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An Extension of Bounded-Input–Bounded-Output Stability

***Abstract*—A new type of stability definition is formulated that is quite similar to bounded-input–bounded-output stability, but offers certain conceptual advantages.**

The concept of bounded-input–bounded-output (BIBO) stability is one of the most useful in modern control theory. Recently Willems,[1] among others,[2] has attempted to formalize this notion by using the idea of extended spaces. Loosely stated, Willems' definition of BIBO stability is as follows.[3] One begins with two linear vector spaces X_e and Y_e, termed the extended input space and the extended output space, respectively, and an operator A mapping X_e into Y_e, termed the system operator. Then a subspace X of X_e and a subspace Y of Y_e are delineated as the spaces of "bounded inputs" and "bounded outputs," respectively. These subspaces X and Y are actually normed linear spaces[4] with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The operator A (or a system whose behavior is characterized by A) is *BIBO stable* if 1) A maps X into Y, and 2) there is a constant $K \geq 0$ such that $\|Ax\|_Y \leq K\|x\|_X$, for all x in X.

While the above definition is elegant in many ways, it has some drawbacks. The main drawback is that, while the subspaces X and Y are normed, the spaces X_e and Y_e are not, and as a result have much less "structure." Further, condition 2) requires that, if A is a linear operator, then not only is A a mapping from X into Y, but A is also a continuous mapping from X into Y.

In this correspondence a new type of stability (called V stability for lack of a better name) is defined. This new type of stability is not in any way claimed to be "superior" to that of Willems. Indeed, such a comparison between two different types of stability is meaningless. It is felt, however, that V stability should be studied by researchers in the field to determine if it has any useful consequences, or perhaps to decide that it is a useless concept.

Let X and Y be Banach spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and let D be a dense subspace of X. Suppose A is a linear mapping from D into Y. Then A (or a system represented by A) is *V stable* if there exists a linear extension A_e of A, such that the domain of A_e is all of X and the range of A_e is in Y.

What are the implications of a linear system being V stable, and why can V stability be expected to be a useful concept? In BIBO stability, the problem is to define "boundedness" of inputs and outputs, and this is primarily why extended spaces are introduced.[1] It is logical to call an input or output "bounded" if it belongs to some preselected normed linear space, the rationale being that the norm of any element in a normed linear space is always finite. With this in mind, one selects normed linear spaces X and Y as sets of bounded inputs and bounded outputs, and calls an operator A "stable" if A maps X into Y. The problem is to have $A: X \to Y$ as a conclusion, rather than as a premise. Further, if $A: X \to Y$ is the conclusion, then what is the premise? Willems[1] chooses to imbed X and Y in larger spaces X_e and Y_e, and starts out with $A: X_e \to Y_e$ as his premise; we start with $A: D \to Y$ as our premise. Crudely speaking, Willems starts with large spaces and shrinks them, while we start with small spaces and expand them. In either case, the statement $A: X \to Y$ has the same type of significance in terms of BIBO stability.

Some examples are now discussed to bring out the details of this definition. Needless to say, if $D = X$ is some problem, then A is automatically V stable.

Example 1

Suppose there is a constant $K \geq 0$ such that $\|Ax\|_Y \leq K\|x\|_X$, for all x in D. Then A is V stable because the extension A_e can be constructed as follows. Suppose $x \in X$, and let $\{x_n\}$ be a sequence in D converging to x (we know such a sequence exists, since D is dense in X). Then $\{x_n\}$ is a Cauchy sequence in D, and by the boundedness property above, $\{Ax_n\}$ is a Cauchy sequence in Y. Since Y is complete, there is an element y in Y such that $\{Ax_n\}$ converges to y. Define $A_e x = y$; it is easily verified that this definition of $A_e x$ is independent of the sequence $\{x_n\}$. Since A_e is a linear mapping from X into Y, A is V stable.

Example 2

Consider the most commonly cited non-BIBO stable system, namely a simple differentiator. In this case, let X be $C[0, 1]$, the Banach space of all continuous functions over $[0, 1]$, with the supremum norm; let $Y = X$, and let D be $C^1[0, 1]$, the space of all continuously differentiable functions over $[0, 1]$ (but with the same norm as on X). Clearly D is dense in X. Define A by the relation

$$(Af)(t) = \frac{df(s)}{ds}\Big|_{s=t}, \quad \text{for } t \in [0, 1], \quad \text{for } f(\cdot) \in X.$$

A little thought will show that A cannot be linearly extended so as to have all of X as its domain. Hence A is not V stable.

Example 3: (V-stable but unbounded operator)

Let $X = D$ be l^1, the Banach space of all absolutely summable sequences, and let Y be C^0, the Banach space of all sequences converging to zero, with the supremum norm. Define a linear mapping A as follows. Given a sequence $\{p_n\}$ in l^1, the sequence $\{q_n\} = A(\{p_n\})$ is defined by $q_n = np_n$. It is now shown that A maps all of l^1 into C^0. Suppose $\{p_n\} \in l^1$. Then, since $\{p_n\}$ is absolutely summable, $p_n \to 0$ faster than $1/n$ as $n \to \infty$, i.e., $\lim_{n\to\infty} np_n = 0$. To see this, assume the contrary, namely that np_n does not approach 0 as $n \to \infty$. Then, for some $\epsilon > 0$, we can find a sequence of integers $\{n_i\}$ such that

$$n_i |p_{n_i}| \geq \epsilon, \quad i = 1, \cdots, \infty.$$

But this implies

$$\sum_{n=1}^{\infty} |p_n| \geq \sum_{i=1}^{\infty} |p_{n_i}| \geq \epsilon \sum_{i=1}^{\infty} \frac{1}{n_i} = \infty$$

which is a contradiction, since $\{p_n\}$ was assumed to belong to l^1. Since $q_n = np_n$, we have $\lim_{n\to\infty} q_n = 0$, or $\{q_n\} \in C^0$. Hence A maps all of l^1 into C^0 and is V stable. However, it is obvious that A is unbounded in the sense that there is no constant $K \geq 0$ such that $\|Ap\|_{C^0} \leq K\|p\|_{l^1}$, for all $p \in l^1$. This example, together with Example 1, shows that V stability is a weaker property than continuity.

In view of all these examples, it is very clear that every linear operator that is BIBO stable according to Willems is also V stable but the converse is not always true. Hence V stability is a weaker property than BIBO stability.

The weak points of V stability are 1) at present, V stability consists only of a definition and no results, and 2) V stability is only defined for linear operators. This second point merits further comments. In the definition of V stability for a linear operator A, its extension A_e was also required to be linear. Now suppose A is no longer assumed to be linear. Then what

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[3] While the basic ideas of the definition are due to Willems,[1] much of the terminology is due to the present author.

[4] For functional analytic results used, see N. Dunford and J. T. Schwartz, *Linear Operators*. New York: Interscience, 1959, pt. I.

restrictions should be placed on the extensions of A? Obviously, some restrictions are necessary. Otherwise, given $A:D \to Y$, nothing prevents one from arbitrarily assigning values in Y corresponding to points in $X - D$, and proclaiming this to be an "extension" of A, thereby making a farce out of the definition. It is hoped that future researchers can overcome all these difficulties. It is difficult to tell just yet whether V stability is a useful concept, but it does appear to deserve further investigation.

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Stability Criterion for a Feedback System with Backlash—An Extension of Frequency-Domain Condition by the Multiplier Method

Abstract—**A stability criterion for a feedback system with backlash is obtained by applying the passivity theorem of Zames in the derivative domain. The result is a generalization of the well-known criterion of the Popov type.**

I. Introduction

The application of the Lyapunov theory and/or Popov theory to the stability of a feedback system with backlash was found in [1]–[4].

Recently it was shown that the stability problem of a feedback system with an instantaneous nonlinearity is reduced to the problem of whether or not a positive real function of a specified class (called multiplier) exists [5], [6].

In this correspondence, we shall apply the passivity theorem of Zames to a feedback system with backlash in the derivative domain and derive a sharper stability criterion than those in [1]–[4].

II. System Description

Fig. 1. shows the feedback system under consideration, where A is a nonlinear amplifier, L a linear subsystem, and B a backlash element. It is assumed that the following assumptions hold.

1) Nonlinear characteristic $\varphi_0(\sigma)$ of A is a single-valued continuous piecewise-differentiable function, for which $\varphi_0(0) = 0$ and there is a constant $k > 0$ such that the inequality $0 \leq (\sigma_1 - \sigma_2)(\varphi_0(\sigma_1) - \varphi_0(\sigma_2)) \leq k(\sigma_1 - \sigma_2)^2$ is valid for all σ_1 and σ_2. Moreover, it is assumed that there is a constant k_∞ such that

$$\lim_{|\sigma| \to \infty} \varphi_0(\sigma)/\sigma = k_\infty, \quad 0 \leq k_\infty \leq k. \tag{1}$$

2) Linear subsystem L is time invariant, nonanticipative, and strictly stable. The input–output relation is given by

$$\theta(t) = w(t) + \int_0^t h(t - \tau) u(\tau)\, d\tau, \quad t \geq 0 \tag{2}$$

where $h(t)$, which is called the impulse response of L, is piecewise continuous and a) $h(t) \in L_1(0, \infty) \cap L_2(0, \infty)$; b) $\int_t^\infty h(\tau)\, d\tau \in L_1(0, \infty) \cap L_2(0, \infty)$; c) $\sum_{i=1}^\infty |h_i| + \int_0^\infty |\dot{h}(t)|\, dt < \infty$, where h_i denotes the jump of $h(t)$ at the point of discontinuity $t = t_i$. The term $w(t)$, which is called zero-input response of L, is a differentiable function satisfying d) $w(t) \in L_2(0, \infty)$, $\dot{w}(t) \in L_2(0, \infty)$; and e) $w(t) \to 0$, $t \to \infty$.

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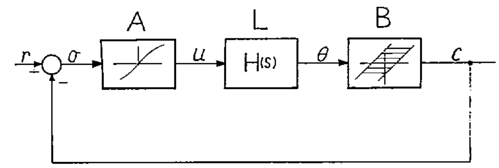

Fig. 1. Feedback system with backlash.

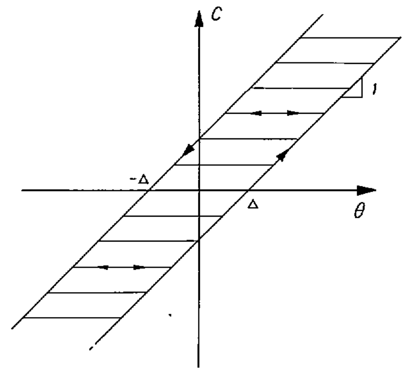

Fig. 2. Input–output characteristic of backlash.

3) It is assumed that the backlash element is inertialess and its input–output relation $c(t) = (B\theta)(t)$ is given as follows (see Fig. 2) [4]:

$$\dot{c}(t) = \begin{cases} \dot{\theta}(t), & \text{if } \begin{cases} \theta(t) - c(t) = \Delta, & \dot{\theta}(t) \geq 0 \\ \theta(t) - c(t) = -\Delta, & \dot{\theta}(t) \leq 0 \end{cases} \\ 0, & \text{if } \begin{cases} \theta(t) - c(t) = \Delta, & \dot{\theta}(t) < 0 \\ \theta(t) - c(t) = -\Delta, & \dot{\theta}(t) > 0 \\ |\theta(t) - c(t)| < \Delta. \end{cases} \end{cases} \tag{3}$$

4) An input belonging to the class R is given by $r(t) = r_0 + r_1(t)$, $t \geq 0$, where r_0 is a constant and $r_1(t)$ is a continuous function satisfying a) $r_1(t) \in L_2(0, \infty)$ and b) $r_1(t) \to 0$ as $t \to \infty$.

Note that the behavior of the feedback system of Fig. 1 is characterized by[1]

$$\begin{aligned} \theta(t) &= w(t) + (h * u)(t) \\ u(t) &= \varphi_0(\sigma(t)), \quad \sigma(t) = r(t) - c(t) \\ c(t) &= (B\theta)(t). \end{aligned} \tag{4}$$

III. Stability Criterion

Theorem

Let $H(i\omega)$ be the Fourier transform of $h(t)$. Suppose that

$$\begin{gathered} \mathrm{Re}\left[\frac{1 + qi\omega - Z_1(i\omega)}{i\omega}\left(H(i\omega) + \frac{1}{k}\right)\right] \geq \delta > 0, \quad \forall \omega \\ H(0) + \frac{1}{k} > 0 \end{gathered} \tag{5}$$

holds for some $q \geq 0$, $\delta > 0$, and some complex function $Z_1(i\omega)$, whose inverse Fourier transform $z_1(t)$ satisfies a) $z_1(t) \geq 0$, $\dot{z}_1(t) \leq 0$, $t \geq 0$; b) $\int_0^\infty z_1(t)\, dt < 1$; c) $\int_t^\infty z_1(\tau)\, d\tau \in L_1(0, \infty) \cap L_2(0, \infty)$. Then $\sigma(t)$ and $\varphi_0(\sigma(t))$ are bounded on $[0, \infty)$ and there are final values $\lim_{t\to\infty} \sigma(t) = \sigma_\infty$, $\lim_{t\to\infty} \varphi_0(\sigma(t)) = \varphi_0(\sigma_\infty)$, for all initial values of L, any nonlinear amplifier satisfying 1), and any input belonging to the class R.

Remark: It can be easily shown that $Z(s) = (1 + qs - Z_1(s))/s$ is a positive real function, or more exactly, $-\pi/2 \leq \arg(Z(i\omega)) \leq 0$ is valid for

[1] $(h * u)(t) = \int_0^t h(t - \tau) u(\tau)\, d\tau.$