

# ALMOST PERIODIC SZEGŐ COCYCLES WITH UNIFORMLY POSITIVE LYAPUNOV EXPONENTS

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ABSTRACT. We exhibit examples of almost periodic Verblunsky coefficients for which Herman's subharmonicity argument applies and yields that the associated Lyapunov exponents are uniformly bounded away from zero. As an immediate consequence of this result, we obtain examples of almost periodic Verblunsky coefficients for which the associated probability measure on the unit circle is pure point.

## 1. INTRODUCTION

The study of probability measures on the real line or the unit circle via the associated orthogonal polynomials and the recursions these polynomials obey is a classical topic. On the other hand, it has only recently been fully realized that the two cases, the real line and the unit circle, are intimately related. While connections of this flavor have been known for some time, a systematic investigation has been carried out only in the past eight years. This is particularly pronounced in part two of Barry Simon's recent monograph [6, 7]. Another recent development in the area of orthogonal polynomials, which is also an important theme in [6, 7], is that spectral theory methods may yield useful insights into orthogonal polynomial questions.

In particular, it is a natural question how the coefficients that appear in the recursions for the orthogonal polynomials are related to the measure under consideration. These coefficients are called Jacobi coefficients in the real line case and Verblunsky coefficients in the unit circle case. Certain classes of coefficients are well understood. For example, there is an exhaustive study of the case of periodic coefficients; see [7, Chapter 11] for the unit circle case. It is known that for periodic Jacobi or Verblunsky coefficients, the associated measure consists of an absolutely continuous piece and a discrete point piece. One often says that "the essential spectrum is purely absolutely continuous."

Interesting classes of coefficients that generalize the periodic case are given by decaying perturbations of periodic sequences and by almost periodic sequences. We will focus here on the almost periodic case; decaying perturbations of periodic coefficients are studied in depth in [3].

In the real line case, there is a huge literature on almost periodic recursion coefficients; the interested reader may use [1] and references therein as a starting point. On the other hand, much less is known in the unit circle case. A central

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role in the Jacobi case is played by Herman's subharmonicity estimate from [5]. It may be used to exhibit examples of almost periodic Jacobi coefficients for which the associated measure is pure point. No analogues of this were known in the unit circle case and our purpose here is to present a class of almost periodic Verblunsky coefficients whose associated measure is pure point. Given the existing theory, it suffices to find examples whose associated Szegő cocycles have uniformly positive Lyapunov exponents.

Consequently, we will recall the definition of Szegő cocycles and Lyapunov exponents in Section 2, exhibit almost periodic examples with uniformly positive Lyapunov exponents in Section 3, and explain in Section 4 how the existing theory then yields pure point measures.

## 2. SZEGŐ COCYCLES AND LYAPUNOV EXPONENTS

Suppose that  $(\Omega, \mu)$  is a probability measure space and  $T : \Omega \rightarrow \Omega$  is ergodic with respect to  $\mu$ . A measurable map  $A : \Omega \rightarrow \text{GL}(2, \mathbb{C})$  gives rise to a so-called cocycle, which is a map from  $\Omega \times \mathbb{C}^2$  to itself given by  $(\omega, v) \mapsto (T\omega, A(\omega)v)$ . This map is usually denoted by the same symbol. When studying the iterates of the cocycle, the following matrices describe the dynamics of the second component:

$$A_n(\omega) = A(T^{n-1}\omega) \cdots A(\omega).$$

Assuming  $\log \|A\| \in L^1(\mu)$  and

$$\inf_{n \geq 1} \frac{1}{n} \int_{\Omega} \log \|A_n(\omega)\| d\mu(\omega) > -\infty,$$

then, by Kingman's subadditive ergodic theorem, the following limit exists,

$$(1) \quad \gamma = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \log \|A_n(\omega)\| d\mu(\omega),$$

and we have

$$\gamma = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A_n(\omega)\|$$

for  $\mu$ -almost every  $\omega \in \Omega$ . The number  $\gamma$  is called the Lyapunov exponent of  $A$ .

We will be interested in the particular case of Szegő cocycles, which arise as follows. Denote the open unit disk in  $\mathbb{C}$  by  $\mathbb{D}$ . For a measurable function  $f : \Omega \rightarrow \mathbb{D}$  with

$$\int_{\Omega} \log(1 - |f(\omega)|) d\mu(\omega) > -\infty$$

and  $z \in \partial\mathbb{D}$ , the cocycle  $A^z : \Omega \rightarrow \text{GL}(2, \mathbb{C})$  is given by

$$(2) \quad A^z(\omega) = (1 - |f(\omega)|^2)^{-1/2} \begin{pmatrix} z & -\overline{f(\omega)} \\ -f(\omega)z & 1 \end{pmatrix}.$$

The Lyapunov exponent of  $A^z$  will be denoted by  $\gamma(z)$ . The complex numbers  $\alpha_n(\omega) = f(T^n\omega)$ ,  $n \geq 0$ , appearing in  $A^z(T^n\omega)$  are called Verblunsky coefficients.

Szegő cocycles play a central role in the analysis of orthogonal polynomials on the unit circle with ergodic Verblunsky coefficients; compare [6, 7] (see in particular [7, Section 10.5] for more information on Lyapunov exponents of Szegő cocycles). As pointed out in the introduction, one of the major themes of [6, 7] is to work out in detail the close analogy between the spectral analysis of Jacobi matrices, or more specifically discrete one-dimensional Schrödinger operators, and that of CMV

matrices. Indeed, a large portion of the second part, [7], is devoted to carrying over results and methods from the Schrödinger and Jacobi setting to the OPUC setting.

Sometimes the transition is straightforward and sometimes it is not. As discussed in the remarks and historical notes at the end of [7, Section 10.16], one of the results that Simon did not manage to carry over is Herman's result on uniformly positive Lyapunov exponents for a certain class of almost periodic Schrödinger cocycles [5] (see also [2, Section 10.2]), which is proved by a beautiful subharmonicity argument.

In the next section we present one-parameter families of almost periodic Szegő cocycles for which we prove uniformly positive Lyapunov exponents using Herman's argument for an explicit region of parameter values.

### 3. EXAMPLES WITH UNIFORMLY POSITIVE LYAPUNOV EXPONENTS

Consider the 1-torus  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  equipped with Lebesgue measure and  $\mathbb{Z}_2$  equipped with the probability measure that assigns the weight  $\frac{1}{2}$  to each of 0 and 1. Let  $\Omega = \mathbb{T} \times \mathbb{Z}_2$  be the product space and  $\mu$  the product measure. Fix some irrational  $\alpha \in \mathbb{T}$ . The transformation  $T : \Omega \rightarrow \Omega$  is given by  $T(\theta, j) = (\theta + \alpha, j + 1)$ . It is readily verified that  $T$  is ergodic with respect to  $\mu$ .

For  $\varepsilon \in (0, 1)$  and  $k \in \mathbb{Z} \setminus \{0\}$ , we define  $f : \Omega \rightarrow \mathbb{D}$  by

$$(3) \quad f(\theta, j) = \begin{cases} (1 - \varepsilon^2)^{1/2} e^{2\pi i k \theta} & j = 0, \\ (1 - \varepsilon^2)^{1/2} e^{-2\pi i k \theta} & j = 1. \end{cases}$$

Clearly,  $f$  is a measurable function from  $\Omega$  to  $\mathbb{D}$  and satisfies  $\log[(1 - |f|^2)^{-1/2}] \in L^1(\mu)$ . Thus, the Lyapunov exponents  $\gamma(z)$ ,  $z \in \partial\mathbb{D}$  exist and we wish to bound them from below.

**Theorem 1.** *For  $(\Omega, \mu, T)$  as above and  $f$  given by (3), we have the estimate*

$$\inf_{z \in \partial\mathbb{D}} \gamma(z) \geq \log \frac{(1 - \varepsilon^2)^{\frac{1}{2}}}{\varepsilon}.$$

*In particular, if  $\varepsilon \in (0, \frac{1}{\sqrt{2}})$ , the Lyapunov exponent  $\gamma(\cdot)$  is uniformly positive on  $\partial\mathbb{D}$ .*

*Proof.* We consider the case  $k > 0$ ; the case  $k < 0$  is completely analogous. Fix any  $z \in \partial\mathbb{D}$ . By the definition (2) of  $A^z$  and the definition (3) of  $f$ , we have

$$A^z(\theta, j) = \begin{cases} \varepsilon^{-1} \begin{pmatrix} z & -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k \theta} \\ -(1 - \varepsilon^2)^{1/2} e^{2\pi i k \theta} z & 1 \end{pmatrix} & j = 0, \\ \varepsilon^{-1} \begin{pmatrix} z & -(1 - \varepsilon^2)^{1/2} e^{2\pi i k \theta} \\ -(1 - \varepsilon^2)^{1/2} e^{-2\pi i k \theta} z & 1 \end{pmatrix} & j = 1. \end{cases}$$

Let us conjugate these matrices as follows (cf. [4, Equation (4.10)]). Define

$$C^z(\theta, j) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & j = 0, \\ \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} & j = 1. \end{cases}$$

For  $j = 0$ , we have

$$\begin{aligned}
\varepsilon C^z(\theta, j) A^z(\theta, j) C^z(\theta, j-1)^{-1} &= \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k \theta} \\ -(1-\varepsilon^2)^{1/2} e^{2\pi i k \theta} z & 1 \end{pmatrix} \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix} \\
&= \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k \theta} z & 1 \\ z & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k \theta} \end{pmatrix} \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix} \\
&= \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k \theta} z^{1/2} & z^{1/2} \\ z^{1/2} & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k \theta} z^{1/2} \end{pmatrix},
\end{aligned}$$

while for  $j = 1$ , we have

$$\begin{aligned}
\varepsilon C^z(\theta, j) A^z(\theta, j) C^z(\theta, j-1)^{-1} &= \\
&= \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} z & -(1-\varepsilon^2)^{1/2} e^{2\pi i k \theta} \\ -(1-\varepsilon^2)^{1/2} e^{-2\pi i k \theta} z & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k \theta} & z \\ 1 & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k \theta} z \end{pmatrix} \\
&= \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k \theta} z^{1/2} & z^{3/2} \\ z^{-1/2} & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k \theta} z^{1/2} \end{pmatrix},
\end{aligned}$$

Thus,

(4)

$$C^z(\theta, j) A^z(\theta, j) C^z(\theta, j-1)^{-1} = \frac{z^{1/2}}{\varepsilon} \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k \theta} & z^j \\ z^{-j} & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k \theta} \end{pmatrix}.$$

We have  $A_n^z(\theta, j) = A^z(\theta + (n-1)\alpha, j+n-1) \cdots A^z(\theta, j)$ , which, by (4), is equal to

$$C^z(\theta, j+n-1)^{-1} \prod_{m=n-1}^0 \left( \frac{z^{1/2}}{\varepsilon} \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k(\theta+m\alpha)} & z^{(j+m \bmod 2)} \\ z^{-(j+m \bmod 2)} & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k(\theta+m\alpha)} \end{pmatrix} \right) C^z(\theta, j-1).$$

Since  $C^z(\theta, j)$  is always unitary and  $w = e^{2\pi i \theta}$  and  $z^{1/2}$  both have modulus one, we find that

$$\begin{aligned}
\|A_n^z(\theta, j)\| &= \varepsilon^{-n} \left\| \prod_{m=n-1}^0 \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k(\theta+m\alpha)} & z^{(j+m \bmod 2)} \\ z^{-(j+m \bmod 2)} & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k(\theta+m\alpha)} \end{pmatrix} \right\| \\
&= \varepsilon^{-n} \left\| \prod_{m=n-1}^0 \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k(2\theta+m\alpha)} & z^{(j+m \bmod 2)} e^{2\pi i k \theta} \\ z^{-(j+m \bmod 2)} e^{2\pi i k \theta} & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k m \alpha} \end{pmatrix} \right\| \\
&= \varepsilon^{-n} \left\| \prod_{m=n-1}^0 \begin{pmatrix} -(1-\varepsilon^2)^{1/2} e^{2\pi i k m \alpha} w^{2k} & z^{(j+m \bmod 2)} w^k \\ z^{-(j+m \bmod 2)} w^k & -(1-\varepsilon^2)^{1/2} e^{-2\pi i k m \alpha} \end{pmatrix} \right\|.
\end{aligned}$$

The  $w$ -dependence of the matrix in the last expression is analytic and hence the log of its norm is subharmonic. Therefore,

$$\begin{aligned}
\int_{\Omega} \log \|A_n^z(\theta, j)\| d\mu(\theta, j) &= \frac{1}{2} \int_{\mathbb{T}} \log \|A_n^z(\theta, 0)\| d\theta + \frac{1}{2} \int_{\mathbb{T}} \log \|A_n^z(\theta, 1)\| d\theta \\
&\geq n \log \frac{(1-\varepsilon^2)^{\frac{1}{2}}}{\varepsilon}.
\end{aligned}$$

Since

$$\gamma(z) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \log \|A_n^z(\theta, j)\| d\mu(\theta, j),$$

the result follows.  $\square$

In Theorem 1 we considered functions given by simple exponentials. Since we obtained explicit terms which bound the Lyapunov exponents uniformly from below, it is possible to add small perturbations to the function and retain uniform positivity of the Lyapunov exponents. For example, given an integer  $k \geq 1$  and  $\lambda, a_{-k}, \dots, a_{k-1} \in \mathbb{C}$ , we set

$$(5) \quad f_{\lambda}(\theta, j) = \begin{cases} (1 - \varepsilon^2)^{1/2} \left( e^{2\pi i k \theta} + \lambda \sum_{l=-k}^{k-1} a_l e^{2\pi i l \theta} \right) & j = 0, \\ (1 - \varepsilon^2)^{1/2} \left( e^{-2\pi i k \theta} + \lambda \sum_{l=-k}^{k-1} a_l e^{-2\pi i l \theta} \right) & j = 1. \end{cases}$$

Since we need  $f_{\lambda}$  to take values in  $\mathbb{D}$ , we have to impose an upper bound on the admissible values of  $\lambda$ . Clearly, once  $\varepsilon \in (0, 1)$ ,  $k \geq 1$  and the numbers  $a_l \in \mathbb{C}$  are chosen, there is  $\lambda_0 > 0$  such that for  $\lambda$  with modulus bounded by  $\lambda_0$ , the range of  $f_{\lambda}$  is contained in  $\mathbb{D}$ .

**Theorem 2.** *Let  $(\Omega, \mu, T)$  be as above. For every  $\varepsilon \in (0, \frac{1}{\sqrt{2}})$ ,  $k \in \mathbb{Z}_+$ , and  $\{a_l\}_{l=-k}^{k-1} \subset \mathbb{C}$ , there is  $\lambda_1 > 0$  such that for every  $\lambda$  with  $|\lambda| < \lambda_1$ , there is  $\gamma_- > 0$  for which the Lyapunov exponent  $\gamma(\cdot)$  associated with  $f_{\lambda}$  given by (5) satisfies*

$$\inf_{z \in \partial \mathbb{D}} \gamma(z) \geq \gamma_-.$$

*Proof.* The smallness condition  $|\lambda| < \lambda_1$  needs to address two issues. First, the range of the function  $f_{\lambda}$  must be contained in  $\mathbb{D}$ , so we need  $\lambda_1 \leq \lambda_0$ . Second, the explicit strictly positive uniform lower bound obtained in the proof of Theorem 1 for the case  $\lambda = 0$  changes continuously once the perturbation is turned on. Thus, it remains strictly positive for  $|\lambda|$  small enough. Notice that the degree requirements for the perturbation in (5) are such that the subharmonicity argument from the proof of Theorem 1 goes through without any changes.  $\square$

#### 4. DISCUSSION

In the previous section we proved a uniform lower bound for the Lyapunov exponents associated with strongly coupled almost periodic sequences of Verblunsky coefficients. A few remarks are in order.

The Verblunsky coefficients take values in the open unit disk and the unit circle has to be regarded as the analogue of infinity in the Schrödinger case. Thus, just as the coupling constant is sent to infinity in the application of Herman's argument in the Schrödinger case, the coupling constant is sent to one in our study. Notice that we need a rather uniform convergence to the unit circle, whereas one may have zeros in the Schrödinger case. In particular, while Herman's argument applies to all non-constant trigonometric polynomials in the Schrödinger case, we only treat small perturbations of simple exponentials.

Another limitation of our proof is that it requires the consideration of the product  $\mathbb{T} \times \mathbb{Z}_2$ . It would be nicer to have genuine quasi-periodic examples (with  $\inf_{|z|=1} \gamma(z) > 0$ ), that is, generated by minimal translations on a finite-dimensional

torus. Our attempts to produce such examples have run into trouble with analyticity issues. It would be of interest to produce quasi-periodic examples or to demonstrate why  $\inf_{|z|=1} \gamma(z) = 0$  for all of them.

As explained by Simon in [7, Theorem 12.6.1], as soon as one knows that  $\gamma(z)$  is positive for (Lebesgue almost) every  $z \in \mathbb{D}$ , one can immediately deduce that for  $\mu$ -almost all elements of  $\Omega$ , Lebesgue almost all Aleksandrov measures associated with the sequence of Verblunsky coefficients in question are pure point. This is applicable to our examples for  $\varepsilon \in (0, \frac{1}{\sqrt{2}})$  and  $|\lambda|$  small enough.

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