REAL MULTIPLICATION THROUGH EXPLICIT CORRESPONDENCES

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Abstract. We describe a method to compute equations for real multiplication on the divisors of genus two curves via algebraic correspondences. We implement our method for various examples drawn from the algebraic models for Hilbert modular surfaces computed by Elkies–Kumar. We also compute a correspondence over the universal family over the Hilbert modular surface of discriminant 5 and use our equations to prove a conjecture of A. Wright on dynamics over the moduli space of Riemann surfaces.

1. Introduction

Abelian varieties, their endomorphisms and their moduli spaces play a central role in modern algebraic geometry and number theory. Their study has important applications in a broad array of fields including cryptography, dynamics, geometry, and mathematical physics. Of particular importance are the abelian varieties with extra endomorphisms (other than those in $\mathbb{Z}$). In dimension one, elliptic curves with complex multiplication have been studied extensively. In this paper, we focus on curves of genus two whose Jacobians have real multiplication by a real quadratic ring $\mathcal{O}$.

For such a curve C, we describe a method for computing the action of real multiplication by $\mathcal{O}$ on the divisors of C. In particular, we determine equations for an algebraic correspondence on C, i.e., a curve Z with two maps $f, g : Z \to C$ such that the induced endomorphism $T = g_* \circ f^*$ of $\mathrm{Jac}(C)$ generates $\mathcal{O}$. We then certify the real multiplication of $\mathcal{O}$ on $\mathrm{Jac}(C)$ by computing the action of T on one-forms. Combined with standard equations for the group law on $\mathrm{Jac}(C)$, our techniques immediately lead to an algebraic description the action of $\mathcal{O}$ on degree zero divisors of C.

This paper completes a research program initiated in [EK] and [KM]. Let $\mathcal{M}_{g,n}$ denote the moduli space of smooth genus g curves with n marked points, and, for each totally real order $\mathcal{O}$, denote by $\mathcal{M}_{g,n}(\mathcal{O})$ the locus of curves whose Jacobians admit real multiplication by $\mathcal{O}$. The paper [EK] describes a method for parametrizing the Humbert surface $\mathcal{M}_2(\mathcal{O}) = \mathcal{M}_{2,0}(\mathcal{O})$ for real quadratic $\mathcal{O}$ as well as its double cover, the Hilbert modular surface $Y(\mathcal{O})$. It also carries out the computation for $\mathcal{O} = \mathcal{O}_K$, the ring of integers of every real quadratic field K of discriminant less than 100, producing equations for the corresponding Hilbert modular surfaces. The paper [KM] describes a method for computing the action of $\mathcal{O}$ on the one-forms of curves in $\mathcal{M}_2(\mathcal{O})$, and uses it in particular to compute algebraic models for Teichmüller curves $\mathcal{M}_2$. Using these techniques one can furnish

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equations defining curves $C \in \mathcal{M}_2(\mathcal{O})$, rigorously prove that $\mathrm{Jac}(C)$ admits real multiplication by $\mathcal{O}$ and rigorously compute the action of $\mathcal{O}$ on the one-forms of C. In this paper we solve the problem of describing the action of $\mathcal{O}$ on $\mathrm{Jac}(C)$ as algebraic morphisms by computing the action on the divisors of C, a problem which is not easily addressed by the methods in [EK] and [KM]. Using equations for correspondences, one can give proof, which is independent of [EK] and [KM], that $\mathrm{Jac}(C)$ admits real multiplication by $\mathcal{O}$ and rigorously compute the action of $\mathcal{O}$ on one-forms. However, to use the methods in this paper, one needs to first find equations for C (e.g. using [EK]).

Example in discriminant 5. To demonstrate our method, consider the genus two curve

$$C : u^2 = t^5 - t^4 + t^3 + t^2 - 2t + 1. \tag{1}$$

Equation 1 was obtained from the equations in [EK]. Using the methods of this paper, we can formulate and prove the following theorem.

Theorem 1. *Let $f : Z \to C$ be the degree two cover of C of Equation 1 defined by*

$$t^2x^2 - x - t + 1 = 0.$$

The curve Z is of genus 6 and admits an additional map $g : Z \to C$ of degree two. The induced endomorphism $T = g_* \circ f^*$ of $\mathrm{Jac}(C)$ is self-adjoint, satisfies $T^2 - T - 1 = 0$ and generates real multiplication by $\mathcal{O}_5$.

In Section 5 we give several other examples of correspondences on particular genus two curves of varying complexity. In Section 6, we describe how to implement our method in families and compute a correspondence for the entire Hilbert modular surface $Y(\mathcal{O}_5)$.

Divisors supported at eigenform zeros. In the universal[1] Jacobian over the space $\mathcal{M}_{g,n}(\mathcal{O})$, there is a natural class of multisection obtained from $\mathcal{O}$-linear combinations of divisors supported at eigenform zeros and marked points. Filip recently showed that such divisors play a pivotal role in the behavior of geodesics in moduli space [F]. As an application of our equations for real multiplication, we prove a theorem about such divisors over $\mathcal{M}_{2,1}(\mathcal{O}_5)$ and establish a conjecture of Wright on dynamics over the moduli space of curves.

In the universal Jacobian over $\mathcal{M}_{2,1}(\mathcal{O}_5)$, let L be the multisection of degree four whose values at the curve C marked at $P \in C$ are divisors of the form

$$(P - Z_1) - T \cdot (Z_2 - Z_1) \in \mathrm{Jac}(C) \tag{2}$$

where T is a self-adjoint endomorphism of $\mathrm{Jac}(C)$ satisfying $T^2 - T - 1 = 0$ and Z_1 and Z_2 are the zeros of a T-eigenform ω on C. The vanishing of L defines a closed subvariety of $\mathcal{M}_{2,1}(\mathcal{O}_5)$:

$$\mathcal{M}_{2,1}(\mathcal{O}_5; L) = \left\{ (C, P) \in \mathcal{M}_{2,1}(\mathcal{O}_5) : \begin{array}{c}\text{some branch of } L \\ \text{vanishes at } (C,P)\end{array} \right\}. \tag{3}$$

One might expect $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ to be a curve in the threefold $\mathcal{M}_{2,1}(\mathcal{O}_5)$ since the relative dimension of the universal Jacobian is two. We use our equations for real multiplication to show that $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ is unexpectedly large.

[1] It would be more accurate to use the term "tautological" rather than "universal" since the various moduli problems we consider are only coarsely representable. We will abuse this terminology and continue to use the term universal throughout this paper.

Theorem 2. *The space $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ is an irreducible surface in $\mathcal{M}_{2,1}$.*

To relate Theorem 2 to dynamics, recall that $\mathcal{M}_{g,n}$ carries a Teichmüller metric and that every vector tangent to $\mathcal{M}_{g,n}$ generates a complex geodesic, i.e., a holomorphic immersion $\mathbb{H} \to \mathcal{M}_{g,n}$ which is a local isometry. McMullen proved that the locus $\mathcal{M}_2(\mathcal{O})$ is the closure of a complex geodesic in the moduli space $\mathcal{M}_2$ of unmarked genus two curves for each real quadratic $\mathcal{O}$ [Mc1]. A corollary of Theorem 2 shows that $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ enjoys the same property.

Theorem 3. *There is a complex geodesic $f : \mathbb{H} \to \mathcal{M}_{2,1}$ with $\overline{f(\mathbb{H})} = \mathcal{M}_{2,1}(\mathcal{O}_5; L)$.*

In other words, there is a dynamically natural way to choose finitely many points on each curve in $\mathcal{M}_2(\mathcal{O}_5)$. This was originally conjectured by Wright, and will be proven by very different means in the forthcoming paper [EMMW].

By Filip's characterization of the behavior of complex geodesics in moduli space [F], every complex geodesic is dense in a subvariety of $\mathcal{M}_{g,n}(\mathcal{O})$ characterized by $\mathcal{O}$-linear relations among divisors supported at eigenform zeros and marked points. Theorem 3 is the first example of such a subvariety where a relation involving a ring strictly larger than $\mathbb{Z}$ appears.

Prior work on equations for real multiplication. Several authors [Wi1, Wi2, Sa1, HS1, HS2, Sa2] have given geometric descriptions of real multiplication based on Humbert's work on Poncelet configurations of conics [Hu, vdG, Ja]. Our method is close to that used in [vW1] to prove that Jacobians of particular genus two curves have complex multiplication. An important distinction is that the method in [vW1] first determines the period lattice exactly, which is typically possible only in the case of Jacobians with complex multiplication. Our method requires only a numerical approximation to the period matrix and can therefore be used with transcendental period lattices like those typically associated to Jacobians with real but not complex multiplication. Another difference is that we systematically use the language of correspondences (which seems implicit in [vW1]) to clarify ideas and make applications to algebraic geometry and arithmetic more accessible. This is essential for our computation of the action of $\mathcal{O}$ on divisors and our applications to dynamics. We also make use of the examples of Jacobians with real multiplication in genus two supplied by [EK] and address challenges to computing correspondences in families.

Outline. In Section 2, we recall some basic facts about Jacobians of curves, their endomorphisms and correspondences. In Section 3, we describe our method for finding the equations of a correspondence associated to a Jacobian endomorphism. In Section 4, we describe how to compute the induced action on one-forms and thereby certify the equations obtained by the method in Section 3. In Section 5, we give several examples of varying complexity of explicit correspondences obtained by our method. In Section 6, we address challenges to implementing our algorithm in families, and describe a correspondence for the entire Hilbert modular surface for discriminant 5. In Section 7, we discuss the applications to dynamics and prove Theorems 2 and 3.

Computer files. Auxiliary files containing computer code to verify the calculations in this paper are available from the arXiv.org e-print archive. To access these, download the source file for the paper. It is a tar archive, which can be extracted to produce not only the LaTeX file for this paper, but also the computer code. The text file README.txt gives a brief guide to the various auxiliary files.

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2. Background

In this section, we recall some general facts about curves, their Jacobians and algebraic correspondences. We will work over the complex numbers. The basic reference for this section is [BL].

Jacobians. Fix a smooth curve C of genus g. The holomorphic one-forms on C form a g-dimensional vector space $\Omega(X)$. Integration gives rise to an embedding $H_1(C,\mathbb{Z}) \to \Omega(C)^*$ and, by the equality of arithmetic and geometric genus for smooth curves, the image of this embedding is a lattice. The quotient

$$\mathrm{Jac}(C) = \Omega(C)^*/H_1(C,\mathbb{Z})$$

is a compact, complex torus called the Jacobian of C. The torus $\mathrm{Jac}(C)$ is principally polarized by the symplectic intersection form on $H_1(C,\mathbb{Z})$.

The Abel-Jacobi map. Let $\mathrm{Pic}^0(C)$ denote the group of degree zero divisors on C up to linear equivalence. Integration gives rise to a multi-valued holomorphic map

$$\widetilde{AJ} : \mathrm{Pic}^0(C) \to \Omega(C)^*$$

which covers a single valued *Abel-Jacobi* map

$$AJ : \mathrm{Pic}^0(C) \to \mathrm{Jac}(C).$$

The Abel-Jacobi map is an isomorphism of abelian varieties.

The theta divisor. Choosing a base point $P_0 \in C$ allows us to define a birational map ϕ between $\mathrm{Pic}^0(C)$ and the gth symmetric power of C via the formula

$$\phi(\{P_1, \dots, P_g\}) = \left(\sum_i P_i\right) - gP_0.$$

The divisor $\{S \in \mathrm{Sym}^g(C) : P_0 \in S\}$ gives rise to a divisor Θ on $\mathrm{Jac}(C)$ called the *theta divisor*.

Pullback and pushforward. Now consider a holomorphic map $g : Z \to C$ between curves. The map g induces a map $\Omega(C) \to \Omega(Z)$ whose dual covers a holomorphic homomorphism

$$g_* : \mathrm{Jac}(Z) \to \mathrm{Jac}(C).$$

Under the identification of Jacobians with degree zero divisors via the Abel-Jacobi map, g_* corresponds to the push-forward of divisors, i.e.,

$$g_*\left(\sum_i P_i - \sum_i Q_i\right) = \sum_i g(P_i) - \sum_i g(Q_i). \tag{4}$$

We call g_* the *pushforward map*. The map g also induces a *pullback map*:

$$g^* : \mathrm{Jac}(C) \to \mathrm{Jac}(Z).$$

The map g^* is dual via principal polarizations to g_*. At the level of divisors, g^* is obtained by summing along fibers, i.e.

$$g^*\left(\sum_i P_i - \sum_i Q_i\right) = \sum_i g^{-1}(P_i) - \sum_i g^{-1}(Q_i). \tag{5}$$

The composition $g_* \circ g^*$ is the multiplication by $\deg(g)$ map on $\mathrm{Jac}(C)$.

Correspondences. A *correspondence* Z on C is a holomorphic curve in $C \times C$. Fix a correspondence Z on C and let $f = \pi_1$ and $g = \pi_2$ be the the two projection maps from Z to C. The correspondence Z gives rise to a Jacobian endomorphism of C via the formula $T = g_* \circ f^*$. From Equations 4 and 5, we see that T acts on divisors of the form $P - Q$ by the formula

$$T(P-Q) = g(f^{-1}(P)) - g(f^{-1}(Q)). \tag{6}$$

Such divisors generate $\mathrm{Pic}^0(C)$, so Equation 6 determines the action of T on $\mathrm{Pic}^0(C)$.

Conversely, every Jacobian endomorphism T arises via a correspondence. To see this, we embed C in $\mathrm{Jac}(C)$ via the map $P \mapsto P - P_0$. When the genus of C is two, the resulting cycle is the theta divisor Θ. Since C generates the group $\mathrm{Jac}(C)$, the restriction of T to C determines T. The restriction $T|_C$ in turn is determined by the graph of T in $C \times \mathrm{Sym}^g(C)$ which can be encoded as the divisor $Z \subset C \times C$. The intersection of Z with $P \times C$ consists of points (P, Q_i) with $Q_1, \ldots, Q_g \in C$ satisfying $T \cdot (P - P_0) = \sum_i Q_i - gP_0$.

3. Computing equations for correspondences

In this section, we describe our method for "discovering" correspondences. The methods in this section are numerical and rely on floating point approximation. Nonetheless, the correspondences we obtain are presented by equations with exact coefficients lying in a number field. In Section 4, we will describe how to certify these equations using only rigorous integer arithmetic in number fields to prove theorems about real multiplication on genus two Jacobians.

Setup. Our starting point is a fixed curve C of genus two *known* to have a Jacobian endomorphism T generating real multiplication by the real quadratic order $\mathcal{O}_D$ of discriminant D. Such curves can be supplied by the methods in [EK]. We assume that C and T are defined over a number field K and that C is presented as a hyperelliptic curve

$$C : u^2 = h(t) \text{ with } h \in K[t], \deg(h) = 5. \tag{7}$$

We fix an embedding $K \subset \mathbb{C}$ so that we can base change to $\mathbb{C}$ and work with the analytic curve C^{an} and the analytic Jacobian $J^{an} = \mathrm{Jac}(C^{an})$. For simplicity we have assumed in this section that h is monic of degree 5 so that C has a K-rational Weierstrass point P_0 at infinity. We discuss below how to handle the sextic case (see Remark 15).

Analytic Jacobians in Magma. The computer system Magma has several useful functions for working with analytic Jacobians and their endomorphisms. An excellent introduction may be found in [vW2], and extensive documentation is available in the Magma handbook [BCFS]. The relevant functions for us are:

(a) AnalyticJacobian (see also BigPeriodMatrix): computes the periods of dt/u and $t\,dt/u$ in $\Omega(C)$, yielding a numerical approximation to the period matrix $\Pi(C^{an})$ and a model for the analytic Jacobian $J^{an} = \mathbb{C}^2/\Pi(C^{an}) \cdot \mathbb{Z}^4$.
(b) EndomorphismRing: computes generators for the endomorphism ring of J^{an}. Each endomorphism T^{an} is presented as a pair of matrices $T^{an}_{\Omega} \in M_2(\mathbb{C})$ and $T^{an}_{\mathbb{Z}} \in M_4(\mathbb{Z})$ satisfying $\Pi(C^{an}) \cdot T_{\mathbb{Z}} = T_{\Omega} \cdot \Pi(C^{an})$ (up to floating point precision).
(c) ToAnalyticJacobian: computes the Abel-Jacobi map by numerical integration.
(d) FromAnalyticJacobian: computes the inverse of the Abel-Jacobi map using theta functions.

Discovering correspondences. Our method for computing equations defining the correspondence Z on C associated to T is as follows.

(1) Compute the analytic Jacobian J^{an} and an endomorphism T^{an} generating real multiplication.
(2) Choose low height points $P_i = (t_i, u_i) \in C^{an}$ with $t_i \in K$. For each i:
(3) Numerically compute points $R_i = (t(R_i), u(R_i))$ and $Q_i = (t(Q_i), u(Q_i))$ in C^{an} such that
$$AJ(Q_i + R_i - 2P_0) = T^{an} \cdot (AJ(P - P_0)).$$
(4) Compute the exact coefficients of the polynomial $F_i(x) = (x - t(Q_i))(x - t(R_i))$ in $K[x]$ using LLL.
(5) Interpolate to determine a polynomial $F \in K(t)[x]$ which specializes to F_i at $t = t(P_i)$ and let Z be the degree two cover defined by F, i.e., with
$$K(Z) = K(C)[x]/(F). \tag{8}$$

To realize Z as a divisor in $C \times C$, we need to compute a square root for $h(x)$ in $K(Z)$. For small examples, this can be done by working in the function field for Z. In general, we revisit steps (3) and (4) and do the following.

(6) For each i, determine $u(Q_i)$ as a K-linear combination of $u(P_i)$ and $u(P_i)t(Q_i)$,
(7) Interpolate to determine a rational function $y \in K(Z)$ which is a $K(t)$-linear combination of u and ux and equals $u(Q_i)$ when specialized to $(t, u, x) = (t(P_i), u(P_i), t(Q_i))$.

Remark 4. Typically we use AnalyticJacobian and EndomorphismRing to carry out step (1), and ToAnalyticJacobian and FromAnalyticJacobian to carry out step (3). The remainder of the algorithm requires only the matrix T^{an}_{Ω} (and not $T^{an}_{\mathbb{Z}}$) which could also be obtained using the algorithm in [KM] rather than EndomorphismRing.

Remark 5. We do not carry out a detailed analysis of the floating point precision needed or the running time of our algorithm. We remark that 400 digits of precision were sufficient for the examples in this paper and that the machine used to perform the computations in this paper (4 GHz, 32 GB RAM) completed the entire sampling and interpolation process for individual correspondences in minutes. For our most complicated example, presented in Theorem 14, CPU time was under two minutes.

To be able to carry out these steps, we need a large supply of sample points, and sufficient precision. As far as the number of sample points needed for interpolation to find the equation of Z, we closely follow the argument of [vW1, Section 3]. There it is observed that the coefficients of F (which are $x_1 + x_2$ and x_1x_2 in the notation of [vW1]) are rational functions in t and have degrees which are bounded by the intersection number of $\alpha(\Theta)$ and 2Θ. In our case, this equals $\mathrm{tr}_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(\alpha^2)$.

Consequently, we choose $\alpha \in \mathcal{O}_D$ for which the trace is minimized: $\alpha = \pm\sqrt{D}/2$ if D is even, and $\alpha = (\pm 1 \pm \sqrt{D})/2$ if D is odd. In practice, since the degrees of the functions involved may be quite a bit smaller than the upper bound, it is more efficient both in terms of time and computer memory to choose a small sample size and attempt to see if the computation of Z succeeds.

Remark 6. One should be able to use variants of this method to compute correspondences for arbitrary endomorphisms of genus two Jacobians.

Remark 7. From the equations for Z and the maps f, g to C, we can compute the action of T on divisors of the form $P - Q$ using Equation 6. We can then use standard equations [CF] for the group law on $\mathrm{Jac}(C)$ to extend this formula to arbitrary divisor classes of degree zero. Similarly, we can compute the algebraic action of an arbitrary element $m + nT \in \mathbb{Z}[T]$ of the real quadratic order using formulas for the group law.

Example. We conclude this section with an example in discriminant 5. Let $K = \mathbb{Q}$ and let C be the genus two curve defined by

$$C : u^2 = h(t) \text{ where } h(t) = t^5 - t^4 + t^3 + t^2 - 2t + 1. \tag{9}$$

The Jacobian of C corresponds to the point $(g, h) = \left(-\frac{8}{3}, \frac{47}{2}\right)$ in the model for $Y(\mathcal{O}_5)$ computed in [EK]. By the method outlined above, we "discover" that the degree two branched cover $f : Z \to C$ defined by

$$F(x) = 0 \text{ where } F(x) = t^2x^2 - x - t + 1 \in K(C)[x] \tag{10}$$

is a correspondence associated to real multiplication by $\mathcal{O}_5$. In fact, setting

$$y = \frac{1}{t^3}u - \frac{t+1}{t^3}ux \in K(Z) \tag{11}$$

we find that y is a square root of $h(x)$ in $K(Z)$, and the map $g(t, u, x) = (x, y)$ defines a second map $Z \to C$. We depict a plane model for Z in Figure 1. In Section 4 we will prove that $T = g_* \circ f^*$ generates real multiplication by $\mathcal{O}_5$, thereby certifying these equations for Z.

Remark 8. The degree of g is two since, fixing x there are two choices for t satisfying Equation 10, and u is determined by (x, y, t) by Equation 11. The genus of Z is six, as can be readily computed in Magma or Maple. It would be interesting to use the tools in this paper to study the *geometry* of correspondences over Hilbert modular surfaces. In particular, one might explore how the geometry of Z varies with C and T, and how Z specializes at curves C lying on arithmetically and dynamically interesting loci such as Teichmüller curves and Shimura curves. For instance, compare Theorems 9 and 16 in discriminant 5 and Theorems 10 and 12 in discriminant 12.

4. Minimal polynomials and action on one-forms

In this section, we describe how to "certify" the equations we discovered by the method in Section 3. We have now determined an equation for a curve Z with an obvious degree two map $f : Z \to C$ given by $f(t, u, x) = (t, u)$. We have also computed equations for a second map $g : Z \to C$ given by $g(t, u, x) = (x, y)$.

We will now describe how to compute the action T_Ω of $T = g_* \circ f^*$ on $\Omega(C)$. Since the representation of the endomorphism ring of $\mathrm{Jac}(C)$ on $\Omega(C)$ is faithful, the minimal polynomial for T is equal to the minimal polynomial for T_Ω. Fixing

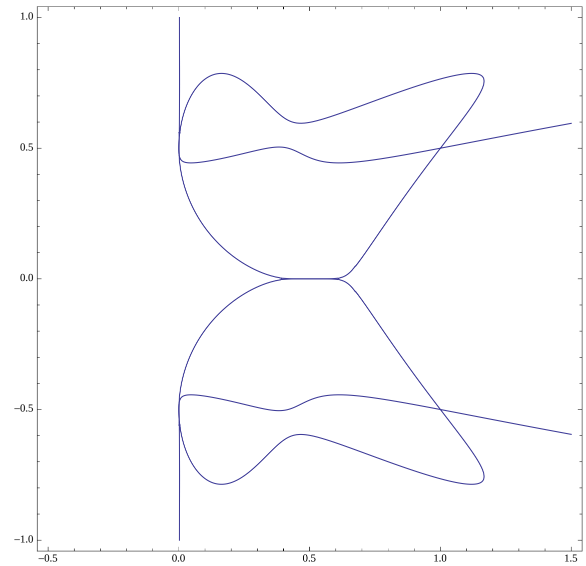

FIGURE 1. The function field of Z (cf. Equations 9 and 10) is generated by x and u. Here we plot Z in the $(1/x, 1/u)$-plane.

$\omega \in \Omega(C)$, to compute $T_\Omega(\omega)$, we first pullback along g and then pushforward along f. The order of composition is reversed since the functor sending C to the vector space $\Omega(C)$ is contravariant, whereas the functor sending C to $\mathrm{Jac}(C)$ is covariant. Pullbacks are straightforward, and the pushforward along f can be computed from the rule

$$f_*(v\,\eta) = \mathrm{tr}(v)\omega \text{ when } \eta = f^*(\omega) \text{ and } v \in K(Z). \tag{12}$$

The trace on the right hand side is with respect to the field extension $K(Z)$ over the subfield isomorphic to $K(C)$ associated to the map f. We now see that

$$T_\Omega(\omega) = f_* \circ g^*(\omega) = \mathrm{tr}(g^*(\omega)/f^*(\omega)) \cdot \omega. \tag{13}$$

The trace on the right hand side of Equation 13 is over the quadratic field extension $K(t,u,x)/K(t,u)$ and can be computed easily from the equations defining f. We return to the example in the previous section.

Example. Let $Z \subset C \times C$ be the correspondence defined by Equations 9, 10 and 11. Let $\omega_1 = dt/u$ and $\omega_2 = t\,dt/u$ be the standard basis for $\Omega(C)$. To compute the action of T_Ω on $\Omega(C)$, we need to work with the function field $K(Z)$ and its derivations. The derivations form a one dimensional vector space over $K(Z)$. It is spanned by both dx and dt, and the relation between them is computed by implicitly differentiating Equation 10. We compute

$$\frac{g^*(\omega_1)}{f^*(\omega_1)} = \frac{dx/y}{dt/u} = \frac{(-2t^4+t^3-t^2)x + (-2t^4+4t^3-t^2-t+1)}{4t^3-4t^2+1}. \tag{14}$$

We now need to compute the trace of the right hand side over $K(t,u)$. From Equation 10, we see that the trace of x is $1/t^2$, and therefore the trace of the right hand side of Equation 14 is $(1-t)$. We conclude that

$$T_\Omega(dt/u) = (1-t)dt/u. \tag{15}$$

Similarly, we compute that $T_\Omega(t\,dt/u) = -dt/u$ and hence the matrix for T_Ω is

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}. \tag{16}$$

The minimal polynomial for M, hence for T_Ω and T as well, is $T^2 - T - 1$. We conclude that T generates a ring $\mathbb{Z}[T]$ isomorphic to $\mathcal{O}_5$.

Rosati involution. The adjoint for T with respect to the Rosati involution is the endomorphism

$$T^* = (g_* \circ f^*)^* = (f_* \circ g^*). \tag{17}$$

By computing the action of T^* on $\Omega(C)$ by the procedure above, we verify that $T^*_\Omega = T_\Omega$ and conclude that $T = T^*$ is self-adjoint.

5. Further examples

We now describe some examples of the results one can obtain via our method. We choose relatively simple curves and small discriminants for purposes of illustration. For instance, the first three examples have Weierstrass points at ∞, and the others have two rational points at ∞. Each of the theorems stated in this section are proved by carrying out an analysis similar to our analysis of the curve defined by Equation 9 in Sections 3 and 4. We provide computer code in the auxiliary files to carry out these analyses.

Our first example collects the information about the curve C defined by Equation 9 and proven above.

Theorem 9. *Let C be the genus two curve defined by $u^2 = t^5 - t^4 + t^3 + t^2 - 2t + 1$ and let $f : Z \to C$ be the degree two branched cover of C defined by*

$$t^2x^2 - x - t + 1 = 0.$$

The curve Z is of genus 6 and admits an additional map $g : Z \to C$ of degree two. The induced endomorphism $T = g_* \circ f^*$ of $\mathrm{Jac}(C)$ is self-adjoint, satisfies $T^2 - T - 1 = 0$ and generates real multiplication by $\mathcal{O}_5$.

Note that Theorem 9 refines Theorem 1 stated in the introduction.

Our next example involves a slightly more complicated (i.e., larger discriminant) quadratic ring.

Theorem 10. *Let C be the curve defined by $u^2 = t^5 - 6t^4 + 15t^3 - 22t^2 + 17t$ and let $f : Z \to C$ be the degree two branched cover defined by*

$$\begin{aligned} t(t^2 - 3t + 1)^2x^2 - (4t^5 - 23t^4 + 46t^3 - 37t^2 + 6t + 17)x \\ +4t(t^4 - 6t^3 + 15t^2 - 22t + 17) = 0. \end{aligned}$$

The curve Z is of genus 12 and admits a map $g : Z \to C$ of degree 5. The induced endomorphism $T = g_* \circ f^*$ of $\mathrm{Jac}(C)$ satisfies $T^2 - 3 = 0$ and generates real multiplication by $\mathcal{O}_{12}$.

Remark 11. The curve C in Theorem 10 corresponds to the point $(e, f) = \left(\frac{34}{27}, -\frac{5}{3}\right)$ on $Y(\mathcal{O}_{12})$ in the coordinates of [EK]. The proof of the theorem proceeds along similar lines as that of Theorem 9. The map $g : Z \to C$ takes (t, u, x) to (x, y) where

$$y = -\frac{t^6 - 5t^5 + 12t^4 - 21t^3 + 32t^2 - 17t - 17}{t^2(t^2 - 3t + 1)^3} xu + \frac{2(t-2)(t^4 - 2t^3 - t^2 - 2t + 17)}{t(t^2 - 3t + 1)^3} u.$$

Our next example illustrates how the method developed in this paper may be used to identify eigenforms and determine points on Teichmüller curves. Recall that W_D is the moduli space of genus two eigenforms for $\mathcal{O}_D$ with a *double zero*, and is a disjoint union of Teichmüller curves [Mc1].

Theorem 12. *Let C be the curve $u^2 = t^5 - 2t^4 - 12t^3 - 8t^2 + 52t + 24$ and let $f : Z \to C$ be the degree two branched cover defined by*

$$16(t-2)(t+1)^2x^2 - (3t^4 + 16t^3 + 12t^2 - 192t - 164)x + (9t^5 - 12t^4 - 140t^3 - 48t^2 + 276t + 16) = 0$$

The curve Z is of genus 11 and admits a map $g : Z \to C$ of degree 5. The induced endomorphism $T = g_* \circ f^*$ of $\mathrm{Jac}(C)$ satisfies $T^2 - 3 = 0$ and generates real multiplication by $\mathcal{O}_{12}$. Moreover, the moduli point corresponding to C on $Y(\mathcal{O}_{12})$ lies on the Weierstrass-Teichmüller curve W_{12}.

Remark 13. This curve corresponds to the point $(e, f) = \left(-\frac{3}{8}, -\frac{1}{2}\right)$ on $Y(\mathcal{O}_{12})$ in the coordinates of [EK]. In [KM], we showed using the Eigenform Location Algorithm that dt/u is an eigenform for $\mathcal{O}_{12}$ to conclude that this point lies on W_{12}. That proof is logically dependent upon [EK] in an essential way. We prove that dt/u is an eigenform in this paper in a way which is logically independent of both [EK] and [KM] (although the statement is not). For completeness, we note the expression for the rational function y on Z needed to define $g : (t, u, x) \mapsto (x, y)$ is

$$y = -\frac{11t^4 - 24t^3 + 12t^2 - 112t - 132}{64(t-2)^2(t+1)^3} xu\sqrt{3} - \frac{15t^5 - 28t^4 - 36t^3 + 288t^2 - 52t - 144}{64(t-2)^2(t+1)^3} u\sqrt{3}.$$

The rest of the verification is carried out in the computer code.

Our next example involves a genus two curve without a rational Weierstrass point. The resulting correspondence is more complicated, but still well within the reach of our method.

Theorem 14. *Let C be the curve $u^2 = t^6 + t^5 + 7t^2 - 5t + 4$, and let $f : Z \to C$ be the degree two branched cover defined by*

$$\begin{aligned} &(3t^3 - t^2 + t + 1)(368t^4 - 597t^3 - 233t^2 + 233t + 41)x^2 \\ &+ x\big(4(199t^4 - 31t^3 - 185t^2 - 33t + 6)u \\ &\qquad + 2(430t^7 - 1601t^6 + 876t^5 - 623t^4 - 338t^3 + 257t^2 - 168t - 65)\big) \\ &+ 4(138t^5 - 153t^4 - 21t^3 + 55t^2 + 3t - 18)u \\ &+ 552t^8 - 1616t^7 - 1435t^6 + 4654t^5 - 3949t^4 + 900t^3 + 1035t^2 - 690t + 21 = 0 \end{aligned} \tag{18}$$

The curve Z is of genus 11 and admits a map g to C of degree 4. The induced endomorphism $T = g_* \circ f^*$ of $\mathrm{Jac}(C)$ satisfies $T^2 - 2 = 0$ and generates real multiplication by $\mathcal{O}_8$.

Remark 15. The curve C in Theorem 14 corresponds to the point $(r,s) = \left(\frac{1}{8}, \frac{59}{32}\right)$ on $Y(\mathcal{O}_8)$ in the coordinates of [EK]. Note that the coefficients of the polynomial F defining Z are not in $K(t)$, in contrast to the case where C is a quintic hyperelliptic curve. This is because the hyperelliptic involution does not preserve the chosen point at infinity P_0, and therefore does not commute with the deck transformation of $f : Z \to C$. Therefore, the "discovery" part of our algorithm in which we compute equations for Z has to be modified slightly. The coefficients of F can be computed by determining a K-linear relation between 1, $t(Q_i)$, $t(Q_i)^2$, $u_i = u(P_i)$ and $u_i t(Q_i)$ by LLL for each i (rather than between the first three quantities as in the quintic case). The coefficients in these relations are values of rational functions specialized at $t_i = t(P_i)$, and we can interpolate to determine these rational functions exactly. A similar modification must be made to solve for $y \in K(Z)$. For brevity, we have omitted the expression for y here, although it is available in the computer files.

6. Correspondences in families

In this section we describe a correspondence on a universal family of genus two curves over the entire Hilbert modular surface $Y(\mathcal{O}_5)$. There is one significant obstacle to implementing the method described in Section 3 in families. Suppose $\{C_\mu : \mu \in U\}$ is a family of curves parametrized by the base U each of which admits real multiplication by $\mathcal{O}$. The method described in Section 3 allows us to compute a correspondence Z_μ over C_μ for any particular $\mu \in U$. However, the first step in computing Z_μ involves a choice of analytic Jacobian endomorphism T^{an}_μ generating $\mathcal{O}$. There are typically two choices for T^{an}_μ with a given minimal polynomial, and it is important to make these choices so that the matrices $T^{an}_{\mu,\Omega}$ vary continuously in μ and the Z_μ's are fibers of a single family.

Our method to overcome this obstacle was to first normalize the entire family so that dt/u and $t\,dt/u$ are eigenforms, using the Eigenform Location Algorithm in [KM]. Then we simply choose T^{an}_μ to have $T^{an}_{\mu,\Omega}$ equal to a constant diagonal matrix. Having consistently chosen T^{an}_μ in this way, we compute Z_μ for various values of μ and interpolate to determine a correspondence over the entire family. The result is the following theorem.

Theorem 16. *For $(p,q) \in \mathbb{C}^2$, let $C(p,q)$ be the curve defined by the equation*

$$u^2 = t^6 + 2pt^5 + 10qt^3 + 10q^2 t - 5(p-1)q^2 \tag{19}$$

and let $f : Z(p,q) \to C(p,q)$ be degree two branched cover defined by

$$\begin{aligned}
&(2t-p)\big(4t + (3+\alpha)p\big)x^2 \\
&+ \big(2(-\alpha-1)u + \alpha(2t^3 - 2pt^2 + p^2 t + 2q) - (6t^3 - 6pt^2 - p^2 t - 10q)\big)x \\
&- 2\big((1-\alpha)t - p\big)u + \alpha(2t^4 - p^2t^2 + 6qt - 4pq) \\
&- (2t^4 - 2pt^3 + 3p^2t^2 - 10qt + 10pq) = 0
\end{aligned} \tag{20}$$

where $\alpha = \sqrt{5}$. For generic $(p,q) \in \mathbb{C}^2$, the curve $Z(p,q)$ is of genus 8 and admits a holomorphic map g to $C(p,q)$ of degree 3. The endomorphism $T = g_* \circ f^*$ of $\mathrm{Jac}(C(p,q))$ is self-adjoint and generates real multiplication by $\mathcal{O}_5$.

The only complication in the proof of Theorem 16 is that we need to work in the function field over the base field $\mathbb{Q}(p,q)$ rather than $\mathbb{Q}$. We provide computer code in the auxiliary files to carry out the certification as in our previous examples. For brevity, we have omitted the lengthy expression for y in the map $g : (t, u, x) \mapsto (x, y)$; it is available from the computer files. For that choice of y, the endomorphism T has minimal polynomial $T^2 - T - 1$. (Replacing y with $-y$ gives rise to an endomorphism with minimal polynomial $T^2 + T - 1$.)

Remark 17. The coordinates for $Y(\mathcal{O}_5)$ in Theorem 16 are related to the coordinates (m, n) appearing in [EK] by

$$(p, q) = \big(m^2/5 - n^2, (m - \alpha n)(5n^2 - m^2)(5n^2 - m^2 + 5)/125\big). \tag{21}$$

In particular, our coordinates are quadratic twists of those appearing in [EK]. This is because they are adapted to the eigenform moduli problem, not the real multiplication moduli problem. The field of definition of a point (p, q) is the field of definition of the eigenforms dt/u and $t\,dt/u$, which need not agree with the field of definition of real multiplication. In fact, these moduli problems are isomorphic over $\mathbb{Q}(\sqrt{5})$, but not over $\mathbb{Q}$. This also explains the appearance of $\alpha = \sqrt{5}$ in the equation defining $Z(p, q)$.

7. Divisors supported at eigenform zeros

We now turn to the applications in dynamics for our equations for real multiplication stated in the introduction. Recall that L is the multisection in the universal Jacobian over $\mathcal{M}_{2,1}(\mathcal{O}_5)$ whose values at the pointed curve (C, P) are divisors of the form in Equation 2. Our goal is to prove that the locus $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ defined by the vanishing of L is an irreducible surface in $\mathcal{M}_{2,1}$ and that $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ is the closure of a complex geodesic for the Teichmüller metric.

Marking eigenform zeros. We start by passing to a cover $\mathcal{M}_{2,1}(\mathcal{O}_5)$ on which we can describe the multisection L in terms of sections. To that end we define $\mathcal{M}_2^{\mathrm{ze}}(\mathcal{O}_5)$ to be the space of pairs (C, Z) where $C \in \mathcal{M}_2(\mathcal{O}_5)$ and $Z \in C$ is a zero of an eigenform for real multiplication by $\mathcal{O}_5$. Similarly we define $\mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5)$ to be the pointed version consisting of triples (C, P, Z) with $(C, P) \in \mathcal{M}_{2,1}(\mathcal{O}_5)$ and $(C, Z) \in \mathcal{M}_2^{\mathrm{ze}}(\mathcal{O}_5)$. Here we are allowing $Z = P$.

The space $\mathcal{M}_2^{\mathrm{ze}}(\mathcal{O}_5)$ is birational to the Hilbert modular surface $Y(\mathcal{O}_5)$. To see this, fix $\gamma \in \mathcal{O}_5$ satisfying $\gamma^2 - \gamma - 1 = 0$. A point $(C, Z) \in \mathcal{M}_2^{\mathrm{ze}}(\mathcal{O}_5)$ determines a self-adjoint endomorphism $T_\gamma(C, Z)$ of $\mathrm{Jac}(C)$ by the requirement that the line of one-forms on C vanishing at Z are γ-eigenforms for $T_\gamma(C, Z)$. The map $(C, Z) \mapsto (\mathrm{Jac}(C), T_\gamma(C, Z))$ is birational. In particular, $\mathcal{M}_2^{\mathrm{ze}}(\mathcal{O}_5)$ is an irreducible surface.

Sections. Let η be the hyperelliptic involution on C. We can now define a section L_γ of the universal Jacobian over $\mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5)$ by the formula

$$L_\gamma(C, P, Z) = (P - Z) - T_\gamma(C, Z) \cdot (\eta(Z) - Z) \in \mathrm{Jac}(C). \tag{22}$$

Let $\mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5; L_\gamma)$ denote the locus in $\mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5)$ where L_γ vanishes. Similarly, we define $T_{1-\gamma}(C, Z)$, $L_{1-\gamma}$ and $\mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5; L_{1-\gamma})$ by replacing γ with its Galois conjugate $1 - \gamma$. From the definition of the multisection L, it is clear the map forgetting Z sends the union of $\mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5; L_\gamma)$ and $\mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5; L_{1-\gamma})$ onto $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$. In fact, each of these spaces individually maps onto $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ since the sections L_γ and $L_{1-\gamma}$ are related by $L_\gamma(C, P, Z) = L_{1-\gamma}(C, P, \eta(Z))$. We record this fact in the following proposition.

Proposition 18. *The space $\mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5; L_\gamma)$ maps onto $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$.*

We will now use our equations for real multiplication to show that $\mathcal{M}^{\mathrm{ze}}_{2,1}(\mathcal{O}_5; L_\gamma)$ is a section of $\mathcal{M}^{\mathrm{ze}}_{2,1}(\mathcal{O}_5) \to \mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5)$.

Proposition 19. *For each $(C, Z) \in \mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5)$, there is a unique solution $P \in C$ to the equation $L_\gamma(C, P, Z) = 0$.*

Proof. The uniqueness is easy and does not require our equations for real multiplication. If P_1, P_2 are solutions to $L_\gamma(C, P, Z) = 0$, then $P_1 - P_2$ is a principal divisor. Since the smooth genus two curve C admits no degree one rational map, we must have $P_1 = P_2$.

The locus in $\mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5)$ consisting of pairs (C, Z) which admit a solution to $L_\gamma(C, P, Z) = 0$ is closed. This follows from the fact that $\mathcal{M}^{\mathrm{ze}}_{2,1}(\mathcal{O}_5) \to \mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5)$ is a projective map and, since $\mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5)$ is irreducible, it is enough to check that the generic pair $(C, Z) \in \mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5)$ admits such a solution.

Recall the notation of Theorem 16 and its proof in the auxiliary files. For generic $(p, q) \in \mathbb{C}^2$, we have a genus two curve $C(p, q)$, a self-adjoint endomorphism $T(p, q)$ of $\mathrm{Jac}(C(p, q))$ satisfying $T(p, q)^2 - T(p, q) - 1 = 0$, and a $T(p, q)$-eigenform $\omega(p, q) = t\, dt/u$ with eigenvalue $\gamma = (1 + \alpha)/2$. To mark a zero on $\omega(p, q)$, we choose z a square root of $5(1 - p)$ and set

$$Z(z, q) = (0, zq) \in C(p, q). \tag{23}$$

Counting dimensions, we see that the (z, q)-plane parametrizes an open subset of $\mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5)$ by the formula $(z, q) \mapsto (C(z, q), Z(z, q))$. We further define

$$P(z, q) = \big(2(1 - p), z(8 - 16p + 8p^2 + 5q)/\alpha\big) \in C(z, q). \tag{24}$$

Using our equations for the correspondence defining $T(z, q)$ and Equation 6, we compute the divisor $T(z, q) \cdot (\eta(Z(z, q)) - Z(z, q))$. Combined with standard formulas for the group law on $\mathrm{Jac}(C)$ (which have been implemented in Magma), we verify that $L_\gamma(C(z, q), P(z, q), Z(z, q)) = 0$. We include code in the auxiliary files to verify this equation. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. The locus $\mathcal{M}^{\mathrm{ze}}_{2,1}(\mathcal{O}_5; L_\gamma)$ is biregular to the irreducible surface $\mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5)$ by Proposition 19, and maps onto $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ by a map of finite degree. Therefore $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ is an irreducible surface in $\mathcal{M}_{2,1}$. □

Complex geodesics in moduli space. We now prove Theorem 3 about geodesics in $\mathcal{M}_{2,1}$ which is a corollary of Theorem 2. We refer the reader to the survey articles [Wr] and [Z] for background on geodesics in the moduli space of curves.

Proof of Theorem 3. Fix a curve $C \in \mathcal{M}_2(\mathcal{O}_5)$ and an $\mathcal{O}_5$-eigenform ω on C. The form ω generates a complex geodesic $f_\omega : \mathbb{H} \to \mathcal{M}_2$ with $f_\omega(i) = C$ and $f'_\omega(i)$ tangent to $\mathcal{M}_2(\mathcal{O}_5)$. By [Mc1], the image of f_ω is contained in $\mathcal{M}_2(\mathcal{O}_5)$. We choose C and ω generically so that $\overline{f_\omega(\mathbb{H})} = \mathcal{M}_2(\mathcal{O}_5)$ (cf. [Mc2]).

The values of f_ω are related to C by Teichmüller mappings. In particular, there is a distinguished holomorphic one-form ω_τ (up to scale) on $C_\tau = f_\omega(\tau)$ and a homeomorphism $C \to C_\tau$ which is affine for the singular flat metrics $|\omega|$ and $|\omega_\tau|$. The zeros of ω are in bijection with those of ω_τ via the Teichmüller mapping and, by [Mc1], ω_τ is also an $\mathcal{O}_5$-eigenform. We conclude that there is a holomorphic zero marked lift

$$f^{\mathrm{ze}}_\omega : \mathbb{H} \to \mathcal{M}^{\mathrm{ze}}_{2}(\mathcal{O}_5) \tag{25}$$

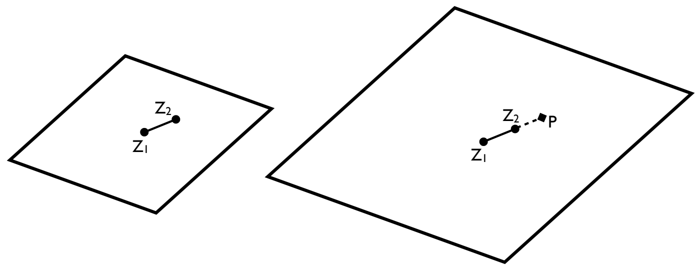

FIGURE 2. Genus two eigenforms for $\mathcal{O}_5$ can be built out of a parallelogram $U \subset \mathbb{C}$ and the similar parallelogram γU by a connected sum. The resulting form has zeros at $Z_1 = 0$ and $Z_2 = t$ and the marked point $P = \gamma t$ satisfies $L_\gamma(C, P, Z_1) = 0$.

whose composition with the map forgetting Z equals f_ω. Composing f_ω^{ze} with the section $\mathcal{M}_2^{\mathrm{ze}}(\mathcal{O}_5) \to \mathcal{M}_{2,1}^{\mathrm{ze}}(\mathcal{O}_5; L_\gamma)$ and the map forgetting Z, we obtain a map

$$f_\omega^P : \mathbb{H} \to \mathcal{M}_{2,1}$$

which is a section of f_ω over $\mathcal{M}_{2,1} \to \mathcal{M}_2$.

There are several ways to conclude that f_ω^P is a complex geodesic. The map f_ω^P is a section over the complex geodesic f_ω, and such sections are complex geodesics by a well-known argument relying on the equality of the Kobayashi and Teichmüller metrics on $\mathcal{M}_{g,n}$. Alternatively, for $(C_\tau, P_\tau) = f_\omega^P(\tau)$ we have an $\mathcal{O}_5$-eigenform ω_τ and a zero Z_τ of ω_τ satisfying $L_\gamma(C_\tau, P_\tau, Z_\tau) = 0$. We conclude that the relative periods

$$\int_{Z_\tau}^{P_\tau} \omega_\tau, \text{ and } \gamma \int_{Z_\tau}^{\eta(Z_\tau)} \omega_\tau$$

differ by an absolute period of ω_τ. Consequently, the Teichmüller mapping from $C \to C_\tau$ sends P to P_τ and f_ω^P is a complex geodesic.

Thus we have a complex geodesic f_ω^P in $\mathcal{M}_{2,1}$ such that $\overline{f_\omega^P(\mathbb{H})}$ lies in $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ and maps onto $\mathcal{M}_2(\mathcal{O}_5)$. Since both $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ and $\mathcal{M}_2(\mathcal{O}_5)$ are irreducible surfaces, we must have $\overline{f_\omega^P(\mathbb{H})} = \mathcal{M}_{2,1}(\mathcal{O}_5; L)$. $\square$

Polygons and marked points. McMullen described how to polygonally present eigenforms for $\mathcal{O}$ in genus two [Mc2]. Set $\gamma = (1 + \sqrt{5})/2$ to be the golden mean. Eigenforms in discriminant 5 are obtained from a parallelogram $U \subset \mathbb{C}$ centered at 0 and the similar parallelogram $\gamma U \subset \mathbb{C}$ by gluing opposing sides on each parallelogram and performing a connected sum along a straight line interval I connecting 0 and $t \in U$. The form dz is invariant under these gluing maps and the resulting quotient $(C, \omega) = (U \#_I \gamma U, dz)/\sim$ is an $\mathcal{O}_5$-eigenform. Wright's conjecture of the existence of a dynamically natural way to mark curves in $\mathcal{M}_2(\mathcal{O}_5)$ posited in particular that one could mark the eigenform $(U \#_I \gamma U)/\sim$ at the point $P = \gamma t$ in the polygon γU (see Figure 2).

One way to see that the algebraically presented locus $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ in Theorem 3 equals the locus polygonally presented by Wright is by first checking that they agree somewhere, e.g. at the regular decagon eigenform which is the limit of $(C(z,q), P(z,q))$ as $q \to 0$ in our parametrization. The period relations imposed by the vanishing of L then imply that the points marked in $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$ coincide with Wright's polygonal description at a nearby generic point. Therefore, the algebraic and polygonal descriptions agree along an entire complex geodesic which is dense in $\mathcal{M}_{2,1}(\mathcal{O}_5; L)$.

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