becomes complicated. On the other hand, the geometric optics description of the wavefield as a singular (high-frequency) incident field plus a smoother (lower-frequency) remainder is no longer valid with *any* modifications when the medium attains a certain degree of roughness.

Second, some degree of smoothness of the profile is required to ensure that the relation between coefficients and band-unlimited surface response is differentiable. The precise number of derivatives of density and velocity (two) follows from arguments similar to those given in Symes (1983), (1985). Without these restrictions, which happen to be precisely those necessary to ensure the validity of the first two terms of the geometric optics expansion, the formal linearizations considered in this paper are not actually derivatives, i.e. do not approximate the coefficient-data relationship locally. The necessity of this restriction flows from the appearance of wave velocity as an unknown in this problem. Small changes in wave velocity give rise to small ("phase") shifts in the travel time map, which have large effect on the relation between high-frequency components in data and coefficient perturbations. These high-frequency components must therefore be constrained *a priori*, in order that the relationship be differentiable. In effect, the requirement that the best-fit-to-data be a smooth optimization problem, hence amenable to variants of Newton's method, imposes *intrinsic resolution limits* on the estimates of density and incompressibility.

Gray (1981) seems to have been the first to point out (a version of) this phase shift difficulty. It is discussed in detail in Coen and Symes (1981), Section 5.

The restriction to precritical data is also necessary to ensure the differentiability of the profile/data relation. In fact, near critical incidence, the high-frequency content of the data changes from enormous to insignificant as the ray parameter passes through its critical value. For fixed ray parameters very near the critical value, this catastrophe can occur as a result of a small perturbation in the wave velocity. Therefore, to keep the data smoothly dependent on the model, we must restrict our attention to the precritical regime.

In the geophysical literature, the fact that only the precritical part of the surface trace depends differentiably on the medium is often expressed by sentiments such as "... the Born approximation ... is not adequate in the evanescent zone." (Clayton and Stolt (1981), p. 1563)

Finally, the smoothness restriction turns out to be necessary even in the *definition* of the precritical data set. In a way, this is expected, since the notion of precritical reflection is entirely ray-theoretical. The way in which the requirement arose was a surprise to the authors, however.

The precritical data set is defined in terms of the Radon transform ("slant-stack"). The naive definition of the Radon transform, common in the geophysical literature, is senseless, i.e. given by a generally divergent integral, at sufficiently large but still precritical incidence, and so cannot be used in forming a precritical projection. Instead, we define a *truncated Radon transform* given by convergent integrals in the entire precritical regime, whose values have the same high-frequency asymptotics as the naive definition would lead one to expect. The argument which establishes this result depends on the geometric optics decomposition, and more generally on certain related estimates of the reflected field, which in turn depend on bounds on some derivatives of the reference coefficients.

Partly due to practical considerations, the definition of the slant-stack (Radon transform) common in the geophysical literature incorporates finite limits of integration. Our suggestion that the truncation ought to be tapered by a smooth cutoff function which depends explicitly on the ray parameter, seems to be new. It is also necessary if the high-frequency asymptotics based on plane-wave analysis are to be retained.

Throughout this work we assume infinite differentiability of the reference profiles, whereas we merely require the coefficient perturbations to have square-integrable derivatives. For application to the nonlinear inverse problem, and for analysis of numerical methods, the smoothness required for the reference and perturbations must be equilibrated. The technical details of this extension will be reported elsewhere, and do not materially alter the conclusions presented here.

4. The Truncated Radon Transform and the Plane Wave Equations

This section is devoted to the proof of the first main result of our paper, that is, that the truncated Radon transform

$$\delta \bar{U}(\mathbf{p}, z, \tau) = \int d\mathbf{x} \, \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) \delta u_3(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x})$$

is identical to the solution of the plane-wave equations (2.4), except for a smooth (low-frequency) error, provided that the cutoff function η is chosen properly.

The principal tool for our analysis is the Radon transform ("slant stack"). The usual geophysical convention for the definition of the Radon transform is:

$$U(\mathbf{p}, z, \tau) = \int d\mathbf{x} \, u_3(\mathbf{x}, z, \tau + \mathbf{x} \cdot \mathbf{p}) \quad (4.1)$$

(See e.g. Chapman, (1978), p. 495.)

This definition requires modification for several reasons. An obvious difficulty is that, even when the impulsive source is replaced by a localized but smooth pressure source, the convergence of the integral (4.1) is in doubt, as the integral extends over $\{-\infty < x_1, x_2, < \infty\}$. In fact, for the integral to converge for general precritical \mathbf{p} requires hypotheses on the behaviour of ρ and λ as $z \rightarrow \infty$ in addition to boundedness to ensure the decay of u_3 at a sufficient rate as $|\mathbf{x}|, t \rightarrow +\infty$. Since all of our considerations are local, we wish to avoid making such hypotheses.

Suppose u solves the problem (2.1, b', c) with highly localized pressure source f . In view of the assumed global bounds on $\log \rho$, $\log \lambda$, we can show rather easily that, for any finite τ , there exists a $p_{\max} > 0$ so that for $p < p_{\max}$, the support of the integrand in (4.1) is bounded. Therefore convergence is assured for smooth u_3 , i.e. regular pressure sources, and is easy to justify for singular sources, i.e. $f(\mathbf{x}, t) = -\delta(\mathbf{x})\delta(t)$.

In Santosa and Symes (1985), Appendix, it is shown that if u_3 is the vertical component of the point-source impulse response, i.e. the solution of (2.1a, b, c) then its Radon transform (4.1) is the solution of the *plane-wave (impulse-response) equations*

$$\begin{aligned}
\rho \partial_t^2 U &= \partial_z \Lambda \partial_z U \\
\partial_z U(p, 0, t) &= -\delta(t) \\
U &\equiv 0, \quad t < 0
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
\Lambda(p, z) &= \lambda(z)(1 - c^2(z)p^2)^{-1} \\
c(z) &= \lambda^{-\frac{1}{2}}(z)\rho^{\frac{1}{2}}(z)
\end{aligned}$$

and p is so small that $c(z)p < 1$ for all $z > 0$

We would like to extend the definition (4.1) of the Radon transform to the entire Δ -precritical set. Unfortunately this extension is generally impossible, as shown by the following example, for which we are indebted to Paul Sacks.

We consider a medium with ($0 < h \ll z_0$)

$$c(z) = \begin{cases} 1, & 0 \leq z \leq z_0 \\ c > 1, & z_0 + h < z \end{cases}$$

with a smooth monotone transition between z_0 and $z_0 + h$. The support of the impulse-response is depicted in Figure 5 (see also Aki and Richards (1980), p. 213). The direct and (internally) refracted arrivals (singularities) are contained in the region $\{t \geq |x|\}$. The head wave region, given to $O(h)$ by

$$\{(x, t): |x| > \frac{1}{c}, \sqrt{1 + \frac{1}{c^2}} - \frac{1}{c^2} + \frac{1}{c} |x| < t < |x|\}$$

contains smoother signals, which decay as $|x|^{-2}$ (Aki and Richards, p. 212). Thus the integral (4.1) over any plane which has an unbounded intersection with this head wave region is (absolutely) logarithmically divergent.

For this example, we can compute the quantities Z_Δ, S_Δ etc. with acceptable accuracy by letting $h \rightarrow 0$, i.e. assume that the velocity is piecewise constant. Then

$$Z_{\Delta}(p) = \begin{cases} z_0, & \frac{\sqrt{1-\Delta^2}}{c} \leq p \leq \sqrt{1-\Delta^2} \\ \infty, & 0 \leq |p| < \frac{\sqrt{1-\Delta^2}}{c} \end{cases}$$

$$S_{\Delta}(p) = \begin{cases} z_0 \sqrt{1-p^2}, & \frac{\sqrt{1-\Delta^2}}{c} \leq p \leq \sqrt{1-\Delta^2} \\ \infty, & 0 \leq p < \frac{\sqrt{1-\Delta^2}}{c} \end{cases}$$

Since any plane $\{t = \tau + p \cdot x\}$ has unbounded intersection with the head wave region containing a nonvoid cone if $\frac{1}{c} < p < 1$, we are prevented from extending the Radon transform to the precritical set without modification.

Since our object is to study the high-frequency part of the response F and its linearization, we can settle for a quantity, defined in terms of the response, which differs from the solution of the plane-wave equations in the precritical region by a smooth (i.e. low-frequency) error, and is identical to the Radon transform for small p .

A suitable approximation is given by the *truncated Radon transform*

$$\tilde{U}(p, z, \tau) = \int \int dx \, \eta(x, z, \tau + p \cdot x, p) u_3(x, z, \tau + p \cdot x)$$

where η is a suitable *cutoff function*.

We shall assume that η is infinitely differentiable, although an Hermite cubic piecewise polynomial is adequately smooth.

In Appendix A we derive the *modified plane-wave equations* satisfied by \tilde{U} :

$$\begin{aligned} (\rho \partial_r^2 - \partial_z \wedge \partial_z) \tilde{U} &= H \\ \partial_z \tilde{U}(p, 0, t) &= -\delta(t) \\ \tilde{U} &\equiv 0, \quad t < 0 \end{aligned} \quad (4.3)$$

The precise form of the inhomogeneous term H is unimportant: it is essential for the following argument only to know that H is a sum of terms of the forms

$$Q(z, p) \int dx \, P_1(\nabla, \partial_t) \eta(x, z, \tau + p \cdot x, p) P_2(\nabla) u_i(x, z, \tau + p \cdot x) \quad (4.4a)$$

and

$$\partial_z Q(z, p) \int_{-\infty}^{\tau} d\tau' \int dx P_1(\nabla) \eta(x, z, \tau' + p \cdot x, p) P_2(\nabla) u(x, z, \tau' + p \cdot x) \quad (4.4b)$$

Here Q is an algebraic combination of ρ , λ , and p , and P_1 and P_2 are differential operators. It is crucial that P_1 has no constant term. This is evident from the explicit construction of H given in Appendix A. As a consequence, the integrands in (4.4a, b) are nonvanishing only where η is nonconstant.

As the first step in the proof of our first main result (2.2), we shall establish that the inhomogeneous term H in (4.3) is smooth in the strip

$$[0, Z_{\Delta}(p)] \times \mathbb{R} = \{(z, \tau): -\infty \leq \tau \leq \infty, \quad 0 \leq z \leq Z_{\Delta}(p)\}$$

provided that the cutoff function η :

- (i) vanishes for large $|x|$ for each (t, z, p) ;
- (ii) for each p , vanishes in a neighborhood of the refracted (turned) ray with ray parameter p , if there is such a ray (terminology explained below).

Granted this result, we see immediately that U and \tilde{U} differ by a smooth function in the region $\{(z, \tau): 0 \leq z \leq Z_{\Delta}(p), -\infty \leq \tau \leq 2S_{\Delta}(p) - S(z, p)\}$. In fact, the difference $V = \tilde{U} - U$ satisfies

$$\begin{aligned} (p \partial_t^2 - \partial_z \wedge \partial_z) V &= H \\ \partial_z V(p, 0, t) &\equiv 0 \\ V &\equiv 0, \quad t \ll 0 \end{aligned} \quad (4.5)$$

Since all of the data, i.e. the r.h.s. in the wave equation, the boundary data, and the initial data, are smooth in the region $\{(z, \tau): \tau < 2S_{\Delta}(p) - S(z, p)\}$, it follows that V is smooth in this region also (see Courant and Hilbert (1962), pp. 471 ff.). Thus U and \tilde{U} have the same high-frequency asymptotics, which is the desired conclusion.

To see how η should be chosen so that H is indeed smooth, we assume that

the forward light cone of the origin is given by an equation

$$t = \psi_i(x, z)$$

with ψ_i (the incident phase) smooth and $\nabla\psi_i \neq 0$, except at $x=0, z=0$.

The meaning of this assumption is that no caustics develop on the rays of geometric optics issuing from the origin. Since this assumption is violated in almost all situations of practical interest, the argument which follows has suggestive value only. Nonetheless, it will lead us to the correct choice of d_1, d_2 . In Santosa and Symes (1985b), we give a completely rigorous proof of the smoothness of H for our choice of cutoff, valid for general smooth ρ, λ , regardless of the presence of caustics.

Under the no-caustics hypothesis, Hadamard's construction (Courant and Hilbert (1962) pp. 740 ff.) leads to the progressing wave expansion for the field u :

$$\begin{aligned} u(x, z, t) = & a_{-1}(x, z) \delta'(t - \psi(x, z)) \\ & + a_0(x, z) \delta(t - \psi(x, z)) \\ & + a_1(x, z, t) H(t - \psi(x, z)) \end{aligned}$$

where the coefficients a_i are determined by solving certain transport equations, and are smooth if ρ, λ are smooth.

A term of the form (4.4a), for instance, becomes a sum of terms of the form

$$Q(z, p) \int dx P_1(\partial_t, \nabla) \eta(x, z, \tau - p \cdot x, p) a(x, z, \tau) P_2(\nabla) \delta(\tau + p \cdot x - \psi_i(x, z)).$$

where a is smooth.

The delta distribution can be integrated out, leaving a smooth integrand evaluated on the smooth hypersurface $\{t = \psi_i\}$, provided that the phase gradient is not stationary, i.e. we do *not* have

$$\nabla_x(\tau + p \cdot x - \psi_i(x, z)) = p - \nabla\psi_i(x, z) = 0 \quad (4.6a)$$

on the intersection of the zero phase surface

$$x: \psi_i(x, z) = \tau + p \cdot x \quad (4.6b)$$

with the support of the integrand

$$\{(x, z, \tau): (|\nabla\eta|^2 + |\partial_t\eta|^2)(x, z, \tau) > 0\} \quad (4.6c)$$

for z, τ ranging over the region of interest

$$C_{\Delta}(p) := \{(z, \tau): 0 \leq z \leq Z_{\Delta}(\tau), s(z, p) \leq \tau \leq 2S_{\Delta}(p) - S(z, p)\}$$

For instance

$$\begin{aligned} & \int d\mathbf{x} f(\mathbf{x}, z, \tau) \delta(\tau + \mathbf{p} \cdot \mathbf{x} - \eta_i(\mathbf{x}, z)) \\ &= |\mathbf{p} - \nabla \psi_i(\mathbf{x}, z)|^{-1} f(\mathbf{x}, z, \psi_i(\mathbf{x}, z) - \mathbf{p} \cdot \mathbf{x}) \end{aligned}$$

under these conditions for smooth f . See Gel'fand and Shilov (1958), pp. 209-246, for instance.

The construction of ψ_i , for which the above conditions are satisfied, depends on the geometry of bicharacteristics for the wave equation ("ray theory"), which we review briefly.

The Hamiltonian h associated to the wave equation may be taken as

$$h(\mathbf{x}, z, t, \xi, \zeta, \tau) = \frac{1}{2} (\tau^2 - c^2(z)(|\xi|^2 + \zeta^2))$$

The *bicharacteristic strips* are solutions of Hamilton's equations:

$$\begin{aligned} \dot{t} &= \partial_{\tau} h = \tau \\ \dot{\mathbf{x}} &= \partial_{\xi} h = -c^2(z)\xi \\ \dot{z} &= \partial_{\zeta} h = -c^2(z)\zeta \\ \dot{\xi} &= -\partial_{\mathbf{x}} h = 0 \\ \dot{\zeta} &= -\partial_z h \\ \dot{\tau} &= -\partial_t h = 0 \end{aligned}$$

The forward light cone ($t = \psi_i$) is made up of the union of *rays* emanating from the origin, which are the (\mathbf{x}, z, t) -projections of the *null* ($h \equiv 0$) bicharacteristic strips passing over $(0, 0, 0)$ and for which $\dot{t} = \tau > 0$. Normalizing $\tau = 1$, we obtain that along the rays,

$$\xi = -\nabla \psi_i, \quad \zeta = -\partial_z \psi_i \quad (4.7)$$

This is the principal result of Hamilton-Jacobi theory.

Since $\dot{t} = 1$, $\cdot = \frac{d}{dt}$. Because ξ is constant along rays, we can parameterize them by ξ (this consequence of the x -independence of c is Snell's law). Since $h \equiv 0$ along the

bicharacteristics, we obtain

$$\zeta = -\sqrt{c^{-2}(z) - |\xi|^2}$$

So long as $c(z)|\xi| \leq 1$, we obtain from the differential equations for a point (x, z, t) on the ray parameterized by ξ :

$$x = \int_0^z dz' \frac{c(z')\xi}{\sqrt{1 - c^2(z')|\xi|^2}} = : X(z, \xi)$$

$$t = \int_0^z \frac{dz'}{c(z')\sqrt{1 - c^2(z')|\xi|^2}} = : T(z, \xi)$$

We introduce the vertical travel time

$$S(z, \xi) = \int_0^z dz' \frac{\sqrt{1 - c^2(z')|\xi|^2}}{c(z')}$$

(which is, of course, just the travel time functional associated with the plane wave equations) and obtain the extremely useful identity: for (x, z, t) on a ray emanating from the origin in the pre-critical region,

$$S(z, \xi) + \xi \cdot x = t \quad (4.8)$$

It will be useful to have an extension of the function S beyond the precritical region. Along the ray with ray parameter ξ , define

$$s(t, \xi) = t - \xi \cdot x(t) \quad (4.9)$$

If $|c(z(t))||\xi| \leq 1$ for $0 \leq t \leq t_0$, then

$$S(z(t), \xi) = s(t, \xi), \quad 0 \leq t \leq t_0$$

As each bicharacteristic lies entirely over the ("0-precritical") set

$$\{(x, z): c(z)|\xi| \leq 1\}$$

we have

$$\dot{s} = 1 - \xi \cdot \dot{x} = 1 - c^2|\xi|^2 \geq 0$$

i.e. s is monotone nondecreasing, and increases strictly interior to the precritical region.

It will also be convenient to introduce functions which give the horizontal distance traveled along a precritical ray as a function of depth: for $0 \leq z \leq Z_\Delta(p)$,

$$X_\Delta(p) = |X(Z_\Delta(p), p)|$$

We assert the existence of a positive number $D_\Delta(p)$ with the property:

$$(4.10) \quad \text{the ray with ray parameter } p \text{ contains no point } (x, z, t) \text{ with } z \leq Z_\Delta(p) \text{ and } X_\Delta(p) \leq |x| < 2X_\Delta(p) - |X(z, p)| + D_\Delta(p) \text{ (see Figure 6).}$$

It follows that there exists a smooth cutoff function η with the required properties (i) and (ii), so that

$$\eta \equiv 1 \quad \text{on the set}$$

$$\{(x, z, t): 0 \leq z \leq Z_\Delta(p), \quad |x| \leq 2X_\Delta(p) - |X(z, p)|, \quad -\infty < t < \infty\} \quad (4.11)$$

Then η is constant ($\equiv 1$ or $\equiv 0$) in a neighborhood of all components of the intersection of the ray with ray parameter p and the slab $\{z \leq Z_\Delta(p)\}$.

The smoothness of H in $C_\Delta(p)$ follows immediately from (4.10). If $(\tau, z) \in C_\Delta(p)$ such that (4.6b) holds, then the point

$$(x, z, t = \tau + p \cdot x)$$

lies on a ray through the origin (since the light cone $\{t = \psi_i\}$ is a union of rays). If (4.6a) holds, then this ray is the one with ray parameter p . However since $z \leq Z_\Delta(p)$, (4.11) now contradicts (4.6c). Thus the set specified by the conditions (4.6) is void, and we conclude that H is smooth.

We now establish the existence of a positive $D_\Delta(p)$ for which (4.10) holds.

Since along the ray

$$\frac{d|x|}{dt} = -c^2(z) \frac{x \cdot p}{|x|} > 0,$$

we can use $x := |x|$ as a coordinate along the ray. Set

$$X_R(p) = \inf\{x: z(x) \leq Z_\Delta(p), \quad x > X_\Delta(p)\}$$

Possibly $X_R(p) = \infty$, i.e. the ray never returns to the slab $0 \leq z \leq Z_\Delta(p)$. In this case (4.10) will hold with $D_\Delta(p)$ chosen to be any convenient positive number. Note that this is the case for sufficiently small p .

Otherwise, by symmetry consideration

$$\zeta(X_R(p)) = -\zeta(X_\Delta(p))$$

Since $\zeta^2 = c^{-2} - p^2$, and $c(Z_\Delta(p))p = \sqrt{1 - \Delta^2}$,

$$\begin{aligned} 2(c^{-2}(Z_\Delta(p)) - p^2)^{\frac{1}{2}} &= \frac{\Delta p}{\sqrt{1 - \Delta^2}} \\ &= |\zeta(X_R(p)) - \zeta(X_\Delta(p))| \\ &= \left| \int_{X_R(p)}^{X_\Delta(p)} dx \frac{d\zeta}{dx} \right| = \left| \int_{X_R(p)}^{X_\Delta(p)} dx \frac{c'}{c} \frac{\zeta^2}{x \cdot p} \right| \\ &\leq \frac{(X_R(p) - X_\Delta(p))}{X_\Delta(p) \cdot p} \left\{ \sup_z (|c'(z)|) \frac{1}{c_*} \left(\frac{1}{c_*^2} - p^2 \right) \right\} \end{aligned}$$

where $c_* = \inf_z c(z)$. Since $c_* \leq c(Z_\Delta(p))$,

$$\frac{1}{c_*^2} - p^2 \leq \frac{1}{c_*} - \frac{1 - \Delta^2}{c_*}.$$

Thus

$$X_R(p) - X_\Delta(p) \geq K p^2 X_\Delta(p) > 0 \quad (4.12)$$

where K is a positive function of Δ and of global bounds on $\log c$ and its derivative.

Now choose $D_\Delta(p)$ to be less than the r.h.s. of (4.12).

Suppose x, z, t lies on the ray with ray parameter p . Then either $|x| \leq X_\Delta(p)$, $|x| \geq X_R(p)$, or $z > Z_\Delta(p)$. In the first and third cases (x, z) manifestly lies outside of the set described by (4.10). In the second case, note that

$$\begin{aligned}
|x| &= X_R(p) + p \int_z^{Z_\Delta(p)} dz' \frac{c(z')}{\sqrt{1 - c^2(z')p^2}} \\
&= 2X_\Delta(p) - |X(z, p)| + (X_R(p) - X_\Delta(p)) \\
&> 2X_\Delta(p) - |X(z, p)| + D_\Delta(p)
\end{aligned}$$

from (4.12).

This completes the proof of (4.10), hence of the smoothness of H .

We now turn to the proof of the first main result. In the notation of section 2 and (4.5), we see that

$$M(\delta\rho, \delta\lambda) = \partial_t \delta V|_{z=0}$$

where δV solves the perturbational equation

$$(\rho\partial_t^2 - \partial_z \wedge \partial_z)\delta V = -(\delta\rho\partial_t^2 - \partial_z \delta \wedge \partial_z)V + \delta H \quad (4.13a)$$

$$\partial_z \delta V \equiv 0 \quad \text{for } z = 0 \quad \text{or } t \ll 0. \quad (4.13b)$$

Because of the results on V and H previously derived, we see that the r.h.s. is a sum of terms either of the form

$$a(z)F(z, \tau, p) \quad (4.14)$$

or of the form

$$b(z, p) \int dx P_1(\nabla, \partial_t) \eta(x, z, \tau + p \cdot x, p) P_2(\nabla) \delta u_i(x, z, \tau + p \cdot x) \quad (4.15)$$

where $a(z)$ is $\delta\rho(z)$, $\delta\lambda(z)$, or $\delta\lambda'(z)$, $b(z, p)$ is smooth, $F(z, \tau, p)$ is smooth in $\{(z, \tau): z \leq Z_\Delta(p), \tau \leq 2S_\Delta(p) - S(z, p)\}$, and $P_1(\nabla), P_2(\nabla)$ are differential operators as before.

We aim to show that

$$M(\delta\rho, \delta\lambda) = O(\zeta^{-N}), \quad \zeta \rightarrow \infty$$

for any N , when

$$\delta\rho, \delta\lambda \sim \chi(z)e^{i\zeta z}$$

with χ a smooth envelope. This will follow if we show a similar result for the trace of the solution of (4.13) with the r.h.s. replaced by either of the forms (4.14) or (4.15).

Accordingly, suppose first that W solves

$$(\rho(z)\partial_\tau^2 - \partial_z \Lambda(z)\partial_z) W(z, \tau) = \chi(z)e^{i\zeta z} F(z, \tau)$$

with F smooth in $\{(z, \tau): z \leq Z_\Delta(p), \tau \leq 2S_\Delta(p) - S(z, p)\}$ (we have suppressed p where convenient).

We shall show that, for any smooth function $\Phi(\tau)$ which vanishes for $\tau > 2S_\Delta(p)$, and satisfies

$$\int_0^{2S_\Delta(p)} d\tau |\Phi|^2(\tau) \leq 1 \quad (4.16a)$$

we have

$$\int d\tau \Phi(\tau) \partial_\tau W(0, \tau) = O(\zeta^{-N}) \quad (4.16b)$$

The estimates (4.16) imply that $\partial_\tau W(0, \cdot)$ is $O(\zeta^{-N})$ in the mean-square sense, which is adequate for our purposes.

To establish (4.16), we use a Green's identity argument, which will be used again in Section 5: suppose that Q solves the backwards or "adjoint" Neumann problem

$$(\rho\partial_\tau^2 - \partial_z \Lambda \partial_z) Q = 0$$

$$\partial_z Q(0, \tau) = \Phi(\tau); \quad Q \equiv 0, \quad \tau > 2S_\Delta(p)$$

Then integration by parts shows that

$$\begin{aligned} & \Lambda(0, p) \int d\tau \Phi(\tau) \partial_\tau W(0, \tau) \\ &= \int_0^\infty dz \int d\tau \partial_\tau Q(z, \tau) \left[\rho(z)\partial_\tau^2 - \partial_z \Lambda(z, p)\partial_z \right] W(z, \tau) \\ &= \int_0^\infty dz \int d\tau \partial_\tau Q(z, \tau) \chi(z) e^{i\zeta z} F(z, \tau) \end{aligned} \quad (4.17)$$

The provenance of F assures that $F(z, \tau) = 0$ for $0 \leq z \leq Z_\Delta(p)$, $\tau < S(z, p)$, whereas $Q \equiv 0$ for $\tau > 2S_\Delta(p) - S(z, p)$ by domain-of-dependence. Thus the integrand vanishes outside the double light cone

$$C_{\Delta}(p) := \{(z, \tau) : 0 \leq z \leq Z_{\Delta}(p) : 0 \leq \tau \leq 2S_{\Delta}(p) - S(z, p)\}$$

For convenience, we also assume that $\chi(z) \equiv 0$ for z in some interval about $z = 0$; the boundary terms thus eliminated in the following integrations-by-parts can be estimated by a slightly more sophisticated argument.

Write $e^{i\zeta z} = (i\zeta)^{-N} \partial_z^N e^{i\zeta z}$ under the integral sign and integrate by parts N times in ζ to obtain for the r.h.s. of (4.17)

$$= \zeta^{-N} \sum_{j=0}^N \int_0^{\infty} dz \int d\tau G_j^{(1)}(z, \tau) e^{i\zeta z} \partial_z^j \partial_{\tau} Q(z, \tau)$$

where in the sequel $G_j^{(K)}$ will denote functions smooth in $C_{\Delta}(p)$ and vanishing for $\tau \leq S(z, p)$, $0 \leq z \leq Z_{\Delta}(p)$. Now eliminate all z -derivatives of Q higher than the first by repeated application of the wave equation in the form

$$\partial_z^2 Q = \frac{\rho}{\Lambda} \partial_{\tau}^2 Q - \partial_z \log \Lambda \partial_z Q$$

to obtain

$$\begin{aligned} &= \zeta^{-N} \sum_{j=0}^N \left\{ \int_0^{\infty} dz \int d\tau G_j^{(2)}(z, \tau) e^{i\zeta z} \partial_{\tau}^{j+1} Q(z, \tau) \right. \\ &\quad \left. + \int_0^{\infty} dz \int d\tau G_j^{(3)}(z, \tau) e^{i\zeta z} \partial_{\tau}^j \partial_z Q(z, \tau) \right\} \\ &= \zeta^{-N} \int_0^{\infty} dz \int d\tau e^{i\zeta z} \left(G^{(4)}(z, \tau) \partial_{\tau} Q(z, \tau) + G^{(5)}(z, \tau) \partial_z Q(z, \tau) \right) \end{aligned}$$

after integration by parts in τ . Now the *a priori* estimate for the Neumann problem

$$\int_{C_{\Delta}(p)} dz d\tau (|\partial_{\tau} Q|^2 + |\partial_z Q|^2) \leq K \int_0^{2S_{\Delta}(p)} d\tau |\Phi|^2 \quad (4.18)$$

and the Cauchy-Schwartz inequality immediately give (4.17).

The *a priori* estimate (4.18) follows from some basic estimates for the one-dimensional Neumann problem which are proved in the spirit of Symes (1985), Section 1.

This estimates the contribution to $M(\delta\rho, \delta\lambda)$ from that part of the r.h.s. of (4.13) which looks like (4.4). To estimate the contribution from terms of the form (4.15), we see we need to

estimate integrals

$$\int d\mathbf{x} P_1(\nabla, \partial_t) \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) P_2(\nabla) \delta u_i(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x})$$

for $(z, \tau) \in C_\Delta(p)$ and so we need some information concerning the 3-dimensional perturbational field δu . Under the standing no-caustics hypothesis, the necessary information is embodied in the asymptotic series

$$\delta u(\mathbf{x}, z, t) = \left(\sum_{k=0}^{\infty} b_k(\mathbf{x}, z, t) \zeta^{-k} \right) e^{i\zeta \phi_r(\mathbf{x}, z, t)} + \sum_{k=1} d_k(\mathbf{x}, z, t) \zeta^{-k} e^{i\zeta z} \quad (4.19)$$

which is derived in Appendix B.

We shall indulge in formal manipulation of this series. As stated above, a completely rigorous proof of our result, which takes full account of the presence of caustics, is given in Santos and Symes (1985b). Also, as in Appendix B, we shall take advantage of the method of images to account for reflections from the free surface $\{z = 0\}$.

Substituting (4.19) in (4.15), we see that we need to show

$$\int d\mathbf{x} g(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) e^{i\zeta \phi_r(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x})} = 0(\zeta^{-N}) \quad (4.20a)$$

where g is smooth and supported in the set

$$\{(\mathbf{x}, z, t, p) : (z, t - \mathbf{p} \cdot \mathbf{x}) \in C_\Delta(p), (|\nabla \eta|^2 + |\partial_t \eta|^2)(\mathbf{x}, z, t, p) > 0\} \quad (4.20b)$$

This is equivalent to the non-existence of stationary points of the phase in (4.20a) in the set (4.20b). These stationary points are the solutions of

$$0 = \nabla_z \phi_r(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}) + \partial_t \phi_r(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}) p \quad (4.21)$$

From Appendix B, we have the following equations for the phase gradient:

$$\rho |\partial_t \phi_r|^2 - \lambda |\nabla \phi_r|^2 \equiv 0 \quad (4.22a)$$

$$\partial_t \phi_r = \frac{1}{2\partial_z \psi_i} \quad (4.22b)$$

$$\nabla \phi_r = \mathbf{e}_z - (2\partial_z \psi_i)^{-1} \nabla \psi_i \quad (4.22c)$$

where $\mathbf{e}_z = (0, 0, 1)^T$, (4.22b), (4.22c) hold only on the incident wavefront $\{t = \psi_i\}$, and (4.22a) is simply the eikonal equation. As indicated in Appendix B, the phase ϕ_r is uniquely determined by (4.22). The construction is entirely ray-theoretical, however, and is possible even when caustics

are present. To clarify the nature of the reflected phase, then, and to provide the ray-geometric step in the rigorous proof of Santosa and Symes (1985b), we reinterpret (4.22) in purely ray-theoretic terms.

Over a point, (x, z, t) for which $t = \psi_i(x, z)$, the phase point

$$(x, z, t, \nabla \psi_i(x, z), 1)$$

lies on a null bicharacteristic passing over the origin:

$$(x_i(t), z_i(t), t, \xi_i(t), \varsigma_i(t), 1)$$

We construct the initial conditions for the *reflected ray* according to (4.22) and to the Hamilton Jacobi identity $\xi_r = \nabla_x \phi_r$, $\varsigma_r = \partial_z \phi_r$: with the t -component of momentum normalized to 1 as before, we obtain

$$\begin{aligned} \xi_r(t) &= \xi_i(t) \\ \varsigma_r(t) &= -\varsigma_i(t) \end{aligned} \tag{4.23}$$

(In deriving these equalities remember that the incident phase has the form $t - \psi_i$, so that $\xi_i = -\nabla_x \psi_i$, $\varsigma_i = -\partial_z \psi_i$).

The reflected null bicharacteristic is the solution of Hamilton's equations the initial conditions (4.23), and the reflected ray is its (x, z, t) -projection. These are further reflected from the free surface $\{z = 0\}$.

Now we see that (4.21) means that $(x, z, \tau + p \cdot x)$ lies on a reflected ray with ray parameter $p (= \xi_r)$, which according to (4.23) and the Hamilton equation $\dot{\xi} = 0$ comes from reflection of an incident ray with the same ray parameter.

Suppose first that the reflection point, i.e. the point at which (4.23) holds, lies in $\{z > Z_\Delta(p)\}$. Then the incident ray must include a point $(x_0, Z_\Delta(p), t_0)$ with $|x_0| = X_\Delta(p)$; see figure 6. The reflected ray must also include a point $(x_1, Z_\Delta(p), t_1)$, necessarily with $|x_1| > X_\Delta(p)$; in fact choose $|x_1|$ to be the largest such, so that the entire ray segment from t_1 to $\tau + p \cdot x$ lies in $[0, Z_\Delta(p)]$. Now

$$t_1 = s(t_1, p) + p \cdot x_1$$

where s is defined by (4.9) along the broken trajectory obtained by combining the incident and reflected rays in the obvious way. In particular, s is nondecreasing, and since $Z_\Delta(p)$ is strictly precritical

$$s(t_1, p) > s(t_0, p) = S_\Delta(p)$$

Similarly,

$$\begin{aligned} s(\tau + p \cdot x, p) &= s(t_1, p) + \int_z^{Z_\Delta(p)} dz' \frac{\sqrt{1 - c^2(z')p^2}}{c(z')} \\ &= s(t_1, p) + S_\Delta(p) - S(z, p) \end{aligned}$$

But

$$\tau = t - p \cdot x = s(\tau + p \cdot x, p) > 2S_\Delta(p) - S(z, p)$$

which puts $(x, z, \tau + p \cdot x)$ outside the support (4.20b) of the integrand in (4.20a).

On the other hand, suppose that the reflection point (x_r, z_r, t_r) lies in $\{z \leq Z_\Delta(p)\}$ (see figure 6). Let R_p be the set specified by

$$\begin{aligned} (x, z, t) \in R_p \text{ if } (x, z, t) \text{ lies on a reflected ray originating at a} \\ \text{reflection point } (x_r, z_r, t_r): z, z_r \leq Z_\Delta(p), \text{ and } t - p \cdot x \leq 2S_\Delta(p) - S(z, p) \end{aligned}$$

That is, R_p is the union of all reflected ray segments in the slab $\{z \leq Z_\Delta(p)\}$ along which $s \leq 2S_\Delta(p) - S(z, p)$. Note that R_p is a compact set, since $t - p \cdot x$ increases strictly along any ray in $z \leq Z_\Delta(p)$.

We claim that R_p is disjoint from that part of the incident ray lying in $\{|x| > X_\Delta(p)\}$, i.e. the refracted ray. This is obvious, of course, if the incident ray is not refracted. Otherwise, suppose that for some reflection point (x_r, z_r, t_r) with $z_r \leq Z_\Delta(p)$, the reflected ray contains a point (x_1, z_1, t_1) on the incident (refracted) ray. Suppose that the turning point on the refracted ray occurs at depth $z_{crit} (> Z_\Delta(p))$. Then

$$|x_1| = 2X(z_{crit}, p) - |X(z_1, p)|$$

$$t_1 = 2T(z_{crit}, p) - T(z_1, p)$$

so

$$\begin{aligned} t_1 - p \cdot x_1 &= 2(T(z_{crit}, p) - p \cdot X(z_{crit}, p)) - (T(z_1, p) - p \cdot X(z_1, p)) \\ &= 2S(z_{crit}, p) - S(z_1, p) \\ &> 2S_\Delta(p) - S(z_1, p) \end{aligned}$$

whence $(x, z_1, t_1) \notin R_p$.

It follows that we can construct a function η which satisfies

$$\begin{aligned} \eta &\equiv 1 \text{ on the union of } R_p \text{ and the set (4.11);} \\ \eta &\equiv 0 \text{ near the refracted ray and for large } |x| \end{aligned}$$

In particular, no reflected ray passes through a point $(x, z, \tau + p \cdot x)$ with $z \leq Z_\Delta(p)$ and $\tau \leq 2S_\Delta(p) - S(z, p)$ at which η is non-constant. Therefore the stationary points in the phase of (4.22a) are disjoint from the support (4.20b) of the integrand, whence follows the estimate (4.22a).

We note that, at the surface $z = 0$, a point $(x, 0, t)$ on a reflected ray with $z_r < Z_\Delta(p)$ obviously satisfies $|x| < 2X_\Delta(p)$ (see figure 7). Thus the intersection $R_p \cap \{z = 0\}$ is the same as the intersection of (4.11) with $\{z = 0\}$, so we can choose η so that additionally

$$\begin{aligned} \eta(\cdot, 0, \cdot, \cdot) &\equiv 1 \text{ on } \{(x, t) : |x| \leq 2X_\Delta(p)\} \\ \eta(\cdot, 0, \cdot, \cdot) &\equiv 0 \text{ on } \{(x, t) : |x| \geq 2X_\Delta(p) + D_\Delta(p)\} \end{aligned}$$

This completes the proof of the result (2.2), for the case of caustic-free incident wavefront.

For the general case, see Santosa and Symes 1985b.

5. High-frequency Asymptotics for the Plane-Wave Equations

Because of the result of the last section, the computation of the high-frequency asymptotics of the perturbational field $\delta \mathbf{u}$ reduces (partly) to that of the solution δU of the plane wave equations. These are hyperbolic systems in one space dimension, and their high-frequency behaviour is relatively easy to compute. We shall use the details of this intermediate result in the proof in Section 6 of our remaining major results.

Denote by U and δU the solutions of the problems (2.3) and (2.4), which we display again for convenience:

$$\begin{aligned}
 \rho \ddot{U} - \partial_z \Lambda \partial_z U &= 0 \\
 \partial_z U(0, \tau) &= \delta(\tau) \\
 U &\equiv 0, \quad \tau < 0 \\
 \rho \delta \ddot{U} - \partial_z \Lambda \partial_z \delta U &= -\delta \rho \ddot{U} + \partial_z \delta \Lambda \partial_z U \\
 \partial_z \delta U(0, \tau) &\equiv 0 \\
 \delta U &\equiv 0, \quad \tau < 0.
 \end{aligned} \tag{5.1a}$$

In this section, we shall generally suppress dependence on p .

Recall that ρ and Λ are smooth. Then U is singular only along the wavefront

$$\tau = S(z) = \int_0^z c^{-1}$$

where $c = \rho^{-\frac{1}{2}} \Lambda^{\frac{1}{2}}$ is the wave speed. Define the *admittance* A by

$$A = c \Lambda^{-1}.$$

Then U admits a progressing wave expansion

$$\begin{aligned}
 U(z, \tau) &= c(0) A^{\frac{1}{2}}(z) \left\{ H(\tau - S(z)) \right. \\
 &+ \left[\frac{1}{4} c(0) \partial_z \log A(0) + \frac{1}{4} c(z) \partial_z \log A(z) \right. \\
 &\left. \left. - \frac{1}{8} \int_0^z d\zeta \frac{c(\zeta)}{c(0)} (\partial_z \log A(\zeta))^2 \right] (\tau - S(z)) H(\tau - S(z)) \right\} \\
 &+ R(z, \tau)
 \end{aligned} \tag{5.2}$$

where $R(z, \tau) = O((\tau - S(z))^2)$, and $R \equiv 0$ for $\tau < S(z)$.

This expansion is completely standard; the general principles are explained for instance in Courant and Hilbert, pp. 618 ff. For a detailed calculation involving the same wave equation with different boundary conditions, see Symes (1981), Theorem A and preceding discussion.

More concisely, we collect the last two terms in (5.2), thus:

$$U(z, \tau) = c(0) \tilde{A}^{\frac{1}{2}}(z) H(\tau - S(z)) + J(z, \tau).$$

We shall not need the detailed properties of J in this paper; see however the discussion in Part 7.

A convenient way to obtain the decomposition of $\partial_t \delta U|_{z=0}$ mentioned in Section 2 (2.5) is through the integration-by-parts identity (Green's Theorem)

$$\int_{z \geq 0} dz \int dt \left[(\partial_t w_1) W w_2 + (\partial_t w_2) W w_1 \right] = \int dt \Lambda(0) (\partial_t w_1 \partial_z w_2 + \partial_t w_2 \partial_z w_1) \quad (5.3)$$

valid when w_1 and w_2 are smooth and have supports which intersect in a bounded set, and W is the wave operator

$$W = \rho \partial_t^2 - \partial_z \Lambda \partial_z$$

We shall use (5.3) also with certain distribution arguments. This (ab-) use can be justified by easy limiting arguments, which we omit.

The principal identity is obtained by setting in (5.3)

$$\begin{aligned} w_1 &= \delta U \\ w_2(z, t) &= U(z, \tau - t) \end{aligned}$$

Note that w_2 depends on $\tau \geq 0$ as a parameter.

Now (5.1), (5.2), and (5.3) imply

$$\begin{aligned} \Lambda(0) \partial_t \delta U(0, \tau) &= \int_{z \geq 0} dz \int dt \partial_t U(z, \tau - t) \left[\delta \rho(z) \partial_t^2 U(z, t) - \partial_z (\delta \Lambda(z) \partial_z U(z, t)) \right] \\ &= \int_{z \geq 0} dz \int dt \left(\tilde{A}^{\frac{1}{2}}(z) \delta(\tau - t - S(z)) + \partial_t J(z, \tau - t) \right) \\ &\quad (\delta \rho(z) (\tilde{A}^{-\frac{1}{2}}(z) \delta'(t - S(z) + \partial_t^2 J(z, t)) - \partial_z (\delta \Lambda(z) \partial_z (\tilde{A}^{\frac{1}{2}}(z) H(t - S(z)) + J(z, t)))) \end{aligned}$$

where we have written

$$\tilde{A} = c(0)^2 A.$$

Expanding the expression in (5.4), we get

$$\begin{aligned}
& \Lambda(0) \partial_t \delta U(0, \tau) = \\
& \int dz \int dt \left\{ \tilde{A}^{\frac{1}{2}}(z) \delta(\tau - t - S(z)) + \partial_t J(z, \tau - t) \right\} \left\{ \delta \rho(z) (\tilde{A}^{\frac{1}{2}}(z) \delta'(t - T(z)) + \partial_t^2 J(z, t) \right. \\
& - \tilde{A}^{\frac{1}{2}}(z) c^{-2}(z) \delta \Lambda(z) \delta'(t - S(z)) - \tilde{A}^{\frac{1}{2}}(z) c^{-1}(z) (\partial_z \delta \Lambda(z) - \delta \Lambda(z) \left(\frac{c'(z)}{c(z)} - \frac{1}{2} \frac{A'(z)}{A(z)} \right)) \cdot \delta(t - S(z)) \\
& - \tilde{A}^{\frac{1}{2}}(z) \left(\frac{1}{2} \partial_z \delta \Lambda(z) \frac{A'(z)}{A(z)} + \delta \Lambda(z) \left(\frac{1}{2} \frac{A''(z)}{A(z)} - \frac{1}{4} \left(\frac{A'(z)}{A(z)} \right)^2 \right) \cdot H(t - S(z)) \right. \\
& \left. \left. - \partial_z \delta \Lambda(z) \partial_z J(z, t) - \delta \Lambda(z) \partial_z^2 J(z, t) \right) \right\} \\
& = \int dz \int dt \left\{ \tilde{A}(z) (\delta \rho(z) - c^{-2}(z) \delta \Lambda(z)) \delta(\tau - t - S(z)) \cdot \delta'(t - S(z)) \right. \\
& + \tilde{A}(z) c^{-1}(z) \partial_z \delta \Lambda(z) \delta(\tau - t - S(z)) \cdot \delta'(t - S(z)) + \tilde{A}(z) c^{-1}(z) \partial_z \delta \Lambda(z) \delta(\tau - t - S(z)) \cdot \delta(t - S(z)) \} \\
& + (\text{remainder})
\end{aligned}$$

where (remainder) represents the remaining terms, which are less singular than the terms listed explicitly in the last formula, in a sense to be made precise below.

To compute the leading integral, introduce

$$X = t - S(z)$$

$$Y = \tau - t - S(z)$$

Then

$$\frac{\partial(X, Y)}{\partial(t, z)} = \begin{pmatrix} 1 & -c^{-1} \\ -1 & -c^{-1} \end{pmatrix}$$

$$\frac{\partial(t, z)}{\partial(X, Y)} = -\frac{1}{2} \begin{pmatrix} -1 & 1 \\ c & c \end{pmatrix}$$

$$\det \left| \frac{\partial(t, z)}{\partial(X, Y)} \right| = c$$

so the integral becomes

$$\begin{aligned}
& \int \int dX dY \frac{c}{2} \left\{ [\tilde{A} (\delta \rho - c^{-2} \delta \Lambda)] \delta(Y) \delta'(X) + \tilde{A} c^{-1} \partial_z \delta \Lambda \delta(Y) \delta(X) \right\} \\
& = \left\{ -\frac{1}{2} \frac{\partial}{\partial X} (c \tilde{A} (\delta \rho - c^{-2} \delta \Lambda)) + \frac{1}{2} \tilde{A} \partial_z \delta \Lambda \right\}_{X=Y=0}
\end{aligned}$$

Now,

$$\frac{\partial}{\partial X} = \frac{\partial z}{\partial X} \frac{\partial}{\partial z} + \frac{\partial t}{\partial X} \frac{\partial}{\partial t} = -\frac{1}{2} c(z) \frac{\partial}{\partial z}$$

for functions depending only on z , and

$$\begin{aligned} c\tilde{A}(\delta\rho - c^{-1}\delta\Lambda) &= c(0)^2 \left(\frac{\delta\rho}{\rho} - \frac{\delta\Lambda}{\Lambda} \right) \\ \tilde{A} \partial_z \delta\Lambda &= c(0)^2 c \Lambda^{-1} \partial_z \delta\Lambda \\ &= c(0)^2 \left[c \partial_z \left(\frac{\delta\Lambda}{\Lambda} \right) - c \frac{\delta\Lambda}{\Lambda} \frac{\partial_z \Lambda}{\Lambda} \right] \end{aligned}$$

Finally, $X = Y = 0$ if $t = S(z)$, $\tau = 2S(z)$, i.e., $z = S^{-1}(\frac{\tau}{2})$, whence the integral becomes

$$\left\{ \frac{1}{4} c^2(0) c \partial_z \left(\frac{\delta\rho}{\rho} + \frac{\delta\Lambda}{\Lambda} \right) + \frac{1}{2} c^2(0) c \frac{\delta\Lambda}{\Lambda} \frac{\partial_z \Lambda}{\Lambda} \right\} (S^{-1}(\frac{\tau}{2}))$$

which suggest the introduction of new independent variables

$$\begin{aligned} r &= \partial_z \log \rho, \quad e = \partial_z \log \Lambda \\ \delta r &= \partial_z \frac{\delta\rho}{\rho}, \quad \delta e = \partial_z \frac{\delta\Lambda}{\Lambda}. \end{aligned}$$

Regarding $\partial_t \delta U(0, \tau)$ as a functional of δr and δe rather than of $\delta\rho$ and $\delta\Lambda$, we obtain

$$\Lambda(0) \partial_t \delta U(0, \tau) = \frac{1}{4} c^2(0) c (\delta r + \delta e) (S^{-1}(\frac{\tau}{2})) + (\text{remainder}). \quad (5.6)$$

Here the terms denoted by (remainder) in (5.6) include the quantity

$$\left\{ \frac{1}{2} c^2(0) c \frac{\delta\Lambda}{\Lambda} \frac{\partial_z \Lambda}{\Lambda} \right\} S^{-1}(\frac{\tau}{2}) = \left\{ \frac{1}{2} c^2(0) c(z) e(z) \int_0^z \delta e \right\} \Big|_{z=S^{-1}(\frac{\tau}{2})} \quad (5.7)$$

together with the terms represented by (remainder) in (5.5). As there are quite a few of these latter, we shall only analyse a representative sample.

One of the terms not represented explicitly in (5.5) is

$$\begin{aligned} &\int_0^\infty dz \int dt \tilde{A}^{-\frac{1}{2}}(z) \delta(\tau - t - S(z)) \delta\rho(z) \partial_t^2 J(z, t) \\ &= \int_0^\infty dz \int dt \tilde{A}^{-\frac{1}{2}}(z) \delta(\tau - t - S(z)) \delta\rho(z) \left\{ \frac{1}{4} (c(0) \partial_z \log A(0) + \frac{1}{4} c(z) \partial_z \log A(z) \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8} \int_0^z d\varsigma \frac{c(\varsigma)}{c(0)} [\partial_z \log A(\varsigma)]^2 \delta(t - S(z)) \\
& + \int_0^\infty dz \int_{S(z)}^\infty dt \tilde{A}^{\frac{1}{2}}(z) \delta(\tau - t - S(z)) \delta\rho(z) \partial_t^2 J(z, t) \\
& = b(z) \delta\rho(z) \Big|_{z=S^{-1}(\frac{\tau}{2})} + \int_0^{S^{-1}(\frac{\tau}{2})} dz \tilde{A}^{\frac{1}{2}}(z) \partial_t^2 J(z, \tau - S(z)) \delta\rho(z) \\
& = \int_0^{S^{-1}(\frac{\tau}{2})} dz k(z, \tau) \delta r(z)
\end{aligned}$$

where we have used (5.2) and written b and k for combinations of derivatives of the reference coefficients and of J . In particular, k is a smooth kernel.

Another such term is

$$\begin{aligned}
& - \int dz \int dt \partial_t J(z, \tau - t) \tilde{A}^{\frac{1}{2}}(z) c^{-2}(z) \delta\Lambda(z) \delta'(t - S(z)) \\
& = - \int dz \int dt \partial_t^2 J(z, \tau - z) \tilde{A}^{\frac{1}{2}}(z) c^{-2}(z) \delta\Lambda(z) \delta(t - S(z)) \\
& = \dots = \int_0^{S^{-1}(\frac{\tau}{2})} dz k(z, \tau) \delta e(z)
\end{aligned}$$

after integration by parts in t and manipulations similar to those above.

All other such terms may be analysed in similar fashion. Moreover, the size of the integral kernels appearing in the final expressions may be related to L^2 -norms of various derivatives of ρ, λ , using energy estimates in the spirit of Symes (1985).

We summarize this analysis:

Suppose that r and e (hence ρ and Λ) are given on the depth interval $[0, Z]$. Then for $0 \leq \tau \leq 2S(z)$,

$$\begin{aligned}
\Lambda(0) \partial_t \delta U(0, \tau) &= \frac{1}{4} c^2(0) c(\delta r + \delta e)(S^{-1}(\frac{\tau}{2})) \\
&+ \int_0^{S^{-1}(\frac{\tau}{2})} dz \{k_r(\tau, z) \delta r(z) + k_e(\tau, z) \delta e(z)\}
\end{aligned} \tag{5.8}$$

where

$$\sup |k_r(s, z)|, \sup |k_e(s, z)| \leq F\left(\int_0^z |r|^2 + |r'|^2, \int_0^z (|e|^2 + |e'|^2)\right)$$

Here the suprema are taken over the set

$$\{(s, z) : 0 \leq z \leq S^{-1}(\frac{\tau}{2}), \quad 0 \leq \tau \leq 2S(Z)\}$$

and F is a universal continuous function of two (non-negative) variables.

6. High-frequency Asymptotics of the Linearized Map in the Δ -precritical region

In this section we use the results on plane-wave asymptotics derived in Section 5 to give a complete asymptotic treatment of the precritically projected perturbational point source response. This allows us to complete the proofs of all of the major results discussed in Section 2.

For the coefficients of the Radon-transformed problem we have (recall $p = |p|$)

$$\begin{aligned} r(z, p) &= r(z) = \partial_z \log \rho(z) = \frac{\rho'(z)}{\rho(z)} \\ \bar{e}(z, p) &= \partial_z \log \Lambda(z, p) = \\ &= \partial_z \log \lambda(z) - \partial_z \log (1 - c^2(z)p^2) \\ &= \frac{\lambda'(z)}{\lambda(z)} + p^2 \left(1 - \frac{\lambda(z)}{\rho(z)} p^2 \right)^{-1} \left(\frac{\lambda'(z)}{\rho(z)} - \frac{\lambda(z)\rho'(z)}{\rho^2(z)} \right) \\ &= \frac{\lambda'(z)}{\lambda(z)} + p^2 \left(1 - \frac{\lambda(z)}{\rho(z)} p^2 \right)^{-1} \frac{\lambda(z)}{\rho(z)} \left(\frac{\lambda'(z)}{\lambda(z)} - \frac{\rho'(z)}{\rho(z)} \right) \end{aligned}$$

Introduce

$$l(z) = \frac{\lambda'(z)}{\lambda(z)}$$

Then

$$e(z, p) = l(z)(1 - c^2(z)p^2)^{-1} - r(z)c^2(z) |\xi|^2 (1 - c^2(z)p^2)^{-1}$$

whereas

$$c(z) = \exp \frac{1}{2} \int_0^z (l - r)$$

For the perturbations, we obtain

$$\begin{aligned} \delta e(z, p) &= \delta l(z) + p^2(1 - c^2(z)p^2)^{-1} c^2(z)(\delta l(z) - \delta r(z)) \\ &\quad + (p^4(l(z) - r(z))(\delta l(z) - \delta r(z))c^2(z)(1 - c^2p^2)^{-2} \\ &= (1 - c^2(z)p^2)^{-1} (1 + p^4 c^2(z)(1 - c^2(z)p^2)^{-2} (l(z) - r(z)) \delta l(z) \\ &\quad - p^2 c^2(z)(1 - c^2(z)p^2)^{-1} (1 + p^2(1 - cp^2)^{-1} (l(z) - r(z)) \delta r(z)) \end{aligned}$$

Set

$$\beta(r, p) = p^2 c^2(z)(1 - c^2(z)p^2)^{-1} [1 + p^2(l(z) - r(z))(1 - c^2(z)p^2)^{-1}]$$

Then

$$\delta r(z) + \delta c(z, p) = (1 + \beta(z, p))\delta l(z) + (1 - \beta(z, p))\delta r(z) \quad (6.1)$$

In using the expressions derived in the last section, we need to replace the velocity c by the vertical wave speed at incidence p :

$$\begin{aligned} v(z, p) &= p(z)^{\frac{1}{2}} \Lambda(z, p)^{\frac{1}{2}} \\ &= c(z)(1 - c^2(z)p^2)^{-\frac{1}{2}} \end{aligned} \quad (6.2)$$

We remind the reader that ρ and λ have been normalized so that

$$\lambda(0) = \rho(0) = c(0) = 1.$$

Thus

$$v(0, p) = (1 - p^2)^{-\frac{1}{2}}. \quad (6.3)$$

Then from (5.8), (6.2) and (6.3) we have

$$\begin{aligned} L(\delta r, \delta l)(\tau, p) &= \frac{1}{4}(1 - p^2)^{-\frac{1}{2}} [v((1 + \beta)\delta l + (1 - \beta)\delta r)](S^{-1}(\frac{\tau}{2}, p)) \\ &\quad + \int_0^{S^{-1}(\frac{\tau}{2}, p)} dz \{K_r(\tau, z, p)\delta r(z) + K_l(\tau, z, p)\delta l(z)\}. \end{aligned}$$

Now let T be the Δ -precritical region for ρ, λ , as in Section 2:

$$T = \{(\tau, p) : 0 \leq \tau \leq 2S_\Delta(p), 0 \leq p \leq P_{\max}\}$$

Define for $(\delta r, \delta l) \in (L^2[0, Z])^2$

$$\begin{aligned} E(\delta r, \delta l)(\tau, p) &= \frac{1}{4}(1 - p^2)^{-\frac{1}{2}} [v((1 + \beta)\delta l + (1 - \beta)\delta r)](S^{-1}(\frac{\tau}{2}, p)) \\ K(\delta r, \delta l)(\tau, p) &= \int_0^{S^{-1}(\frac{\tau}{2}, p)} dz \{K_r(\tau, z, p)\delta r(z) + K_l(\tau, z, p)\delta l(z)\} \end{aligned}$$

so that

$$L(\delta r, \delta l) = (E - K)(\delta r, \delta l).$$

This is the decomposition (2.5), our second main result. The property claimed of K , namely

$$\kappa(L) \leq C \kappa(E)$$

follows from well-known facts about (matrix) Volterra integral operators with smooth kernels, together with bounds for E . We now establish the latter. In order to make the computations tractable, we will first investigate the operator E with the weighted norm

$$\begin{aligned} & \| |E(\delta r, \delta l)| \|_{L^2(T, (1-p^2) dr, dp)}^2 \\ &= \frac{1}{16} \int_T \int dp \, v^2((1+\beta)\delta l + (1-\beta)\delta r)^2 (\delta^{-1}(\frac{\tau}{2}, p)) \\ &= \frac{1}{8} \int_{p \leq \sqrt{1-\Delta^2}} dp \int_0^{Z_\Delta(p)} dz \, v(z) ((1+\beta(z, p))\delta l(z) + (1-\beta(z, p))\delta r(z))^2 \end{aligned}$$

where Z_Δ is defined, as before:

$$Z_\Delta = \min [Z, \inf \{z \geq 0 : c(z)p = \sqrt{1-\Delta^2}\}].$$

Now, Z_Δ is monotone in p , hence has a (possibly discontinuous) monotone inverse $P_\Delta(z)$ defined on $0 \leq z \leq Z$. Interchanging the order of integrations, we have

$$\begin{aligned} \| |E(\delta r, \delta l)| \|^2 &= \int_0^Z dz \{E_{ll}(z, \Delta) \delta l^2(z) \\ &+ 2E_{lr}(z, \Delta) \delta l(z) \delta r(z) + E_{rr}(z, \Delta) \delta r^2(z)\} \end{aligned}$$

where we define

$$\begin{aligned} E_{ll}(z, \Delta) &= \frac{1}{8} \int_{p \leq P_\Delta} dp \, v(z, p) (1+\beta(z, p))^2 \\ E_{lr}(z, \Delta) &= \frac{1}{8} \int_{p \leq P_\Delta} dp \, v(z, p) (1-\beta^2(z, p)) \\ E_{rr}(z, \Delta) &= \frac{1}{8} \int_{p \leq P_\Delta} dp \, v(z, p) (1-\beta(z, p))^2. \end{aligned}$$

Let

$$\begin{aligned} \mu(\Delta) &= \inf_{\substack{z \in [0, Z] \\ x_l^2 + x_r^2 = 1}} \{ (E_{ll}(z, \Delta) x_l^2 + 2E_{lr}(z, \Delta) x_l x_r + E_{rr}(z, \Delta) x_r^2) \\ &= \inf_{z \in [0, Z]} \lambda_- E(z). \end{aligned}$$

where λ_- is the smaller eigenvalue of the matrix $\mathbf{E}(z)$

$$\mathbf{E} = \begin{pmatrix} E_{ll} & E_{lr} \\ E_{lr} & E_{rr} \end{pmatrix}$$

Then,

$$\| |E(\delta r, \delta l)| \|_{L^2(T, (1-p^2)dr, dp)}^2 \geq \mu(\Delta) \| |\delta r, \delta l| \|_{L^2[0, Z]}^2. \quad (6.4)$$

We proceed by evaluating the entries in \mathbf{E} :

$$\begin{aligned} E_{ll}(z, \Delta) &= \frac{1}{8} \int_{p \leq P_{\Delta}(z)} dp \, c(z)(1-c^2(z)p^2)^{-\frac{1}{2}} [1+p^2c^2(z)(1-c^2(z)p^2)^{-1} \\ &\quad (1+p^2(l(z)-r(z))(1-c^2(z)p^2)^{-1})^2] \\ &= \frac{\pi}{4} \int_0^{P_{\Delta}(z)} d\sigma \, \sigma \, c(z)(1-c^2(z)\sigma^2)^{-\frac{1}{2}} [1+c^2(z)\sigma^2(1-c^2(z)\sigma^2)^{-1} \\ &\quad (1+\sigma^2(l(z)-r(z))(1-c^2(z)\sigma^2)^{-1})^2] \\ &= \frac{\pi}{4c(z)} \int_0^{\gamma(z)} ds \, \frac{s}{\sqrt{1-s^2}} \left[1 + \frac{s^2}{1-s^2} \left[1 + \frac{l(z)-r(z)}{c^2(z)} \frac{s^2}{1-s^2} \right]^2 \right] \\ &= \frac{\pi}{4c(z)} \int_{\sqrt{1-\gamma^2(z)}}^1 d\tau [1+(\tau^2-1)(1+\alpha(z)(\tau^2-1))]^2 \\ &= \frac{\pi}{4c(z)} \int_{\sqrt{1-\gamma^2(z)}}^1 d\tau (1+\nu(z, \tau))^2 \end{aligned}$$

where

$$\alpha(z) = (l(z)-r(z))c^{-2}(z)$$

$$\nu(z, \tau) = (\tau^2-1)(1-\alpha(z)(\tau^2-1))$$

and $\gamma(z) = c(z)P_{\Delta}(z)$ as defined earlier.

Similarly, we get

$$\begin{aligned} E_{lr}(z) &= \frac{\pi}{4c(z)} \int_{\sqrt{1-\gamma^2(z)}}^1 d\tau (1-\nu^2(z, \tau)) \\ E_{rr}(z) &= \frac{\pi}{4c(z)} \int_{\sqrt{1-\gamma^2(z)}}^1 d\tau (1-\nu(z, \tau))^2 \end{aligned}$$

From these expressions, it is clear that the matrix elements of \mathbf{E} are rational functions of $c(z)$ and $(1-\gamma^2(z))^{\frac{1}{2}}$, with coefficients which are polynomial in $\alpha(z)$. All these coefficients, hence the eigenvalues, are scaled by the factor $\frac{\pi}{4c(z)}$. It follows that the condition number, which governs relative errors, is a function of $\gamma(z)$ and $\alpha(z)$ only, whereas the eigenvalues $\lambda_-(z)$ and $\lambda_+(z)$, which govern absolute errors, are additionally linear in $c^{-1}(z)$.

Suppressing the dependence on z and using the abbreviation $m = \sqrt{1-\gamma^2}$, we obtain for the discriminant

$$\left(\frac{4c}{\pi}\right)^2(E_{ll}E_{rr} - E_{lr}^2) = 4\left[\int_m^1 1^2 \int_m^1 \nu^2 - \left(\int_m^1 \nu^2\right)^2\right] \geq 0.$$

The sharp form of the Cauchy-Schwarz inequality implies that the above quantity is strictly positive, unless the functions ν and 1 are collinear on the interval $[m, 1]$. As ν is non-constant, this is impossible, and we conclude that the matrix \mathbf{E} is invertible, and so likewise the operator E . In particular $\mu(\Delta) > 0$.

Suppressing for the moment the scale factor $\frac{\pi}{4c(z)}$, we obtain for the eigenvalues

$$\lambda_{\pm} = \int_m^1 (1 + \nu^2) \pm \left[\left(\int_m^1 (1 + \nu^2)^2 - 4((1-m) \int_m^1 \nu^2 - \left(\int_m^1 \nu^2\right)^2) \right)^{\frac{1}{2}} \right]$$

To leading order in m^{-1} ,

$$y := \int \nu \sim \begin{cases} \frac{\alpha}{3m^3} & \alpha \neq 0 \\ \frac{1}{m} & \alpha = 0 \end{cases}$$

$$x := \int \nu^2 \sim \begin{cases} \frac{\alpha^2}{7m^7} & \alpha \neq 0 \\ \frac{1}{3m^3} & \alpha = 0 \end{cases}$$

Thus as $m \rightarrow 0$,

$$x \rightarrow +\infty, \quad yx^{-1} \rightarrow 0$$

For the eigenvalues we obtain

$$\begin{aligned} \lambda_{\pm} &= [(1-m) + x] \pm \{[(1-m) + x]^2 - 4[(1-m)x - y^2]\}^{\frac{1}{2}} \\ &= [(1-m) + x] \left\{ 1 \pm \sqrt{1 - \frac{4[(1-m)x - y^2]}{[(1-m) + x]^2}} \right\}, \end{aligned}$$

in particular, for m small (so $x \gg y$),

$$\lambda_{\pm} \simeq [(1-m) + x] \left\{ 1 \pm \left[1 - 2 \frac{(1-m)x - y^2}{[(1-m) + x]^2} \right] \right\}$$

Thus

$$\lambda_+ \sim 2x \simeq \begin{cases} \frac{2\alpha^2}{7m^7} & \alpha \neq 0 \\ \frac{2}{3m^3} & \alpha = 0 \end{cases} \quad (6.5a)$$

$$\lambda_- \simeq 2 \frac{x - y^2}{1 + x} \simeq 2 \quad \text{for all } \alpha.$$

For the condition number of \mathbf{E} , we obtain

$$\kappa = \sqrt{\frac{\lambda_+}{\lambda_-}} \simeq \begin{cases} \frac{\sqrt{7}\alpha^2}{2} m^{-3.5} & \text{for } \alpha \neq 0 \\ \frac{1}{\sqrt{6}} m^{-1.5} & \text{for } \alpha = 0 \end{cases}, \quad \text{as } m \rightarrow 0 \quad (6.5b)$$

This estimate (6.5) gives the maximum-aperture limiting behaviour.

Concerning the limiting behaviour as $m \rightarrow 1$, i.e. $\gamma \rightarrow 0$ (narrow aperture), clearly

$$\nu(\tau) = O(\tau^{-1})$$

so

$$y(m) = \int_m^1 \nu = O((1-m)^2)$$

$$x(m) = \int_m^1 \nu^2 = O((1-m)^3)$$

and

$$E = \frac{4\pi}{c} \begin{bmatrix} (1-m) + 2y + x & (1-m) - x \\ (1-m) - x & (1-m) - 2y + x \end{bmatrix}$$

$$= \frac{2\pi}{c} \gamma \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + O(\gamma^2)$$

as $1-m = \frac{1}{2} \gamma + O(\gamma^2)$, $\gamma \rightarrow 0$. Thus

$$\lambda_+ \sim \frac{4\pi}{c} \gamma + O(\gamma^2)$$

$$\lambda_- \sim O(\gamma^2)$$
(6.6a)

and

$$\kappa \sim O(\gamma^{\frac{1}{2}})$$
(6.6b)

as $\gamma \rightarrow 0$.

We draw a number of conclusions from these bounds, which together constitute our third major set of results.

Suppose first that $\gamma = cP_\Delta$ remains well away from zero for $0 \leq z \leq Z_\Delta$, so that $m = \sqrt{1-\gamma^2}$ remains close to zero (hence Δ close enough to zero since $m \geq \Delta$). Then according to (6.4) and the estimate $\lambda_- \sim 2 > 1$ (6.5a), we have

$$||E(\delta r, \delta l)||^2 \geq (||\delta r||^2 + ||\delta l||^2).$$

Hence, we have absolute stability for the linear map involving plane waves in the precritical region, provided that the slowness aperture remains near its maximal value, in the language of Section 3.

Concerning spectral bounds for E , we can be somewhat more precise by localizing the perturbations $\delta r, \delta l$. If

$$\text{supp } \delta r \cup \text{supp } \delta l \subset [z_1, z_2],$$

then

$$||E(\delta r, \delta l)|| \leq \sup \{ \lambda_+(\mathbf{E}(z)): z_1 \leq z \leq z_2 \} ||(\delta r, \delta l)||$$

and

$$||E(\delta r, \delta l)|| \geq \inf \{ \lambda_-(\mathbf{E}(z)): z_1 \leq z \leq z_2 \} ||(\delta r, \delta l)||$$

Since \mathbf{E} and its spectrum depend smoothly on z , we conclude that for perturbations localized near z , $\kappa(E) \sim \kappa(\mathbf{E}(z))$.

In figures (8a-f) we have plotted the condition number of \mathbf{E} as a function of the product $\gamma = c(z)P_\Delta(z)$ for several values of $\alpha(z)$. Note that

$$\begin{aligned} \alpha(z) &= c^{-2}(z)(l(z) - r(z)) \\ &= c^{-2}(z) \partial_z \log \frac{\lambda(z)}{\rho(z)} \\ &= 2c^{-3}(z) c'(z) \end{aligned}$$

measures the slope of the local wave velocity at depth z .

From the plots (and from the asymptotics of displays (6.5), (6.6)) we conclude that for perturbations localized near z , for fixed α the condition number $\kappa(\mathbf{E})$ attains an optimal value $\kappa_{opt}(\alpha)$ at some optimal $\gamma = \gamma_{opt}(\alpha)$. For $\alpha = 0$, $\kappa_{opt} \simeq 2.8$ is attained at $\gamma_{opt} \simeq .85$.

For $\alpha \neq 0$, apparently the interval of γ for which κ is close to optimal is smaller than for $\alpha = 0$ (see figures 8a-f). This is easy to understand, as when $\alpha(z_0) \neq 0$, there is a point z close to z_0 for which $c(z) > c(z_0)$. Thus a smaller localized perturbation in c is needed to make a near-critical p post-critical for z near z_0 than is the case if c has no slope at z_0 .

We see that $\kappa \rightarrow \infty$ while $\lambda_- \rightarrow 2$ as $\gamma \rightarrow 1$, i.e. as the slowness aperture approaches the full critical range. This appears somewhat counterintuitive: the information content seems to degrade as more data is added! In fact, this ill-conditioning near critical incidence reflects the lack of differentiability of the map F , as mentioned in the introduction — or, otherwise put, the unboundedness of DF , or the “failure of the Born approximation.” The physical source of the explosion is the phase shift caused by small velocity perturbations, as mentioned in Section 3.

The fact that $\lambda_- \rightarrow 2$ hints that some perturbations in ρ, λ do cause well-behaved data perturbations; we shall see below that these are exactly the perturbations preserving the velocity.

On the other hand, the fact that $\kappa \rightarrow \infty$ as $\gamma \rightarrow 0$ also tells us that some perturbations are better represented than others for narrow slowness aperture. From (6.6), $\lambda_+ = 0(1-m) = 0(\gamma)$ for small γ . If we weight the norm used to measure $E(\delta r, \delta l)$ by the local aperture, i.e.

$$\begin{aligned} \|f\| &= \left\| f \right\|_{L^2(T, \gamma^{-1}(S^{-1}(\frac{\tau}{2}))(1-p^2) dp d\tau)} \\ &= \int \int_T dp dT \gamma^{-1}(S^{-1}(\frac{\tau}{2}))(1-p^2) |f(p, \tau)|^2 \end{aligned} \quad (6.7)$$

then the normal operator for E becomes the matrix multiplication operator

$$\gamma^{-1}(z) E(z)$$

and now λ_+ is uniformly bounded away from zero, whereas $\lambda_- = 0(\gamma)$. If we take Δ close enough to 1, then P_Δ becomes independent of z (i.e. the entire aperture is precritical -- the case covered by the work of Coen and others). The limit $\Delta \rightarrow 1$ corresponds to $\gamma \rightarrow 0$, and in this limit one component of the perturbation in ρ, λ corresponding to λ_+ is perfectly represented in the data, whereas an orthogonal component, corresponding to $\lambda_- \rightarrow 0$, is not represented at all.

For more information on these limiting cases we must make use of the eigenvectors of E . For $\gamma \rightarrow 1$ (i.e. $m \rightarrow 0$).

$$E \sim \begin{pmatrix} \frac{1}{3m^3} & \frac{-1}{3m^3} \\ -\frac{1}{3m^3} & \frac{1}{3m^3} \end{pmatrix} + O(\frac{1}{m^2})$$

(for $\alpha = 0$; similar conclusions for $\alpha \neq 0$) so that the eigenvector Q_+ corresponding to λ_+ is

$$Q_+ \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ so } Q_- \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} :$$

Thus perturbations with $\delta r = \delta l$ are the only ones which *do not* yield very large data perturbations near critical angle. Since

$$c(z) = \exp \int_0^z (r - l)$$

$$\delta c(z) = \left(\int_0^z (\delta r - \delta l) \right) c(z)$$

these are exactly those perturbations which *do not* yield a perturbation in the velocity. This precisely justifies our earlier (part 3) assertion that velocity perturbations make this problem ill-conditioned for large apertures, because of the possible transition from pre- to post-critical incidence.

In the other direction, for $\gamma \rightarrow 0$, hence $m \rightarrow 1$, the expression preceding (6.6) shows that

$$Q_+ \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad Q_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

i.e. after scaling by γ^{-1} , the perturbations with $\delta r + \delta l = 0$ produce vanishing data perturbations as $\gamma \rightarrow 0$. These are precisely the perturbations which fix the admittance

$$A = \rho^{\frac{1}{2}} \lambda^{\frac{1}{2}} = \frac{1}{2} \exp \int r + l$$

$$\delta A = (\delta r + \delta l) A$$

As the scaled limit $\gamma \rightarrow 0$ is essentially the problem of a single normally incident plane wave, this is an unsurprising result: it is well-known that only the admittance or an equivalent quantity can be determined from plane wave data at normal incidence (see e.g. Bamberger et al. (1979)).

Finally, suppose that we constrain the problem by a relation between ρ and λ . In fact, this is quite reasonable in many applications. A widely used phenomenological relation in exploration geophysics is, for instance,

$$\rho \simeq k c^{\frac{1}{4}} \tag{6.8}$$

which seems to hold for a wide range of sedimentary rocks with suitable constant k (see Gardner et al. (1974)).

Accordingly, suppose

$$\log \rho(z) = G(\log c(z)) \tag{6.9}$$

identically in E . Then

$$\begin{aligned}\frac{\delta\rho}{\rho} &= G'(\log c) \frac{\delta c}{c} \\ \delta r &= \partial_z \frac{\delta\rho}{\rho} = \partial_z \left(G' \log c \right) \frac{\delta c}{c} \\ &= \frac{1}{2} G'(\log c)(\delta l - \delta r) + G_1(c) \delta c\end{aligned}$$

where $G_1(c)$ is a combination of G , c , and their derivatives.

Since δc is an integral of $\delta r - \delta l$, the second summand plays no role in the leading order asymptotics. To highest order in frequency, then, (6.9) is equivalent to

$$\delta r = \frac{1}{2} G'(\log c)(\delta l - \delta r)$$

meaning that

$$\begin{pmatrix} \delta r \\ \delta l \end{pmatrix} = \delta r \begin{pmatrix} 1 \\ \frac{2}{G'} + 1 \end{pmatrix}$$

if $G'(\log c) \neq 0$, else

$$\delta r = 0$$

In either case, the projection onto the “good” eigenvector $\sqrt{2^{-1}}(1, 1)^T$ is nonzero provided that

$$G'(\log c) \neq -1$$

For the sedimentary relation (6.8), $G' \equiv \frac{1}{4}$, so the acoustic parameters are well-determined

in that case by the scaled data (6.7), even when the aperture becomes small.

7. Implications for the Nonlinear Inverse Problem

In this section we discuss the consequences of our high-frequency perturbational analysis for the nonlinear inverse problem, and for the design of numerical algorithms. We shall be very brief: extended discussion will appear elsewhere.

To begin with, we note that the precritical projection of the linearized map, \tilde{L} , is an isomorphism onto its range. That the solution map for the plane-wave equations, L , is an isomorphism, follows from the decomposition

$$L = E + K,$$

the Volterra nature of K , and the isomorphic character of E , proved in the last section. On the other hand

$$\tilde{L} = L - M$$

and the estimates established for M in section 4 show that M is compact (in fact, smoothing). Therefore \tilde{L} is an isomorphism onto its range if and only if $\tilde{L}(\delta\rho, \delta\lambda) = 0$ implies $\delta\rho = \delta\lambda \equiv 0$. The injectivity of \tilde{L} follows from the fact that, for small p , the truncated Radon transform is the same as the untruncated transform inside the double light cone, as pointed out in section 4. That is, for small p

$$\tilde{L}(\delta\rho, \delta\lambda)(\tau, p) = L(\delta\rho, \delta\lambda)(\tau, p)$$

However L is injective, even when restricted to a small range of p 's (this is just the linearized version of Coen's uniqueness theorem). Thus \tilde{L} is injective, and so it is an isomorphism.

We consider next the case of non-impulsive point sources. Sources distributed also in x may be treated similarly.

Thus suppose that the boundary condition is replaced by

$$\lambda(0) \nabla \cdot u(x, 0, t) = \delta(x) w(t)$$

Then the time-invariance of the equation of motion implies that the precritical projection of the perturbational response is given by

$$w * \tilde{L}$$

$$= w * (E + K + M)$$

From the last section,

$$E(\delta r, \delta l)(\tau, p) = (c_r \delta r + c_l \delta l)(S^{-1}(\frac{\tau}{2}, p), p)$$

for suitable functions $c_r(z, p)$, $c_l(z, p)$. Thus

$$\begin{aligned} & (w * E)(\delta r, \delta l)(\tau, p) \\ &= 2 \int dz w(\tau - 2S(z, p)) c^{-1}(z) (1 - c^2(z) p^2)^{\frac{1}{2}} [c_r(z, p) \delta r(z) + c_l(z, p) \delta l(z)] \end{aligned} \quad (7.1)$$

The general analysis of operators such as (7.1) is beyond the scope of this article. Suppose however that δr and δl are supported in a region $[z_0 - \delta z, z_0 + \delta z]$ on which ρ, λ , hence c , are constant. Then for $|z - z_0| < \delta z$

$$S(z, p) = S(z_0, p) + \frac{\sqrt{1 - c^2(z_0) p^2}}{c(z_0)} (z - z_0)$$

and we obtain

$$(w * E)(\delta r, \delta l)(\tau, p) = \tilde{c}_r(p) \tilde{w}_p * \delta r(\tau) + \tilde{c}_l(p) \tilde{w}_p * \delta l(\tau)$$

where \tilde{c}_r, \tilde{c}_l are coefficients in (7.1) evaluated at $z = z_0$, and

$$\tilde{w}_p(t) = w\left(\frac{\sqrt{1 - c^2(z_0) p^2}}{c(z_0)} \tau\right)$$

If, as is typical in seismology and other applications, w has a *passband*, i.e. a range $\Omega_1 \leq |\omega| \leq \Omega_2$ of frequencies for which

$$|\hat{w}(\omega)| \geq k > 0$$

$$\Omega_1 \leq |\omega| \leq \Omega_2$$

then the passband for \tilde{w}_p has limits

$$\frac{c(z_0)}{\sqrt{1 - c^2(z_0) p^2}} \Omega_i, \quad i = 1, 2. \quad (7.2)$$

If the lower bandlimit of w , i.e. Ω_1 , is sufficiently high, then $w * K$ and $w * M$ are *small* operators, so for perturbations $\delta r, \delta l$ supported near z_0 whose Fourier transforms have most of their energy in the passbands (7.2), the size of $w * \tilde{L}(\delta r, \delta l)$ will bound $(\delta r, \delta l)$.

To make these considerations precise requires relations between the bandlimits and the smoothing properties of K and M , and also arguments which give "passband" results similar to those above when c is nonconstant on the supports of $\delta\rho, \delta l$. These matters will be discussed elsewhere.

Remark. One could paraphrase this result as follows: "The inverse problem is well-posed in the passband of the data." (This paraphrase is due to Professor Norman Bleistein.) One must be rather careful in the interpretation of this paraphrase, however. It applies to the linearized problem only. Thus the reference profiles (usually taken to be the out-of-passband trends in the density and velocity) must be correct in order for small bandlimited data perturbations to correspond to small bandlimited parameter changes. This is obvious, as otherwise even the "passbands" (7.2) in which the perturbations $(\delta r, \delta l)$ are well-determined are not even necessarily the same as they may depend explicitly on the out-of-passband components of the velocity (i.e. the trends).

One would like to pass directly from well-posedness results concerning the linearized problem to similar statements about the nonlinear inverse problem, *via* the Implicit Mapping Theorem. Indeed, under the suitable assumptions concerning the reference profiles (r_0, l_0 have square-integrable derivatives) the linearized map DF actually is the Frechét derivative of F (restricted, of course, to precritical data). Unfortunately, the lower bound for DF is in a weaker norm, so the Implicit Mapping Theorem does not apply. In fact, the nonlinear inverse problem is quite ill-posed, and must be set as an optimization with the additional smoothness conditions on r, l imposed as *a priori* bounds. To conclude that such problems have stable solutions, one also needs control over the second derivative D^2F , which implies even more *a priori* smoothness (constrained upon the second derivatives of r, l in the mean-square sense).

The necessary machinery for establishing well-posed versions of the nonlinear problem is sketched in Symes (1985b), which treats the recovery of the velocity from plane-wave data at a single (precritical) incidence, in the presence of known density. The arguments given there carry over with inessential modifications to the layered acoustic inverse problem. These arguments involve estimation of the second derivative of the precritically projected map which in turn

requires estimates of the remainder in the progressing wave expansion (5.8).

For numerical solution of the linearized precritically projected problem

$$\tilde{L}(\delta r, \delta l) = g$$

or a least-squares version appropriate for inconsistent data:

$$\min ||\tilde{L}(\delta r, \delta l) - g||_{L^2(T)}^2$$

a version of conjugate gradient-iteration seems especially attractive. Note that matrix methods are essentially useless for these problems, as access to the matrix elements involves computing the perturbational acoustic field over an entire basis of perturbations in ρ and λ . One would hope to obtain good estimates of $\delta\rho, \delta\lambda$ at far less cost.

On the other hand, a very good approximate inverse is available at high frequencies. Set

$$C = (E^*E)^{\frac{1}{2}}$$

(This square root is well-defined by the spectral theorem, as E^*E is self-adjoint positive-definite).

Then

$$\begin{aligned} & C^{-1}(\tilde{L}^* \tilde{L})C^{-1} \\ &= C^{-1}(E^*E + E^*(K+M) + (K+M)^*E + (K+M)^*(K+M))C^{-1} \\ &= I + K_1 \end{aligned}$$

where K_1 is compact, as follows from the nature of K and M . Thus the preconditioned conjugate gradient method (see e.g. Golub and Van Loan (1983), section 10.3) with $(E^*E)^{-1}$ as preconditioning matrix is superlinearly convergent (Daniels (1970), section 2).

To carry out such a preconditioned iteration, it is merely necessary to solve equations such as

$$(E^*E)(\delta r, \delta l) = (\delta \hat{r}, \delta \hat{l})$$

This is trivial, however, as E^*E is simply the multiplication operator by the matrix E of section 6. Thus our problem can be effectively preconditioned by inversion of a 2×2 matrix (-valued function)!

In general, the perturbation K_1 is merely compact, not small, and the preconditioned problem is guaranteed to be well-conditioned only on the complement of some finite -(but possibly large -) dimensional subspace. On the other hand, with band-limited (nonimpulsive) sources and appropriate bandlimits on $\delta r, \delta l, K$ and M , hence K_1 , may be regarded as small. Thus we expect the p.c.g. procedure to be very effective for solution of appropriately regularized bandlimited problems.

Note that Clayton and Stolt (1981) suggest (essentially) that use of $(E^*E)^{-1}E^*$, composed with suitable bandlimiting filters, should yield an adequate least-squares solution. This amounts to the first step of preconditioned conjugate gradients. We expect that further steps will improve the output at intermediate frequencies. This is unimportant when the background (r, l) is accurate and the aim is the location of high-frequency energy ("migration problem"), but is crucial when attempting to make genuine changes in the background in solving the nonlinear problem.

The nonlinear version of our problem should be phrased as follows:

$$\min_{\rho, \lambda} \| \Pi_{\Delta}(\rho, \lambda)(F(\rho, \lambda) - g) \|_{L^2}^2 \quad (7.3)$$

where $\Pi_{\Delta}(\rho, \lambda)$ denotes the precritical projector as defined in section 4. The most notable feature of this problem is that the definition of the data, i.e. the projection Π , depends on ρ, λ . That is, the very definition of "precritical" depends on the reference medium.

Here the importance of allowing the "margin of safety" $\Delta > 0$ in the definition of the precritical data set is manifest. In fact, $\Pi_{\Delta}(\rho, \lambda)$ is still a precritical projector for ρ, λ in an entire neighborhood of ρ_0, λ_0 . This allows us to replace the functional (7.3) by an appropriate quadratic model and to use techniques from smooth optimization.

Unfortunately, a large initial error in ρ, λ leads to a large initial error in the projector Π . Such a substantial projection error makes local improvement of (7.3) very difficult. The way out of this impasse is to localize the problem in space-time. On the one hand, the smoothness constraints on ρ, λ necessary for linearization to be valid, allow us to *extrapolate* the velocity c from the surface to a small depth without significant error. Since small depths correspond to small times, we can predict the restriction of Π_{Δ} to small τ with confidence. The restricted residual can

then be made small, which yields an estimate of c which is *improved* to small depth. Once again, smoothness allows us to estimate c to a somewhat larger depth, reliably compute Π_Δ for a somewhat greater range of τ , and obtain substantial decrease in the restricted residual.

It is possible to show that this "layer stripping" process succeeds, in computing a good approximate solution, provided that the residual at the minimum is suitably small. (Otherwise, it computes a local minimum, which is all one can expect in large residual problems).

A similar "layer stripping" nonlinear least-squares procedure is described in Symes (1985b) for the problem of recovering c (constant ρ) from normal incidence plane-wave data.

As indicated above, the bandlimited linearized problem, roughly speaking, yields information only about the components of $\delta\rho$, $\delta\lambda$ in a corresponding "passband." For the nonlinear problem, this means that the out-of-passband components at the solution must be given with considerable accuracy. As the "passband" itself depends on the velocity structure, this is unlikely to be the case for an initial guess.

It has often been said that the redundancy of multidimensional data would overcome the lack of low-frequency content in seismic data. A counterexample to this supposition is presented in Bube, Santosa, and Symes (1985). This counterexample involves constant (and known) background velocity, so that the "passband" is also known. On the other hand, numerical experiments due to Chavent et al. (1985) and McAuley (1985) indicate that high-frequency surface data determines a layered velocity profile, *including its trends* (i.e. "out-of-band" components). These calculations thus appear to combine the *velocity analysis* procedure of exploration seismology with the solution of the inverse problem. Understanding this relationship constitutes the most important currently open problem in the study of inverse problems in wave propagation.

8. Conclusion

We have shown that the high-frequency precritical surface response of an acoustic wavefield to perturbations in the acoustic parameters ρ, λ , may be analysed by means of the truncated Radon transform. We have used the plane-wave equations, satisfied by the Radon-transformed field up to a smooth error, to compute the spectral properties of the perturbational relationship. We have shown that this relationship is well-conditioned in the absence of low-velocity zones, but that the presence of such zones leads to unavoidable instability, which we have quantified precisely. Finally, we have used this detailed picture of the response to suggest analytical and computational approaches to the inverse problem of identifying the acoustic parameters from the response.

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Appendix A. Derivation of the Modified Plane Wave Equations

Temporarily define \tilde{U} by

$$\tilde{U}(\mathbf{p}, \mathbf{z}, \tau) = \int d\mathbf{x} \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) u(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x})$$

Then for $i = 1, 2$

$$\begin{aligned} & \rho(z) \partial_\tau^2 \tilde{U}_i(\mathbf{p}, \mathbf{z}, \tau) \\ &= \int d\mathbf{x} \rho \left\{ \eta \partial_i^2 u_i + 2\partial_i \eta \partial_i u_i + \partial_i^2 \eta u_i \right\}(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \\ &= \int d\mathbf{x} \left\{ \lambda \eta \partial_{z_i} \nabla \cdot \mathbf{u} + \rho(2\partial_i \eta \partial_i u_i + \partial_i^2 \eta u_i) \right\}(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \\ &= \lambda(z) \int d\mathbf{x} \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \left\{ \partial_{z_i} [\nabla \cdot \mathbf{u}(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x})] - p_i \partial_\tau [\nabla \cdot \mathbf{u}(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x})] \right. \\ &\quad \left. + \rho(z) \int d\mathbf{x} (2\partial_i \eta \partial_i u_i + \partial_i^2 \eta u_i)(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \right. \\ &= -\lambda(z) \int d\mathbf{x} (\partial_{z_i} \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) + \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) p_i \partial_\tau) \\ &\quad [(\partial_{z_1} - p_1 \partial_\tau)(u_1(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x})) + (\partial_{z_2} - p_2 \partial_\tau)(u_2(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x})) \\ &\quad \left. + \partial_{z_3} u_3(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x})] + \rho(z) \int d\mathbf{x} (2\partial_i \eta \partial_i u_i + \partial_i^2 \eta u_i)(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \right. \\ &= \lambda(z) \sum_{j=1}^2 \int d\mathbf{x} \left\{ \partial_{z_i} \partial_{z_j} (\eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \right. \\ &\quad \left. + ((\partial_{z_i} \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) p_j + \partial_{z_j} \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) p_i) \partial_\tau \right. \\ &\quad \left. + \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) p_i p_j \partial_\tau^2 \right\} u_j(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}) \\ &\quad + \rho(z) \int d\mathbf{x} (2\partial_i \eta \partial_i u_i + \partial_i^2 \eta u_i)(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \\ &\quad - \lambda(z) \int d\mathbf{x} \left\{ \partial_{z_i} \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) + \eta(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) p_i \partial_\tau \right\} \\ &\quad \partial_{z_i} u_3(\mathbf{x}, \mathbf{z}, \tau + \mathbf{p} \cdot \mathbf{x}) \end{aligned}$$

$$\begin{aligned}
& + \rho(z) \int d\mathbf{x} (2\partial_t \eta \partial_t u_i + \partial_t^2 \eta \cdot u_i)(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) \\
& = \lambda(z) \sum_{j=1}^2 p_i P_j^2 \partial_\tau^2 \tilde{U}_j(p, z, \tau) \\
& \quad - \lambda(z) p_i \partial_\tau \partial_z \tilde{U}_3 \\
& \quad + K_i(p, z, \tau)
\end{aligned} \tag{A.1}$$

whereby K_1 and K_2 are defined.

This system of equations may be solved for $\partial_\tau^2 U_1, \partial_\tau^2 U_2$:

$$\begin{aligned}
\partial_\tau^2 \tilde{U}_1 &= -\frac{\lambda p_1}{\rho - \lambda^2 p^2} \partial_t \partial_z \tilde{U}_3 + \frac{\rho - \lambda p_2^2}{\rho(\rho - \lambda p^2)} K_1 - \frac{\lambda p_1 p_2}{\rho(\rho - \lambda p^2)} K_2 \\
\partial_\tau^2 \tilde{U}_2 &= -\frac{\lambda p_2}{\rho - \lambda^2 p^2} \partial_\tau \partial_z \tilde{U}_3 + \frac{\lambda p_1 p_2}{\rho(\rho - \lambda p^2)} K_1 + \frac{\rho - \lambda p_1^2}{\rho(\rho - \lambda p^2)} K_2
\end{aligned} \tag{A.2}$$

Similarly,

$$\begin{aligned}
& \rho(z) \partial_\tau^2 \tilde{U}_3(p, z, \tau) \\
& = \int d\mathbf{x} \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) \partial_z \lambda(z) \left\{ (\partial_{x_1} - p_1 \partial_\tau)(u_1(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x})) \right. \\
& \quad + (\partial_{x_2} - p_2 \partial_\tau)(u_2(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x})) \\
& \quad \left. + \partial_z u_3(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}) \right\} \\
& \quad + \rho(z) \int d\mathbf{x} (2\partial_t \eta \partial_t u_3 + \partial_t^2 \eta \cdot u_3)(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) \\
& = - \int d\mathbf{x} \left[(\partial_{x_1} \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) + \eta(\mathbf{x}, z, \tau - \mathbf{p} \cdot \mathbf{x}, p) p_1 \partial_\tau) \right. \\
& \quad \partial_z (\lambda(z) u_1(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}) + (\partial_{x_2} \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) + \\
& \quad \left. \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, p) p_2 \partial_\tau) \partial_z (\lambda(z) u_2(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x})) \right]
\end{aligned}$$

$$\begin{aligned}
& + \partial_z \lambda(z) \partial_z \tilde{U}_3(\mathbf{p}, z, \tau) \\
& - \int d\mathbf{x} \partial_z (\lambda(z) u_3(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}) \partial_z \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p})) \\
& + \rho(z) \int d\mathbf{x} (2\partial_t \eta \partial_t u_3 + \partial_t^2 \eta u_3)(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \\
& = - \sum_{j=1}^2 \partial_z \lambda(z) p_j \partial_\tau \tilde{U}_j(\mathbf{p}, z, \tau) \\
& + \partial_z \lambda(z) \partial_z \tilde{U}_3(\mathbf{p}, z, \tau) \\
& + H_3(\mathbf{p}, z, \tau)
\end{aligned}$$

with

$$\begin{aligned}
H_3(\mathbf{p}, z, \tau) = & - \int d\mathbf{x} \left\{ [\partial_{z_1} \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \partial_z (\lambda(z) u_1(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}))] \right. \\
& + [\partial_{z_2} \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \partial_z (\lambda(z) u_2(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}))] \\
& \left. + \partial_z (\lambda(z) u_3(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x})) \partial_z \eta(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p}) \right\} \\
& \rho(z) \int d\mathbf{x} (2\partial_t \eta \partial_t u_3 + \partial_t^2 \eta u_3)(\mathbf{x}, z, \tau + \mathbf{p} \cdot \mathbf{x}, \mathbf{p})
\end{aligned}$$

Integrate (A.2) in τ , using the vanishing of \tilde{U} for $\tau \ll 0$ to obtain $\partial_\tau U_i$, $i = 1, 2$, in terms of $\partial_z U_3$ and K_1, K_2 , then substitute in (A.1) to obtain

$$\rho \partial_\tau^2 U_3 = \partial_z \Lambda \partial_z U_3 + H$$

with

$$\Lambda(z, \mathbf{p}) = \lambda(z)(1 - c^2 p^2)^{-1}$$

and

$$H = H_3 + \lambda \partial_z ((\rho - \lambda p^2)^{-1} \int_{-\infty}^T d\tau' (p_1 K_1 + p_2 K_2))$$

With $U \leftarrow U_3$, these are the modified plane-wave equations of section 4.

Appendix B. Asymptotic Expansion of the Reflected Wave

We derive an asymptotic expansion of the form

$$\begin{aligned} \delta u(x, z, t) \sim & \left(\sum_{k=-1}^{\infty} b_k(x, z, t) \zeta^{-k} \right) e^{i\zeta \phi_r(x, z, t)} \\ & + \left\{ \sum_{k=1}^{\infty} d_k(x, z, t) \zeta^{-k} \right\} e^{i\zeta z} \end{aligned}$$

for (x, z, t) interior to the light cone, where ϕ_r is a solution of the eikonal equation

$$\rho |\partial_t \phi_r|^2 = \lambda |\nabla \phi_r|^2$$

associated to a reflected ray family, and b_k, d_k are slowly varying amplitudes which solve certain transport equations.

Actually we shall simplify the calculations immensely by computing a similar expansion for the perturbation δv in the pressure field

$$v = \lambda \nabla \cdot u$$

which satisfies the scalar wave equation

$$\frac{1}{\lambda} \partial_t^2 v = \nabla \cdot \frac{1}{\rho} \nabla v$$

$$v = \delta(x, t) \quad \text{on } z = 0$$

$$v \equiv 0 \quad t \ll 0$$

and its perturbational version

$$\left\{ \frac{1}{\lambda} \partial_t^2 - \nabla \cdot \frac{1}{\rho} \nabla \right\} \delta v = - \left\{ \frac{\delta \lambda}{\lambda^2} \partial_t^2 - \nabla \cdot \frac{\delta \rho}{\rho^2} \nabla \right\} v$$

$$\delta v \equiv 0 \quad z = 0 \quad \text{or} \quad t \ll 0$$

From the expansion for δv we can pass immediately to an expansion for δu via the relations

$$\begin{aligned}
\rho \partial_t^2 \mathbf{u} &= \nabla v \\
\rho \partial_t^2 \delta \mathbf{u} &= \nabla \delta v - \dot{\epsilon} \rho \partial_t^2 \mathbf{u} \\
&= \nabla \delta v - \frac{\delta \rho}{\rho} \nabla v
\end{aligned}$$

To simplify the calculations still further we will assume that $\rho \equiv 1$ and $\delta \rho = 0$, and adopt as in section 4

$$\frac{\delta \lambda}{\lambda} = \chi(z) e^{i\zeta z}$$

where χ is a smooth envelope function, which is assumed to vanish for small z .

An essential restriction in this derivation is that the ray family emanating from the space-time origin develops no caustics. Under this condition, Hadamard's construction (Courant and Hilbert, (1962, vol. II, pp. 740 ff.) and the method of images yields the progressing wave expansion

$$v = a_{-1} \delta'(t - \psi_i) + a_0 \delta(t - \psi_i) + a_1 H(t - \psi_i)$$

where the a 's are certain explicit functionals of λ .

Define $w = \int^t ds \int^s d\tau v$ to be the second time primitive of v , likewise $\delta w = \int^t ds \int^s d\tau \delta v$.

Then

$$(\partial_t^2 - \lambda \nabla) \delta w = \frac{\delta \lambda}{\lambda} v \quad (\text{B.1})$$

$$\delta w = 0 \quad \text{for } z = 0 \quad \text{or } t \ll 0$$

We adopt for δw the ansatz

$$\delta w = c_0 \delta(t - \psi_i) e^{i\zeta \phi_r} + c_1 H(t - \psi_i) e^{i\zeta \phi_r} + e_1 H(t - \psi_i) e^{i\zeta z}$$

where c_0, c_1, e_1 are asymptotic series in ζ ,

$$\begin{aligned}
c_0 &\approx \sum_{k=1}^{\infty} c_0^k (i\zeta)^{-k} \\
c_1 &\approx \sum_{k=0}^{\infty} c_1^k (i\zeta)^{-k} \\
e_1 &\approx \sum_{k=2}^{\infty} e_1^k (i\zeta)^{-k}
\end{aligned} \quad (\text{B.2})$$

and c_0 is independent of t . Substitution of the *ansatz* in the differential equation (B.1) results in the condition

$$\begin{aligned}
& \{a_{-1}\delta'(t-\psi_i) + a_0\delta(t-\psi_i) + a_1H(t-\psi_i)\}\chi e^{i\zeta z} \\
&= \{[2(\partial_t + \lambda\nabla\psi_i\cdot\nabla)c_1 + \lambda\nabla^2\psi_i c_1 + 2i\zeta\lambda\partial_z\psi_i c_1]\delta(t-\psi_i) + [(\partial_t^2 - \lambda\nabla^2)c_1 - 2i\zeta\lambda\partial_z c_1 \\
&\quad + \zeta^2\lambda c_1]H(t-\psi_i)\}e^{i\zeta z} \\
&\quad + \{[2\lambda\nabla\psi_i\cdot\nabla c_0 + \lambda\nabla^2\psi_i c_0 + 2i\zeta(\partial_t\phi_r + \lambda\nabla\psi_i\cdot\nabla\phi_r)c_0]\delta'(t-\psi_i) \\
&\quad + [-\lambda\nabla^2 c_0 + 2(\partial_t + \lambda\nabla\psi_i\cdot\nabla)c_1 + \lambda\nabla^2\psi_i c_1 \\
&\quad + i\zeta((\partial_t^2\phi_r - \lambda\nabla^2\phi_r)c_0 + 2\lambda\nabla\phi_r\cdot\nabla c_0 \\
&\quad + 2(\partial_t\phi_r + \lambda\nabla\psi_i\cdot\nabla\phi_r)c_1) \\
&\quad - \zeta^2((\partial_t\phi_r)^2 - \lambda|\nabla\phi_r|^2)c_0]\delta(t-\psi_i) \\
&\quad + [(\partial_t^2 - \lambda\nabla^2)c_1 + i\zeta((\partial_t^2\phi_r - \lambda\nabla^2\phi_r)c_1 \\
&\quad + 2(\partial_t\phi_r\partial_t - \lambda\nabla\phi_r\cdot\nabla)c_1) - \zeta^2((\partial_t\phi_r)^2 - \lambda|\nabla\phi_r|^2)c_1]H(t-\psi_i)\}e^{i\zeta\phi_r},
\end{aligned} \tag{B.3}$$

where we have used the eikonal equation for ψ_i to eliminate δ'' terms.

The condition that a distribution of the form

$$f_{-1}\delta'(t-\psi_i) + f_0\delta(t-\psi_i) + f_1H(t-\psi_i)$$

vanish is

$$\left. \begin{aligned} f_{-1} &= 0 \\ f_0 - \partial_t f_{-1} &= 0 \end{aligned} \right\} \text{ on } t = \psi_i \tag{B.4}$$

$$f_1 \equiv 0 \quad t > \psi_i$$

The first condition implies that for $t = \psi_i$

$$\chi a_{-1}e^{i\zeta(z-\phi_r)} = 2\lambda\nabla\psi_i\cdot\nabla c_0 + \lambda\nabla^2\psi_i c_0 + 2i\zeta(\partial_t\phi_r + \lambda\nabla\psi_i\cdot\nabla\phi_r)c_0 \quad \text{on } \{t = \psi_i\}$$

We substitute for c_0 its assumed asymptotic expansion (B.2) and obtain

$$\chi a_{-1}e^{i\zeta(z-\phi_r)} = 2(\partial_t\phi_r + \lambda\nabla\psi_i\cdot\nabla\phi_r)c_0^1 + O(\zeta^{-1}) \quad \text{for } t = \psi_i \tag{B.5}$$

from which we conclude that $\phi_r = z$ when $t = \psi_i$, i.e.

$$\phi_r(x, z, \psi_i, (x, z)) \equiv z$$

Differentiating this relation, we obtain

$$\nabla\phi_r + \partial_t\phi_r\nabla\psi_i = e_z \quad \text{for } t = \psi_i$$

where $e_z = (0, 1)^T$. Thus

$$\begin{aligned}\nabla\psi_i \cdot \nabla\phi_r &= \partial_z\psi_i - \partial_t\phi_r |\nabla\psi_i|^2 \\ &= \partial_z\psi_i - \lambda^{-1}\partial_t\phi_r\end{aligned}$$

using the eikonal equation for ψ_i . Thus

$$\partial_t\phi_r + \lambda\nabla\psi_i \cdot \nabla\phi_r = \lambda\partial_z\psi_i > 0$$

This last holds by virtue of the Hamilton-Jacobi relation

$$\begin{aligned}\dot{z} &= -c^2\partial_z(t - \psi_i) \\ &= \lambda\partial_z\psi_i\end{aligned}$$

valid along the rays associated to the phase $t - \psi_i$, and the assumed absence of turning points (i.e. $\dot{z} > 0$).

Consequently we can solve (B.5) for c_0^1 :

$$c_0^1 = (2\lambda\partial_z\psi_i)^{-1}\chi a_{-1} \quad (\text{B.6})$$

The remaining terms in (B.5) constitute a recursion formula for the coefficients of c_0 : for $k \geq 1$, $t = \psi_i$,

$$c_0^{k+1} = -(2\lambda\partial_z\psi_i)^{-1}(2\lambda\nabla\psi_i \cdot \nabla c_0^k + \lambda\nabla^2\psi_i c_0^k) \quad (\text{B.7})$$

Looking forward, we see that the quantity

$$(\partial_t\phi_r)^2 - \lambda|\nabla\phi_r|^2$$

occurs as the coefficient of ζ^2 in both δ - and H -terms. It seems reasonable to require that this vanish identically, i.e. require that ϕ_r satisfy the eikonal equation. This justifies our calling ϕ_r the reflected phase.

In interpreting the second condition in (B.4), we take advantage of hindsight to see that the values of the coefficients of e_1 are determined directly in terms of the coefficient a_1 and the envelope function χ . This will be shown in the last stages of the calculation. Accordingly we regard e_1 as known at this point. Thus the second condition (B.4) implies

$$\begin{aligned}
& \lambda \nabla^2 c_0 + i\zeta B c_0 + 2\zeta^2 \partial_t \phi_r \lambda \partial_z \psi_i c_0 \\
& + T_i c_1 + 2i\zeta \lambda \partial_z \psi_i c_1 = \chi a_0 \\
& - 2(\partial_t + \lambda \nabla \psi_i \cdot \nabla) c_1 - \lambda \nabla^2 \psi_i c_1 + i\zeta \lambda \partial_z \psi_i c_1
\end{aligned} \tag{B.8}$$

where

$$B = (\partial_t^2 \psi - \lambda \nabla^2 \phi_r) - 2\partial_t (\partial_t \phi_r + \lambda \nabla \psi_i \cdot \nabla \phi_r) - \partial_t \phi_r \lambda \nabla^2 \psi_i - 2\lambda \partial_z$$

and

$$T_i = 2(\partial_t + \lambda \nabla \psi_i \cdot \nabla) + \nabla^2 \psi_i$$

so that T_i is the transport operator along the incident ray family.

Since

$$c_1 = \sum_{k=0}^{\infty} c_1^k (i\zeta)^{-k}$$

the highest order term $(0)(\zeta)$ in (B.8) gives

$$2\lambda \partial_z \psi_i c_1^0 = 2\partial_t \phi_r \lambda \partial_z \psi_i c_0^1$$

so

$$c_1^0 = (\partial_t \phi_r) c_0^1 \quad \text{on } \{t = \psi_i\} \tag{B.9}$$

The next term $(0(1))$ gives

$$B c_0^1 - 2\lambda \partial_t \phi_r \partial_z \psi_i c_0^2 + T_i c_1^0 + 2\lambda \partial_z \psi_i c_1^1 = \chi a_0$$

which determines c_1^1 on $\{t = \psi_i\}$.

The remaining terms in (B.8) yield the recursion

$$\begin{aligned}
& \lambda \nabla^2 c_0^k + B c_0^{k+1} - 2\lambda \partial_t \phi_r \partial_z \psi_i c_0^{k+2} \\
& + T_i c_1^k + 2\lambda \partial_z \psi_i c_1^{k+1} = -\lambda \partial_z \psi_i c_1^{k+1} \\
& - 2(\partial_t + \lambda \nabla \psi_i \cdot \nabla) c_1^k - \lambda \nabla^2 \psi_i c_1^k
\end{aligned}$$

which determines the remaining coefficients of c_1 on $\{t = \psi_i\}$.

The last condition in (B.4) is

$$\begin{aligned}
& \{(\partial_t^2 - \lambda \nabla^2) c_1 + i\zeta T_r c_1\} e^{i\zeta \phi_r} \\
& + \{(\partial_t^2 - \lambda \nabla^2) c_1 - 2i\zeta \lambda \partial_z c_1 + \zeta^2 \lambda c_1\} e^{i\zeta z} \\
& = a_1 \chi e^{i\zeta z} \quad \text{in } \{t \geq \psi_i\}
\end{aligned} \tag{B.10}$$

Here

$$T_r = 2(\partial_t \phi_r \cdot \partial_t - \lambda \nabla \phi_r \cdot \nabla) + (\partial_t^2 \phi_r - \lambda \nabla^2 \phi_r)$$

is the transport operator along the reflected ray family, i.e. the ray family associated to the reflected phase ϕ_r . The ray equations are

$$\dot{t} = \partial_t \phi_r$$

$$\dot{x} = -\lambda \partial_x \phi_r$$

$$\dot{z} = -\lambda \partial_z \phi_r$$

We shall see that these rays are precisely those given by the equal-angles law of reflection, applied to the incident rays (associated with ψ_i) at a horizontal interface.

We temporarily denote $(x, z) \rightarrow \mathbf{x}$. Then:

along the incident rays

$$\dot{\mathbf{x}} = \lambda \nabla \psi_i$$

along the reflected rays

$$\dot{\mathbf{x}} = -\lambda \nabla \phi_r$$

(The sign for the incident rays results from our definition of $t = \psi_i$ as the incident phase).

The eikonal equations read

$$\lambda |\nabla \psi_i|^2 = 1$$

$$\lambda |\nabla \phi_r|^2 = |\partial_t \phi_r|^2$$

Thus the unit velocity vectors along the incident and reflected rays are given respectively by

$$v_i = \lambda^{\frac{1}{2}} \nabla \psi_i, \quad v_r = -\lambda^{\frac{1}{2}} (\partial_t \phi_r)^{-1} \nabla \phi_r$$

Recall that the first condition (B.4) implied that $\phi_r = z$ on the incident wavefront $\{t = \psi_i\}$,

whence by differentiation

$$\begin{aligned} \nabla \phi_r + \partial_t \phi_r \nabla \psi_i &= e_z \\ &= \lambda^{-\frac{1}{2}} \partial_t \phi_r v_i - \lambda^{-\frac{1}{2}} \partial_t \phi_r v_r \end{aligned}$$

Thus $v_r - v_i$ is parallel to e_z at the incident wavefront $\{t = \psi_i\}$, where the incident and reflected rays intersect. This is exactly the equal-angles law of reflection. Note that the normal to the incident wavefront $\{t = \psi_i\}$ is

$$(1, -\nabla\psi_i)$$

whereas the velocity vector along the reflected ray family is

$$(\partial_t \phi_r, -\lambda \nabla \phi_r)$$

Thus the scalar product is

$$\partial_t \phi_r + \lambda \nabla \psi_i \cdot \nabla \phi_r = \lambda \partial_z \psi_i > 0$$

as noted above. Thus the reflected rays are transverse to the incident wavefront, as a consequence of the absence of turning points.

We return to (B.10). The highest order term in (B.10) ($O(\zeta)$) is

$$T_r c_1^0 = 0 \tag{B.11}$$

which is just the usual transport equation along the reflected rays. The initial conditions on $\{t = \psi_i\}$ are given by (B.9). As noted above, the reflected rays are transverse to $\{t = \psi_i\}$, so that the initial value problem (B.11), (B.9) has a unique solution.

The transport operator T_r is a first-order operator along rays. For a suitable choice of parameter τ , we can write

$$T_r = \frac{d}{d\tau} + m$$

We select τ so that $\tau=0$ defines a point on $\{t = \psi_i\}$. We regard all of the functions in (B.10), restricted to a particular reflected ray, as functions of τ , and write

$$\Phi(\tau) = z(\tau) - \phi_r(t(\tau), x(\tau), z(\tau))$$

Then (B.10) is solved for c_1 by

$$\begin{aligned}
c_1(\tau) &= \exp\left(-\int_0^\tau m\right) c_1(0) \\
&+ (i\zeta)^{-1} \int_0^\tau d\sigma \exp\left(-\int_\sigma^\tau m\right) \{(\lambda \nabla^2 - \partial_t^2) c_1 + [(\lambda \nabla^2 - \partial_t^2) c_1 \\
&+ 2i\zeta \lambda \partial_z c_1 + (i\zeta)^2 \lambda c_1 + a_1 \chi] e^{i\zeta \Phi}\}
\end{aligned} \tag{B.12}$$

Note that

$$\frac{d\Phi}{d\tau} = (\partial_t \phi_r \cdot \partial_t - \lambda \nabla \phi_r \cdot \nabla)(z - \phi_r) = -\lambda \partial_z \phi_r$$

Since no turning points occur along the incident ray, a simple symmetry argument shows that no turning point occurs along the deflected ray either, at least until after it reflects from the free surface. Thus $\Phi' \neq 0$, and we can write

$$e^{i\zeta \Phi} = \frac{1}{i\zeta \Phi'} \frac{d}{d\tau} e^{i\zeta \Phi}$$

Consequently for any smooth f ,

$$\begin{aligned}
&\int_0^\tau d\sigma \exp\left(-\int_\sigma^\tau m\right) f(\sigma) e^{i\zeta \Phi(\sigma)} \\
&= \int_0^\tau d\sigma \exp\left(-\int_\sigma^\tau m\right) \frac{f(\sigma)}{i\zeta \Phi'(\sigma)} \frac{d}{d\sigma} e^{i\zeta \Phi(\sigma)} \\
&= \frac{f(\sigma)}{i\zeta \Phi'(\sigma)} \exp\left\{i\zeta \Phi(\sigma) - \int_\sigma^\tau m\right\} \Big|_{\sigma=0}^{\sigma=\tau} \\
&\quad - \frac{1}{i\zeta} \int_0^\tau d\sigma \exp\left(-\int_\sigma^\tau m\right) T_r \left[\frac{f(\sigma)}{\Phi'(\sigma)} \right] e^{i\zeta \Phi(\sigma)}
\end{aligned}$$

The last summand is an integral of the same form. Applying this rule recursively and using $\Phi(0)=0$ (since $t=0$ defines a point of $\{t=PHI\}$ and $\Phi=z-\phi_r$),

$$\begin{aligned}
&\int_0^\tau d\sigma \exp\left(-\int_\sigma^\tau m\right) f(\sigma) e^{i\zeta \Phi(\sigma)} \\
&\sim \frac{e^{i\zeta \Phi(\tau)}}{i\zeta \Phi'(\tau)} \sum_{k=0}^{\infty} (\hat{T}^k f)(\tau) (i\zeta)^{-k} \\
&\quad - \frac{\exp\left(-\int_0^\tau m\right)}{i\zeta \Phi'(0)} \sum_{k=0}^{\infty} (\hat{T}^k f)(0) (i\zeta)^{-k}
\end{aligned}$$

where

$$\hat{T} = -T \frac{1}{\Phi'}$$

Apply this asymptotic expansion to (B.12) to obtain

$$\begin{aligned} \sum_{k=0}^{\infty} c_1^k(\tau)(i\zeta)^{-k} &= \sum_{k=0}^{\infty} \exp\left(-\int_0^\tau m\right) c_1^k(0)(i\zeta)^{-k} \\ &\quad \sum_{k=1}^{\infty} \left[\int_0^\tau d\sigma \exp\left(-\int_\sigma^\tau m\right) ((\lambda \nabla^2 - \partial_t^2) c_1^{k-1})(\sigma) \right] (i\zeta)^{-k} \\ &\quad + \frac{e^{i\zeta\Phi(\tau)}}{(i\zeta)^2 \Phi'(\tau)} \sum_{k=0}^{\infty} \hat{T}^k ((\lambda \nabla^2 - \partial_t^2) e_1 + 2i\zeta \lambda \partial_z e_1 + (i\zeta)^2 \lambda e_1 + a_1 \chi)(\tau) (i\zeta)^{-k} \\ &\quad - \frac{\exp\left(-\int_0^\tau m\right)}{(i\zeta)^2 \Phi'(0)} \sum_{k=0}^{\infty} \hat{T}^k ((\lambda \nabla^2 - \partial_t^2) e_1 + 2i\zeta \lambda \partial_z e_1 + (i\zeta)^2 \lambda e_1 + a_1 \chi)(0) (i\zeta)^{-k} \end{aligned} \quad (\text{B.13})$$

The coefficient of $e^{i\zeta\Phi}$ must vanish identically. Inserting the asymptotic expansion of e_1 we obtain

$$\begin{aligned} 0 &\approx \sum_{k=0}^{\infty} \hat{T}^k ((\lambda \nabla^2 - \partial_t^2) \sum_{j=2}^{\infty} e_1^j (i\zeta)^{-j} + 2\lambda \sum_{j=2}^{\infty} \partial_z e_1^j (i\zeta)^{-j+1} \\ &\quad + \lambda \sum_{j=2}^{\infty} e_1^j (i\zeta)^{-j+2} + a_1 \chi) (i\zeta)^{-k} \\ &= \sum_{k=0}^{\infty} (\hat{T}^k (a_1 \chi) + \sum_{j=2}^{k+2} \hat{T}^{k-j+2} (\lambda e_1^j) + \sum_{j=2}^{k+1} \hat{T}^{k-j+1} (2\lambda \partial_z e_1^j) \\ &\quad + \sum_{j=2}^k \hat{T}^{k-j} (\lambda \nabla^2 - \partial_t^2) e_1^j) (i\zeta)^{-k} \end{aligned}$$

For $k=0$, we obtain

$$a_1 \chi + \lambda e_1^2 = 0$$

which determines e_1^2 .

For $k=1$,

$$\hat{T} a_1 \chi + \hat{T} (\lambda e_1^2) + \lambda e_1^3 + 2\lambda \partial_z e_1^2 = 0$$

which determines e_1^3 .

For $k \geq 2$,

$$\begin{aligned}
0 = & \hat{T}^k(a_1\chi) + \lambda c_1^{k+2} + \sum_{j=2}^{k+1} \hat{T}^{k-j+2} \lambda c_j \\
& + \sum_{j=2}^{k+1} \hat{T}^{k-j+1} 2\lambda \partial_z c_j + \sum_{j=2}^k \hat{T}^{k-j} (\lambda \nabla^2 - \partial_t^2) c_j
\end{aligned}$$

which determines c_1^{k+2} . Thus c_1 is completely determined in terms of $a_1\chi$ and the operator \hat{T}^k , as promised earlier.

In particular, the values of c_1^k on $\tau=0$ are determined. Since the values of c_1^k on $\{t=\psi_i\}$, i.e. $\tau=0$, have already been determined (B.8, B.9, and following), we can rewrite B.13 as

$$\sum_{k=0}^{\infty} c_1^k(\tau)(i\zeta)^{-k} = \sum_{k=1}^{\infty} \int_0^{\tau} d\sigma \exp\left(-\int_{\sigma}^{\tau} m\right) ((\lambda \nabla^2 - \partial_t^2) c_1^{k-1}(\sigma))(i\zeta)^{-k} + \dots$$

where the ellipses represent terms already determined. This equation is in obvious recursive form; since c_1^0 is already determined (B.11), the rest of the coefficients in c_1 follows.

Finally, we extract the desired series for δu by taking the gradient, as indicated above.

Figure 1

- (a) Velocity profile with low-velocity zones
- (b) Δ -precritical slowness P_Δ as function of depth z , $\Delta = .01$
- (c) Δ -precritical two-way time S_Δ as function of slowness p
- (d) Δ -precritical velocity-slowness product γ as function of depth z . Note that $\gamma \ll 1$ in low-velocity zones, otherwise $\gamma \approx 1$. High-frequency features are caused by numerical interpolation errors.
- (e) Δ -precritical depth Z_Δ as function of slowness p

Figure 2

Construction of cutoff radii for the truncated Radon transform at slowness p .

Figure 3

The aperture of the ray depicted is the angle a . The slowness aperture is the depth-dependent angle subtended with the vertex, as illustrated by angle b .

Figure 4

At the indicated depths, the various rays share the same slowness aperture. The aperture required to achieve a given slowness aperture thus depends on depth.

Figure 5

Arrival times vs. offset for a layer over a half-space:

- a = direct arrival
- b = head wave front
- c = postcritical reflections
- d = (section of) integration domain for Radon integral: $t = \tau + p \cdot x$

As the integration domain has an unbounded intersection with the head-wave region (above line b), the Radon integral is logarithmically divergent, even though (τ, p) is a precritical pair.

Figure 6

$$\begin{aligned} a &= \{ |x| = X_\Delta(p) \} \\ b &= \{ |x| = 2X_\Delta(p) + D_\Delta(p) - |X(z, p)| \} \end{aligned}$$

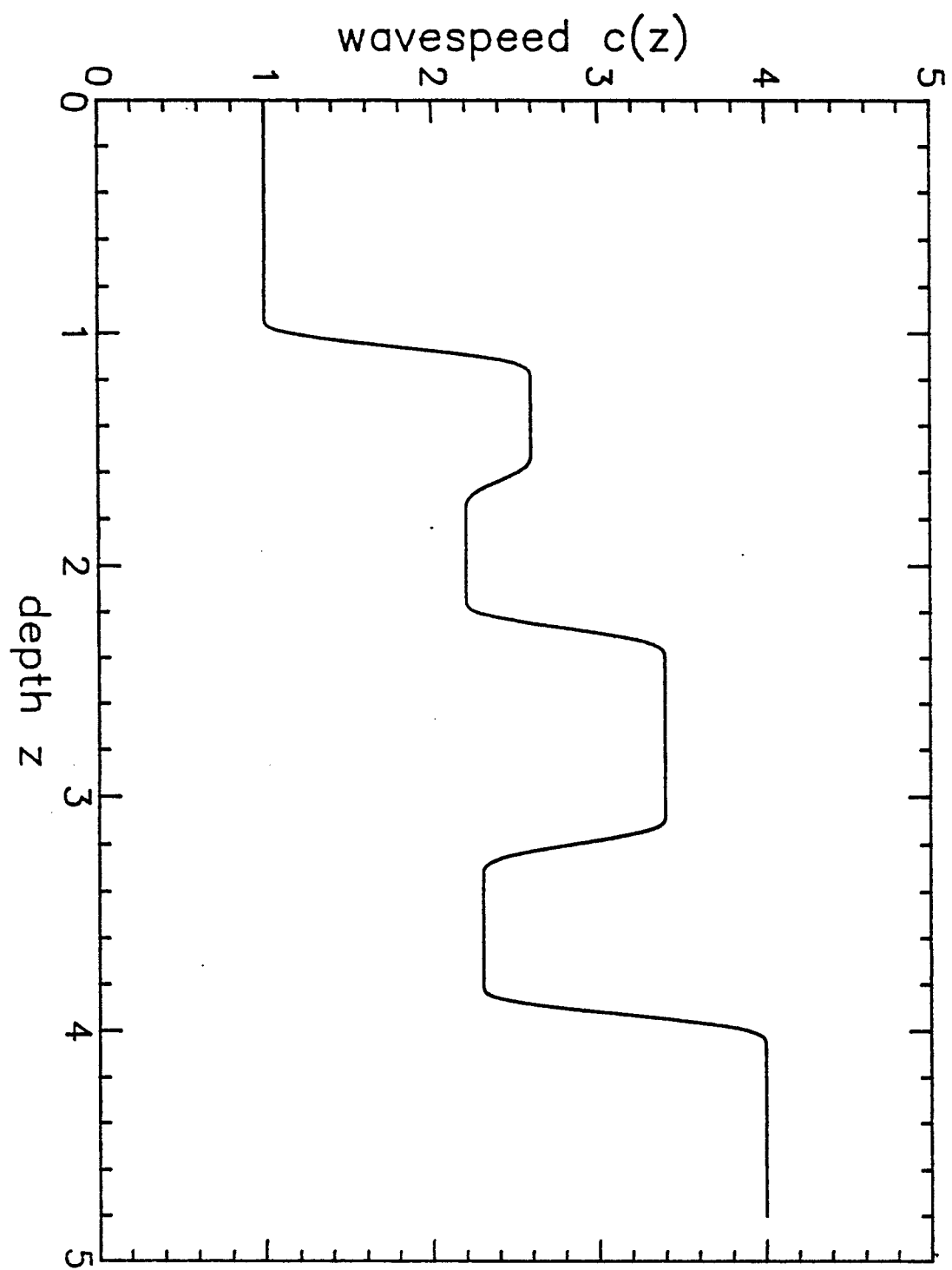
Figure 7

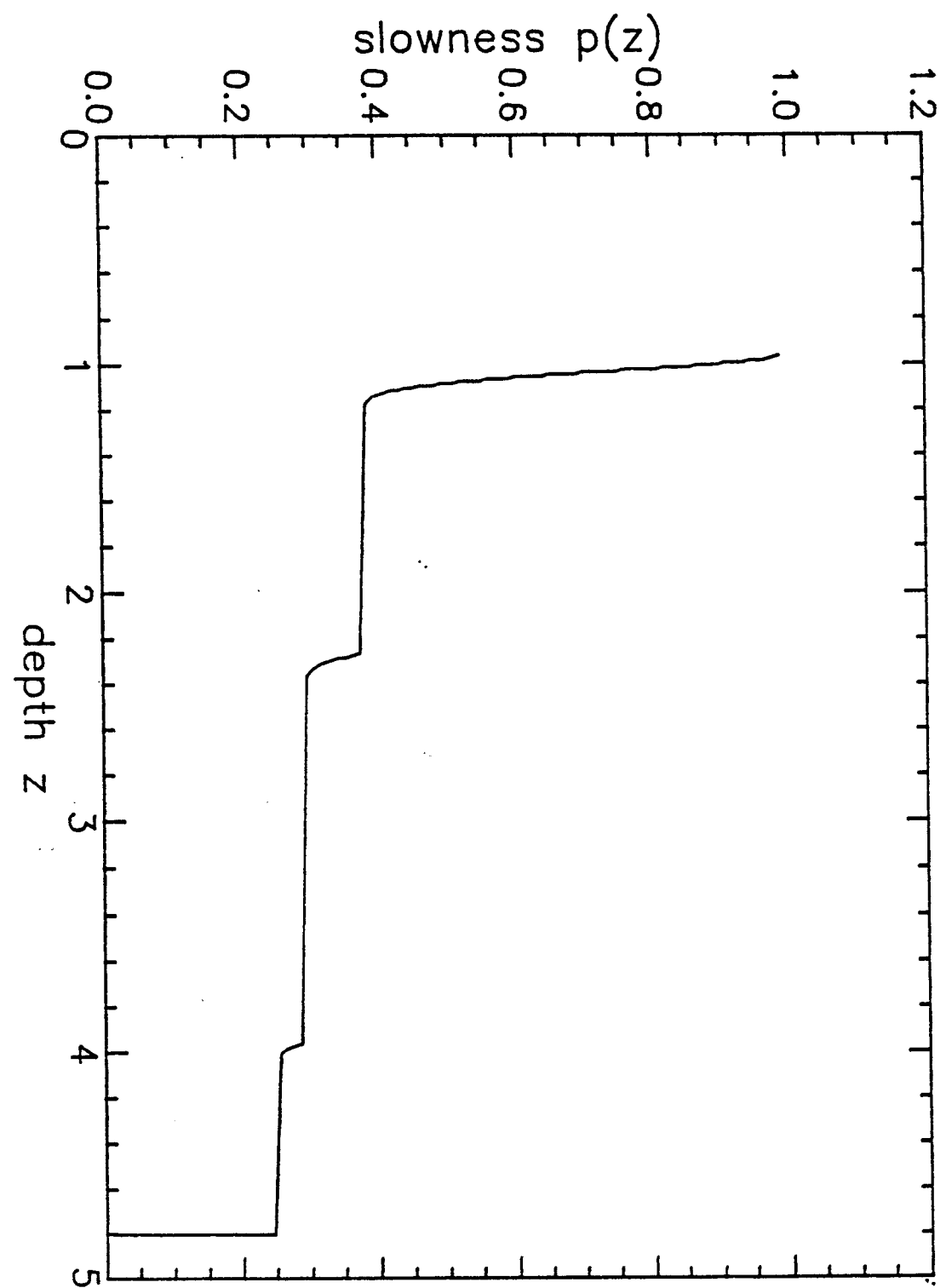
If the reflecting depth z_r satisfies $z_r < Z_\Delta(p)$, then any point $(x, 0, t)$ on the reflected ray with $|x| < 2|X(z_{crit}, p)|$ satisfies $|x| < 2X_\Delta(p)$.

Figure 8

Eigenvalues $\lambda_-(\textit{lambda min})$, $\lambda_+(\textit{lambda max})$, and condition number (condition) of $\mathbb{E}(z)$ plotted as function of γ for various values of $\alpha(z)$:

- (a) $\alpha = 0$
- (b) $\alpha = .1$
- (c) $\alpha = 1$
- (d) $\alpha = 10$
- (e) $\alpha = -1$
- (f) $\alpha = -10$





critical 2-way travel time

