

MATRICIAL QUANTUM GROMOV-HAUSDORFF DISTANCE

DAVID KERR

ABSTRACT. We develop a matricial version of Rieffel’s Gromov-Hausdorff distance for compact quantum metric spaces within the setting of operator systems and unital C^* -algebras. Our approach yields a metric space of “isometric” unital complete order isomorphism classes of metrized operator systems which in many cases exhibits the same convergence properties as those in the quantum metric setting, as for example in Rieffel’s approximation of the sphere by matrix algebras using Berezin quantization. Within the metric subspace of metrized unital C^* -algebras we establish the convergence of sequences which are Cauchy with respect to a larger Leibniz distance, and we also prove an analogue of the precompactness theorems of Gromov and Rieffel.

1. INTRODUCTION

A compact quantum metric space, as defined by Marc Rieffel in [10], is an order-unit space equipped with a certain type of semi-norm, called a Lip-norm, which plays the role of a Lipschitz semi-norm on functions over a compact metric space. The crucial part of the definition of a Lip-norm L on an order-unit space A is the requirement that the metric

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A \text{ and } L(a) \leq 1\}$$

on the state space of A give rise to the weak* topology. By applying Hausdorff distance to state spaces, Rieffel defines a quantum analogue of Gromov-Hausdorff distance and thereby synthetically obtains a complete separable metric space of “isometric” order isomorphism classes of compact quantum metric spaces for which a Gromov-type precompactness theorem holds [10]. The most immediate motivation for introducing a theory of quantum Gromov-Hausdorff distance is the search for an analytic framework for describing, or at least clarifying at a metric level, the type of convergence of spaces that has recently begun to play an important role in string theory (see the introduction to [10] for a discussion and references). The main objects of study thus tend to be C^* -algebras, and so it is natural to ask, as does Rieffel in [10], if it is possible to develop a matricial version of quantum Gromov-Hausdorff distance. This is the goal of the present paper.

The key is to define metrics on matrix state spaces using a Lip-norm just as one does for an order-unit state space as above, only now replacing the modulus by matrix norms. We introduce this definition within a general operator system setting in Section 2. We then define “complete” distance (Section 3) by using Hausdorff distance at the matrix state space level in the same way that Rieffel does with regard to order-unit state spaces in the formulation of quantum Gromov-Hausdorff

distance. In fact many of the constructions and arguments involving quantum Gromov-Hausdorff distance in [10, 11] are naturally suited to our matricial setting and lead to similar estimates, as for instance in the proof of the triangle inequality (Proposition 3.4) and the approximation of the sphere by matrix algebras via Berezin quantization (Example 3.13). On the other hand a completely different approach is required to show that complete distance zero implies “isometric” unital complete order isomorphism (the subject of Section 4), and the proofs of the convergence of sequences of metrized unital C^* -algebras which are Cauchy with respect to “ f -Leibniz complete distance” (Section 5) and our analogue of the Gromov and Rieffel precompactness theorems (Section 6) ultimately rely on some arguments especially attuned to the complete order context.

This work was supported by the Natural Sciences and Engineering Research Council of Canada. I thank Yasuyuki Kawahigashi and the operator algebra group at the University of Tokyo for their hospitality and for the invigorating research environment they have provided. I am grateful to Hanfeng Li for making a careful critical reading of an initial draft that prompted a number of clarifications and corrections, and I also thank the referee for helpful comments and suggestions.

2. LIP-NORMED OPERATOR SYSTEMS AND MATRIX STATE SPACE METRIZATION

We begin by describing our operator system framework. For references see [3, 7, 17]. A (concrete) *operator system* is a closed unital self-adjoint linear subspace of a unital C^* -algebra (for an abstract definition see [3]). Given an operator system X , for each $r > 0$ we will denote by \mathcal{B}_r^X the closed norm ball $\{x \in X : \|x\| \leq r\}$ of radius r . The state space of X will be denoted by $S(X)$. We will denote by X_{sa} the set of self-adjoint elements of X . The unit of X will be written 1 , or sometimes 1_X for clarity. For $x \in X$ we write $\text{Re}(x)$ and $\text{Im}(x)$ to refer to the self-adjoint elements $(x + x^*)/2$ and $(x - x^*)/(2i)$ (the real and imaginary parts of x), respectively.

The C^* -algebra of $n \times n$ matrices over \mathbb{C} will be written M_n . Given operator systems X and Y we say that a linear map $\varphi : X \rightarrow Y$ is *n -positive* if the map $\text{id}_n \otimes \varphi : M_n \otimes X \rightarrow M_n \otimes Y$ is positive, and if $\text{id}_n \otimes \varphi$ is n -positive for all $n \in \mathbb{N}$ then we say that φ is *completely positive*. A completely positive (resp. unital completely positive) linear map will be referred to as a *c.p.* (resp. *u.c.p.*) map. If $\varphi : X \rightarrow Y$ is a unital m -positive map with m -positive inverse for $m = 1, \dots, n$ then φ is a *unital n -order isomorphism*, and if φ is u.c.p. with c.p. inverse then φ is a *unital complete order isomorphism*.

An operator system X is *nuclear* if the identity map on X lies in the point-norm closure of the set of u.c.p. maps from X to itself which factor through matrix algebras.

Given an operator system X and $n \in \mathbb{N}$, there is a bijective linear map from c.p. maps $X \rightarrow M_n$ to positive linear functionals on $M_n \otimes X$ [7, Thm. 5.1] defined as follows. To each c.p. map $\varphi : X \rightarrow M_n$ we associate the positive linear functional σ_φ on $M_n \otimes X$ given by

$$\sigma_\varphi((x_{ij})) = \frac{1}{n} \sum_{1 \leq i, j \leq n} \varphi(x_{ij})_{ij}$$

for all $(x_{ij}) \in M_n(X) \cong M_n \otimes X$. Conversely, to each positive linear functional σ on $M_n \otimes X$ we associate the c.p. map $\varphi_\sigma : X \rightarrow M_n$ given by

$$(\varphi_\sigma(x))_{ij} = n \sigma(e_{ij} \otimes x)$$

for all $x \in X$, where $\{e_{ij} : 1 \leq i, j \leq n\}$ is the set of standard matrix units of M_n . The maps $\varphi \mapsto \sigma_\varphi$ and $\sigma \mapsto \varphi_\sigma$ are mutual inverses and are homeomorphisms with respect to the point-norm topologies (for the space of positive linear functionals this is the weak* topology) as well as with respect to the norm topologies. If $\varphi : X \rightarrow M_n$ is u.c.p. then σ_φ is a state on $M_n \otimes X$. However, if $\sigma \in S(M_n \otimes X)$ then φ_σ need not be unital, nor even contractive, although it is clear that $\|\varphi_\sigma\| \leq n^3$ (see the discussion after Theorem 5.4 in [7]). We denote by $SCP_n(X)$ the collection of c.p. maps $\varphi : X \rightarrow M_n$ such that σ_φ is a state on $M_n \otimes X$, and by $UCP_n(X)$ the subcollection of $SCP_n(X)$ consisting of all u.c.p. maps from X into M_n (the *matrix state spaces*).

We now introduce metrics into our picture via the notion of a Lip-norm, which we recall from [10].

Definition 2.1 ([10, Defns. 2.1 and 2.2]). Let A be an order-unit space. A *Lip-norm* on A is a semi-norm L on A such that

- (1) for all $a \in A$ we have $L(a) = 0$ if and only if a is a scalar multiple of the order unit, and
- (2) the metric ρ_L defined on the state space $S(A)$ by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A \text{ and } L(a) \leq 1\}$$

induces the weak* topology.

A pair (A, L) consisting of an order-unit space A with Lip-norm L is called a *compact quantum metric space*.

Important examples of order-unit spaces are real linear unital subspaces of self-adjoint elements in an operator system, and in fact every order-unit space is isomorphic to one of these, as shown in Appendix 2 of [10]. We can thus apply the above definition in a direct way to our setting. First we introduce some general notation.

Notation 2.2. Let X be an operator system and L a semi-norm on a linear subspace of X or a real linear subspace of X_{sa} . We denote by $\mathcal{D}(L)$ the domain of L , or, if L is permitted to take the value $+\infty$, the set of elements in the domain of L on which L is finite-valued. For $r > 0$ we denote by $\mathcal{D}_r(L)$ the set $\{x \in \mathcal{D}(L) : L(x) \leq r\}$.

Definition 2.3. By a *Lip-normed operator system* we mean a pair (X, L) where X is an operator system and L is a Lip-norm on a dense order-unit subspace of X_{sa} such that $\mathcal{D}_1(L)$ is closed in X_{sa} . If X is a unital C^* -algebra then we will also refer to (X, L) as a *Lip-normed unital C^* -algebra*. Any qualifiers preceding ‘‘Lip-normed’’ will refer to the Lip-norm while those following it will refer to the operator system or C^* -algebra (e.g., *lower semicontinuous Lip-normed nuclear operator system*).

A Lip-norm L on an order-unit space A is said to be *closed* if the set $\{a \in A : L(a) \leq 1\}$ is closed in the completion of A [12, Defn. 4.5]. Thus the requirement in Definition 2.3 that $\mathcal{D}_1(L)$ be closed in X_{sa} is equivalent to asking that L be a closed

Lip-norm. Given any Lip-norm L on an order-unit space A there is a largest lower semicontinuous Lip-norm L^s smaller than L [12, Thm. 4.2], and L^s extends to a closed Lip-norm L^c [12, Prop. 4.4]. The theorem and proposition from [12] cited in the last sentence also show that $\rho_{L^c} = \rho_{L^s} = \rho_L$. Furthermore, the property of being closed passes to order-unit quotients by [10, Prop. 3.3], and it also holds in natural examples of interest—see for instance Example 2.6 and [12, Prop. 3.6]. Thus, in view of the completeness of operator systems, it is natural to assume that our Lip-norms are closed. This will also guarantee that complete distance zero is equivalent to the existence of a unital complete order isomorphism which is *bi-Lip-isometric* in the obvious sense:

Definition 2.4. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems. We will say that a positive unital map $\Phi : X \rightarrow Y$ is *Lip-isometric* if $\Phi(\mathcal{D}(L_X)) \subset \mathcal{D}(L_Y)$ and $L_Y(\Phi(x)) = L_X(x)$ for all $x \in \mathcal{D}(L_X)$. If Φ has a positive inverse then we say that Φ is *bi-Lip-isometric* if both Φ and Φ^{-1} are Lip-isometric.

If we were to define a strict operator system analogue of a Lip-norm then the conditions on the semi-norm L in the following proposition would seem to be the most reasonable. Indeed many examples arise naturally in this way, as Example 2.6 illustrates.

Proposition 2.5. Let L be a semi-norm on an operator system X , permitted to take the value $+\infty$, such that $\mathcal{D}(L)$ is dense in X , $\mathcal{D}_1(L)$ is closed in X_{sa} , and

- (i) $L(x^*) = L(x)$ for all $x \in X$ (adjoint invariance),
- (ii) for all $x \in X$ we have $L(x) = 0$ if and only if $x \in \mathbb{C}1$, and
- (iii) the metric $d_L(\sigma, \omega) = \sup_{x \in \mathcal{D}_1(L)} |\sigma(x) - \omega(x)|$ on $S(X)$ induces the weak* topology.

Then the restriction L' of L to the order-unit space $\mathcal{D}(L) \cap X_{\text{sa}}$ is a Lip-norm, (X, L') is a Lip-normed operator system, and the restriction map from $S(X)$ onto $S(\mathcal{D}(L'))$ is a weak* homeomorphism which is isometric for d_L and $\rho_{L'}$.

Proof. First note that the fact that $\mathcal{D}(L)$ is closed in X immediately implies that $\mathcal{D}(L')$ is closed in X_{sa} . Next, if $x \in X_{\text{sa}}$ and $\epsilon > 0$ then by the density of $\mathcal{D}(L)$ we can find a $y \in \mathcal{D}(L)$ with $\|x - y\| < \epsilon$. Then

$$\|x - \text{Re}(y)\| \leq \|x - y\|/2 + \|(x - y)^*\|/2 < \epsilon$$

while $L'(\text{Re}(y)) = (L(y) + L(y^*))/2 = L(y) < +\infty$, and so $\mathcal{D}(L')$ is dense in X_{sa} . With this fact it is straightforward to show that the restriction map from $S(X)$ onto $S(\mathcal{D}(L'))$ is a weak* homeomorphism. To see that this map is isometric, suppose $\sigma, \omega \in S(X)$ and $\epsilon > 0$. Then we can find an $x \in \mathcal{D}_1(L)$ such that $|\sigma(x) - \omega(x)| \geq d_L(\sigma, \omega) - \epsilon$, and so for some complex number μ of unit modulus we have $\sigma(\mu x) - \omega(\mu x) \geq d_L(\sigma, \omega) - \epsilon$. Since

$$L'(\text{Re}(\mu x)) \leq (L(\mu x) + L(\bar{\mu}x^*))/2 = (L(x) + L(x^*))/2 = L(x) \leq 1$$

we then have

$$\rho_{L'}(\sigma|_{\mathcal{D}(L')}, \omega|_{\mathcal{D}(L')}) \geq \sigma(\text{Re}(\mu x)) - \omega(\text{Re}(\mu x))$$

$$\geq d_L(\sigma, \omega) - \epsilon,$$

from which we infer that the map in question is indeed an isometric weak* homeomorphism. Since condition (ii) in the proposition statement immediately implies condition (1) in Definition 2.1 for L' , it thus follows that L' is a Lip-norm, and so (X, L') is a Lip-normed operator system in view of the density of $\mathcal{D}(L')$ in X_{sa} . \square

In the converse direction, given a Lip-normed operator system (X, L') we can extend L' to a semi-norm L on X such that the conditions and conclusions in the statement of Proposition 2.5 hold. Definition 4.2 and Proposition 4.3 indicate how this can be done.

Example 2.6 (ergodic actions of compact groups). As studied in [13], ergodic actions of compact groups give rise to important examples of Lip-normed unital C^* -algebras, notably noncommutative tori (see Example 6.8). Let γ be an ergodic action of a compact group G on a unital C^* -algebra A . Let e be the identity element of G . We suppose that G is equipped with a length function ℓ , that is, a continuous function $\ell : G \rightarrow \mathbb{R}_{\geq 0}$ such that, for all $g, h \in G$,

- (1) $\ell(gh) \leq \ell(g) + \ell(h)$,
- (2) $\ell(g^{-1}) = \ell(g)$, and
- (3) $\ell(g) = 0$ if and only if $g \neq e$.

The group action γ and the length function ℓ together yield the semi-norm L on A defined by

$$L(a) = \sup_{g \in G \setminus \{e\}} \frac{\|\gamma_g(a) - a\|}{\ell(g)}.$$

It is easily seen that L is adjoint-invariant and that $L(a) = 0$ if and only if $a \in \mathbb{C}1$. Furthermore, by [13, Thm. 2.3] the metric $d_L(\sigma, \omega) = \sup_{x \in \mathcal{D}_1(L)} |\sigma(x) - \omega(x)|$ on $S(A)$ induces the weak* topology, by [13, Prop. 2.2] $\mathcal{D}(L)$ is dense in X , and it is readily verified that $\mathcal{D}_1(L)$ is closed in A (see [10, Prop. 8.1]), so that by Proposition 2.5 we obtain a Lip-normed unital C^* -algebra by restricting L to $\mathcal{D}(L) \cap A_{\text{sa}}$.

Example 2.7 (quotients). Let (X, L) be a Lip-normed operator system, Y an operator system, and $\Phi : X \rightarrow Y$ a unital positive linear map such that $\Phi(\mathcal{D}(L))$ is dense in Y_{sa} (which is automatic if Φ is surjective). Then by [10, Prop. 3.1] L gives rise to a Lip-norm L_Y on $\Phi(\mathcal{D}(L))$ via the prescription

$$L_Y(y) = \inf\{L(x) : x \in \mathcal{D}(L) \text{ and } \Phi(x) = y\}$$

for each $y \in Y$, and the induced map from $(S(Y), \rho_{L_Y})$ to $(S(X), \rho_L)$ is an isometry. Since L_Y is closed by [10, Prop. 3.3], (Y, L_Y) is a Lip-normed operator system. We say that L induces L_Y via Φ .

The following definition captures the observation that Lip-norms define metrics on matrix state spaces in much the same way as they do on state spaces. We will thereby be able to define a matrix version of quantum Gromov-Hausdorff distance by applying Hausdorff distance to the matrix state spaces (Definition 3.2).

Definition 2.8. Let (X, L) be a Lip-normed operator system and $n \in \mathbb{N}$. We define the metric $\rho_{L,n}$ on $UCP_n(X)$ by

$$\rho_{L,n}(\varphi, \psi) = \sup_{x \in \mathcal{D}_1(L)} \|\varphi(x) - \psi(x)\|$$

for all $\varphi, \psi \in UCP_n(X)$,

Note that $\rho_{L,n}$ is indeed a metric since it clearly satisfies the triangle inequality and is symmetric, and it is non-zero at any pair of distinct points owing to the density of $\mathcal{D}(L)$ in X_{sa} . That $\rho_{L,n}$ is finite follows from the norm compactness of $\mathcal{D}_1(L) \cap \mathcal{B}_r^X$ for any $r > 0$ (a consequence of [10, Thm. 4.5] by scaling) along with Proposition 2.11 below.

Proposition 2.9. The diameters of $UCP_n(X)$ relative to the respective metrics $\rho_{L,n}$ are finite and coincide for all $n \in \mathbb{N}$.

Proof. The restriction map from $S(X)$ onto $S(\mathcal{D}(L))$ is evidently a weak* homeomorphism which is isometric with respect to $\rho_{L,1}$ and ρ_L (Definition 2.1), and so the diameter of $S(X)$ with respect to $\rho_{L,1}$ is finite by [10, Thm. 4.5]. Now given $n \in \mathbb{N}$, $\varphi, \psi \in UCP_n(X)$, and $x \in X_{\text{sa}}$ we can find a state σ on M_n such that

$$|(\sigma \circ \varphi)(x) - (\sigma \circ \psi)(x)| = \|\varphi(x) - \psi(x)\|.$$

It follows that the diameter of $UCP_n(X)$ is bounded above by that of $S(X) = UCP_1(X)$. On the other hand $S(X)$ embeds into $UCP_n(X)$ via the map which takes $\sigma \in S(X)$ to $x \mapsto \sigma(x)1_{M_n}$, from which we see that the diameter of $UCP_n(X)$ is at least that of $S(X)$, so that the two are equal. \square

Definition 2.10. Given a Lip-normed operator system (X, L) we define its *diameter* $\text{diam}(X, L)$ to be the common value of the diameters of $UCP_n(X)$ with respect to $\rho_{L,n}$ for $n \in \mathbb{N}$.

The next proposition, in addition to showing the finiteness of the metrics $\rho_{L,n}$ (see the paragraph following Definition 2.8), will also be of use in Sections 3 and 6 since it will enable us to streamline the statement and verification of conditions involving local approximation of elements of bounded Lip-norm.

Proposition 2.11. Let (X, L) be a Lip-normed operator system. Let $x \in \mathcal{D}(L)$ and let r be any number in its spectrum. Then $\|x - r1\| \leq L(x)\text{diam}(X, L)$.

Proof. We can find $\sigma, \omega \in S(X)$ such that $\sigma(x - r1) = \|x - r1\|$ and $\omega(x) = r$, whence

$$\begin{aligned} \|x - r1\| &= |\sigma(x - r1) - \omega(x - r1)| \\ &= |\sigma(x) - \omega(x)| \\ &\leq L(x)\text{diam}(X, L). \end{aligned}$$

\square

Since one of the requirements for a Lip-norm is that the associated metric on the state space give rise to the weak* topology, one would hope that the associated

metrics on the matrix state spaces give rise to the respective point-norm topologies. The following result shows that this is indeed the case.

Proposition 2.12. The metric $\rho_{L,n}$ gives rise to the point-norm topology on $UCP_n(X)$.

Proof. Let

$$U_{\varphi,\Omega,\epsilon} = \{\psi \in UCP_n(X) : \|\varphi(x) - \psi(x)\| < \epsilon \text{ for all } x \in \Omega\}$$

be a basic open set in the point-norm topology, with $\varphi \in UCP_n(X)$, $\epsilon > 0$, and Ω a finite subset of A . For each $x \in \Omega$ pick $y_{x,1}, y_{x,2} \in \mathcal{D}(L)$ with $\|y_{x,1} - \operatorname{Re}(x)\| < \epsilon/6$ and $\|y_{x,2} - \operatorname{Im}(x)\| < \epsilon/6$, and choose $M > 0$ so that $M \geq \max_{x \in \Omega, j=1,2} L(y_{x,j})$. Now if $\psi \in UCP_n(X)$ and $\rho_{L,n}(\varphi, \psi) < (6M)^{-1}\epsilon$ then $\|\varphi(y_{x,j}) - \psi(y_{x,j})\| < \epsilon/6$ for all $x \in \Omega$ and $j = 1, 2$, and hence

$$\begin{aligned} \|\varphi(\operatorname{Re}(x)) - \psi(\operatorname{Re}(x))\| &\leq \|\varphi(\operatorname{Re}(x)) - \varphi(y_{x,1})\| + \|\varphi(y_{x,1}) - \psi(y_{x,1})\| \\ &\quad + \|\psi(y_{x,1}) - \psi(\operatorname{Re}(x))\| \\ &< \epsilon/2 \end{aligned}$$

and similarly $\|\varphi(\operatorname{Im}(x)) - \psi(\operatorname{Im}(x))\| < \epsilon/2$, so that $\|\varphi(x) - \psi(x)\| < \epsilon$. Thus $U_{\varphi,\Omega,\epsilon}$ contains the open $\rho_{L,n}$ -ball centred at φ with radius $(6M)^{-1}\epsilon$, from which it follows that the metric topology is finer than the point-norm topology.

Suppose now that $\mathcal{B}(\varphi, \epsilon)$ is the $\rho_{L,n}$ -ball centred at some $\varphi \in UCP_n(X)$ with radius some $\epsilon > 0$. Note that $\mathcal{D}_1(L) \cap \mathcal{B}_{\operatorname{diam}(S(X))}^X$ is compact, since $\mathcal{D}_1(L)$ is closed by the definition of a Lip-normed operator system and by [10, Thm. 4.5] $\mathcal{D}_1(L) \cap \mathcal{B}_1^X$ is totally bounded, which implies the total boundedness of $\mathcal{D}_1(L) \cap \mathcal{B}_{\operatorname{diam}(S(X))}^X$ via a scaling argument. Hence we can find a finite set $\Omega \subset \mathcal{D}_1(L) \cap \mathcal{B}_{\operatorname{diam}(S(X))}^X$ which is $(\epsilon/3)$ -dense in $\mathcal{D}_1(L) \cap \mathcal{B}_{\operatorname{diam}(S(X))}^X$. Thus if $\psi \in UCP_n(X)$ and $\|\varphi(x) - \psi(x)\| < \epsilon/3$ for all $x \in \Omega$ then $\|\varphi(x) - \psi(x)\| < \epsilon$ for all $x \in \mathcal{D}_1(L) \cap \mathcal{B}_{\operatorname{diam}(S(X))}^X$, and so $\mathcal{B}(\varphi, \epsilon)$ contains the point-norm basic open set

$$\{\psi \in UCP_n(X) : \|\varphi(x) - \psi(x)\| < \epsilon/3 \text{ for all } x \in \Omega\}.$$

We conclude that the metric and point-norm topologies coincide on $UCP_n(X)$. \square

We round out this section by showing that matrix state spaces embed isometrically under quotient maps, as do state spaces in the quantum metric setting. This will be crucial for the application of Hausdorff distance in formulating our matrix version of quantum Gromov-Hausdorff distance.

Proposition 2.13. Let (X, L) be a Lip-normed operator system, Y an operator system, $n \in \mathbb{N}$, $\Phi : X \rightarrow Y$ a unital n -positive map with $\Phi(\mathcal{D}(L))$ dense in Y_{sa} , and L_Y the quotient Lip-norm on Y induced by L via Φ . Then the map $\Gamma : UCP_n(Y) \rightarrow UCP_n(X)$ given by $\Gamma(\varphi) = \varphi \circ \Phi$ is an isometry with respect to $\rho_{L,n}$ and $\rho_{L_Y,n}$.

Proof. Suppose Φ is n -positive and let $\varphi, \psi \in UCP_n(Y)$. Since Φ is Lip-norm-decreasing, we have $\rho_{L_Y,n}(\varphi, \psi) \geq \rho_{L,n}(\varphi \circ \Phi, \psi \circ \Phi)$. For the reverse inequality, let $\epsilon > 0$ and choose $y \in \mathcal{D}_1(L_Y)$ such that

$$\rho_{L_Y,n}(\varphi, \psi) < \|\varphi(y) - \psi(y)\| + \epsilon.$$

We may assume $L(y) < 1$ for otherwise we can replace y with μy for some $\mu < 1$ sufficiently close to 1. Then by definition of the quotient Lip-norm there is an $x \in \mathcal{D}_1(L)$ such that $\Phi(x) = y$, and so

$$\begin{aligned} \rho_{L_Y, n}(\varphi, \psi) &< \|(\varphi \circ \Phi)(x) - (\psi \circ \Phi)(x)\| + \epsilon \\ &\leq \rho_{L, n}(\varphi \circ \Phi, \psi \circ \Phi) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we conclude that $\rho_{L_Y, n}(\varphi, \psi) = \rho_{L, n}(\varphi \circ \Phi, \psi \circ \Phi)$, so that Γ is an isometry with respect to $\rho_{L_Y, n}$ and $\rho_{L, n}$. \square

3. n -DISTANCE AND COMPLETE DISTANCE

The definition of quantum Gromov-Hausdorff distance [10] involves forming a direct sum and considering Lip-norms thereupon which induce the given Lip-norms on the summands. One then takes an infimum of the Hausdorff distances between the state spaces under their isometric embeddings into the state space of the direct sum. We will apply the same procedure here with regard to the matrix state spaces.

Notation 3.1. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems. We denote by $\mathcal{M}(L_X, L_Y)$ the collection of closed Lip-norms on $\mathcal{D}(L_X) \oplus \mathcal{D}(L_Y)$ which induce L_X and L_Y via the quotient maps onto $\mathcal{D}(L_X)$ and $\mathcal{D}(L_Y)$, respectively.

Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems and $L \in \mathcal{M}(L_X, L_Y)$. Since the projection map $X \oplus Y \rightarrow X$ is u.c.p., by Proposition 2.13 we obtain an isometry $UCP_n(X) \rightarrow UCP_n(X \oplus Y)$ with respect to ρ_{L_X} and ρ_L . Similarly, we also have an isometry $UCP_n(Y) \rightarrow UCP_n(X \oplus Y)$. For notational simplicity we will thus identify $UCP_n(X)$ and $UCP_n(Y)$ with their respective images under these isometries.

Definition 3.2. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems. For each $n \in \mathbb{N}$ we define the n -distance

$$\text{dist}_s^n(X, Y) = \inf_{L \in \mathcal{M}(L_X, L_Y)} \text{dist}_H^{\rho_L, n}(UCP_n(X), UCP_n(Y))$$

where $\text{dist}_H^{\rho_L, n}$ denotes Hausdorff distance with respect to the metric $\rho_{L, n}$. We also define the complete distance

$$\text{dist}_s(X, Y) = \inf_{L \in \mathcal{M}(L_X, L_Y)} \sup_{n \in \mathbb{N}} \text{dist}_H^{\rho_L, n}(UCP_n(X), UCP_n(Y)).$$

The reason for defining the complete distance as above and not by taking the supremum over the n -distances is the desire for a closer conceptual and technical affinity with Rieffel's quantum Gromov-Hausdorff distance, whereby a single distance quantity (in our case a supremum of Gromov-Hausdorff distances) is gauged for each $L \in \mathcal{M}(L_X, L_Y)$ and then an infimum taken. Notice that

$$\text{dist}_s(X, Y) \geq \sup_{n \in \mathbb{N}} \text{dist}_s^n(X, Y)$$

and in particular that zero complete distance implies zero n -distance for all $n \in \mathbb{N}$.

Lemma 3.3. If (X, L_X) and (Y, L_Y) are Lip-normed operator systems and $m > n \geq 1$, then $\text{dist}_s^n(X, Y) \leq \text{dist}_s^m(X, Y)$.

Proof. Let $L \in \mathcal{M}(L_X, L_Y)$ and $\varphi \in UCP_n(X)$. Let ω be any state on X , and define $\tilde{\varphi} \in UCP_m(X)$ by setting $\tilde{\varphi}(x) = \varphi(x) + \omega(x)p$ for all $x \in X$, where M_n has been identified with the upper left-hand corner of M_m and p is the unit for the lower right $(m-n) \times (m-n)$ corner. Choose $\psi' \in UCP_m(Y)$ with $\rho_{L,m}(\varphi', \psi') \leq \text{dist}_s^m(X, Y)$. If ψ is the cut-down of ψ' to the upper-left hand $n \times n$ corner of M_m , then viewing it as an element of $UCP_n(X)$ we evidently have

$$\rho_{L,n}(\varphi, \psi) \leq \rho_{L,m}(\varphi', \psi').$$

Hence $\text{dist}_H^{\rho_{L,n}}(UCP_n(X), UCP_n(Y)) \leq \text{dist}_H^{\rho_{L,m}}(UCP_m(X), UCP_m(Y))$, and so we conclude that $\text{dist}_s^n(X, Y) \leq \text{dist}_s^m(X, Y)$. \square

The inequality $\text{dist}_s^n(X, Y) \leq \text{dist}_s^m(X, Y)$ in Lemma 3.3 can be strict. For instance, consider any Lip-normed separable unital C^* -algebra (A, L) such that A is not $*$ -isomorphic to its opposite algebra A^{op} (see [8] for examples of such C^* -algebras, and note that by Proposition 1.1 of [9] we can always Lip-norm a separable unital C^* -algebra). Then we obtain another Lip-normed unital C^* -algebra (A^{op}, L) using the canonical isomorphism between A and A^{op} as order-unit spaces, and $\text{dist}_s^1(A, A^{\text{op}}) = 0$ by Theorem 12.11 of [10]. On the other hand, since A and A^{op} are not $*$ -isomorphic we have $\text{dist}_s^2(A, A^{\text{op}}) \neq 0$ by Corollary 4.11(ii) in the next section.

Proposition 3.4 (triangle inequality). If (X, L_X) , (Y, L_Y) , and (Z, L_Z) are Lip-normed operator systems then

$$\text{dist}_s^n(X, Z) \leq \text{dist}_s^n(X, Y) + \text{dist}_s^n(Y, Z),$$

for all $n \in \mathbb{N}$, and

$$\text{dist}_s(X, Z) \leq \text{dist}_s(X, Y) + \text{dist}_s(Y, Z).$$

Proof. This follows by exactly the same argument used for quantum Gromov-Hausdorff distance in [10], since in the last part of the proof of [10, Thm. 4.3] we can replace the state spaces by matrix state spaces and the reference to [10, Prop. 3.1] by a reference to our Proposition 2.13. \square

In order to build a general framework for estimating distance between quantum metric spaces, Rieffel formulates in [10, Defn. 5.1] the notion of a *bridge*, which we now recall.

Definition 3.5. Let (A, L_A) and (B, L_B) be compact quantum metric spaces. A *bridge* between (A, L_A) and (B, L_B) is a norm-continuous semi-norm N on $A \oplus B$ such that

- (i) $N(1_A, 1_B) = 0$ while $N(1_A, 0) \neq 0$, and
- (ii) for each $a \in A$ and $\delta > 0$ there exists a $b \in B$ such that

$$\max(L_B(b), N(a, b)) \leq L_A(a) + \delta,$$

with the same statement also holding upon interchanging A and B .

Theorem 5.2 in [10] then shows that if N is a bridge between the compact quantum metric spaces (A, L_A) and (B, L_B) then

$$L(a, b) = \max(L_A(a), L_B(b), N(a, b))$$

defines a Lip-norm L on $A \oplus B$ which induces L_A and L_B via the respective quotient maps. Since N is norm-continuous, L will be closed if L_A and L_B are both closed.

If (X, L_X) and (Y, L_Y) are Lip-normed operator systems then by a *bridge between (X, L_X) and (Y, L_Y)* we will mean a bridge between the compact quantum metric spaces $(\mathcal{D}(L_X), L_X)$ and $(\mathcal{D}(L_Y), L_Y)$. We begin by illustrating this notion with a simple example which shows that if we scale a Lip-norm by a factor λ and let $\lambda \rightarrow \infty$ then we obtain convergence to a ‘‘point,’’ just as for ordinary metric spaces.

Example 3.6. Let (X, L) be a Lip-normed operator system. For each $\lambda > 0$ define the Lip-normed operator system (X, L_λ) by setting $L_\lambda = \lambda L$. Let (\mathbb{C}, P) be the ‘‘one-point’’ Lip-normed operator system, with $P(\mu) = 0$ for all $\mu \in \mathbb{C}$. Then

$$\text{dist}_s((X, L), (\mathbb{C}, P)) \leq C\lambda^{-1}$$

where $C = \text{diam}(X, L)$. To show this we define a bridge on $\mathcal{D}(L_X) \oplus \mathbb{C}$ by

$$N_\lambda(x, \mu) = C^{-1}\lambda\|x - \mu 1_X\|.$$

To see that this is indeed a bridge we verify condition (ii) in Definition 3.5 by observing that $N_\lambda(\mu 1_X, \mu) = 0$ for all $\mu \in \mathbb{C}$ while if $x \in \mathcal{D}(L_X)$ then letting r denote the infimum of the spectrum of x we have by Proposition 2.11

$$\|x - r 1_X\| \leq \text{diam}(X, L)L_X(x) = C\lambda^{-1}L_\lambda(x).$$

Let M_λ be the Lip-norm in $\mathcal{M}(L_\lambda, P)$ given by $M_\lambda(x, \mu) = \max(L_\lambda(x), P(\mu), N_\lambda(x, \mu))$. Let $n \in \mathbb{N}$ and let ψ be the unique element in $UCP_n(\mathbb{C})$. If $\varphi \in UCP_n(X)$ and $(x, \mu) \in \mathcal{D}_1(M)$ then

$$\|\varphi(x) - \psi(\mu)\| = \|\varphi(x - \mu 1_X)\| \leq \|x - \mu 1_X\| \leq C\lambda^{-1},$$

yielding the desired complete distance estimate. Hence (X, L_λ) converges to (\mathbb{C}, P) as $\lambda \rightarrow \infty$ for complete distance.

As in the quantum metric setting [10, Prop. 5.4] we can apply the concept of a bridge to show that the complete distance (and hence also the n -distance) is always finite.

Proposition 3.7. If (X, L_X) and (Y, L_Y) are Lip-normed operator systems then

$$\text{dist}_s(X, Y) \leq \text{diam}(X, L_X) + \text{diam}(Y, L_Y).$$

Proof. As in the proof of [10, Prop. 5.4], for arbitrary $\gamma > 0$, $\sigma_0 \in S(X)$, and $\omega_0 \in S(Y)$ we can construct a bridge

$$N(x, y) = \gamma^{-1}|\sigma_0(x) - \omega_0(y)|.$$

Let L be the Lip-norm in $\mathcal{M}(L_X, L_Y)$ given by $L(x, y) = \max(L_X(x), L_Y(y), N(x, y))$. Then if $(x, y) \in \mathcal{D}_1(L)$, $\varphi \in UCP_n(X)$, and $\psi \in UCP_n(Y)$ we can find a $\sigma \in S(M_n)$ such that $\|\varphi(x) - \psi(y)\| = |(\sigma \circ \varphi)(x) - (\sigma \circ \psi)(y)|$ whence

$$\|\varphi(x) - \psi(y)\| \leq |(\sigma \circ \varphi)(x) - \sigma_0(x)| + |\sigma_0(x) - \omega_0(y)| + |\omega_0(y) - (\sigma \circ \psi)(y)|$$

$$\leq \rho_{L_X,1}(\sigma \circ \varphi, \sigma_0) + \gamma + \rho_{L_Y,1}(\omega_0, \sigma \circ \psi).$$

Since γ was arbitrary the proposition follows. \square

Propositions 3.8 and 3.9 yield estimates on the complete distance in situations involving bridges constructed via the norm.

Proposition 3.8. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems, and suppose X and Y are operator subsystems of an operator system Z . Let $\epsilon > 0$ and suppose that $N(x, y) = \epsilon^{-1}\|x - y\|$ defines a bridge between (X, L_X) and (Y, L_Y) . Then $\text{dist}_s(X, Y) \leq \epsilon$.

Proof. Let L be the Lip-norm in $\mathcal{M}(L_X, L_Y)$ given by

$$L(x, y) = \max(L_X(x), L_Y(y), \epsilon^{-1}\|x - y\|)$$

for all $(x, y) \in \mathcal{D}(L_X) \oplus \mathcal{D}(L_Y)$. Let $n \in \mathbb{N}$ and $\varphi \in UCP_n(X)$. By Arveson's extension theorem we can extend φ to a u.c.p. map $\tilde{\varphi} : Z \rightarrow M_n$. Then if $(x, y) \in \mathcal{D}_1(L)$ we have

$$\|\varphi(x) - \tilde{\varphi}(y)\| \leq \|x - y\| < \epsilon$$

and thus $\rho_{L,n}(\varphi, \tilde{\varphi}|_Y) < \epsilon$. We can interchange the roles of X and Y and apply the same argument to conclude that $\text{dist}_s(X, Y) \leq \epsilon$. \square

Proposition 3.9. Let (X, L_X) be a Lip-normed operator system, Y an operator system, $\Phi : X \rightarrow Y$ a surjective unital positive map, and L_Y the quotient Lip-norm induced by L_X via Φ . Let Z be an operator system containing X as an operator subsystem and let $\Gamma : Y \rightarrow Z$ be a unital map such that $\|(\Gamma \circ \Phi)(x) - x\| \leq \epsilon$ for all $x \in \mathcal{D}_1(L)$. If Φ and Γ are n -positive then $\text{dist}_s^n(X, Y) \leq \epsilon$, and if Φ and Γ are completely positive then $\text{dist}_s(X, Y) \leq \epsilon$.

Proof. Given $\eta > 0$ we define the bridge N between (X, L) and (Y, L_Y) by $N(x, y) = \eta^{-1}\|\Phi(x) - y\|$ for all $(x, y) \in X \oplus Y$ (to verify condition (ii) of Definition 3.5, given $x \in X$ and $\delta > 0$ we can choose $y = \Phi(x)$, while if $y \in Y$ and $\delta > 0$ we can take any $x \in X$ such that $\Phi(x) = y$ and $L_X(x) \leq L_Y(y) + \delta$). Let L be the Lip-norm in $\mathcal{M}(L_X, L_Y)$ given by

$$L(x, y) = \max(L_X(x), L_Y(y), N(x, y))$$

for all $(x, y) \in X \oplus Y$. Suppose that Φ is n -positive. Then if $\varphi \in UCP_n(Y)$ we have $\varphi \circ \Phi \in UCP_n(X)$, and so if $L(x, y) \leq 1$ then $\|\Phi(x) - y\| \leq \eta$ so that $\|(\varphi \circ \Phi)(x) - \varphi(y)\| \leq \eta$, whence $\rho_{L,n}(\varphi \circ \Phi, \varphi) \leq \eta$. On the other hand if $\varphi \in UCP_n(X)$ then, extending φ to a u.c.p. map $\tilde{\varphi} : Z \rightarrow M_n$ by Arveson's extension theorem, we have $\tilde{\varphi} \circ \Gamma \in UCP_n(Y)$, and so if $L(x, y) \leq 1$ then

$$\begin{aligned} \|\varphi(x) - (\tilde{\varphi} \circ \Gamma)(y)\| &\leq \|\tilde{\varphi}(x - (\Gamma \circ \Phi)(x))\| + \|(\tilde{\varphi} \circ \Gamma)(\Phi(x) - y)\| \\ &\leq \epsilon + \eta, \end{aligned}$$

yielding $\rho_{L,n}(\varphi, \tilde{\varphi} \circ \Gamma) \leq \epsilon + \eta$. Since η was arbitrary we conclude that $\text{dist}_s^n(X, Y) \leq \epsilon$. In the case that Φ is completely positive we can apply the above argument over all $n \in \mathbb{N}$ to obtain $\text{dist}_s(X, Y) \leq \epsilon$. \square

The following three propositions guarantee approximability by Lip-normed well-behaved finite-dimensional operator systems under conditions on the given operator system or C^* -algebra which hold in a wide range of situations.

Proposition 3.10. Let (X, L_X) be a Lip-normed nuclear operator system. Then for every $\epsilon > 0$ there is a Lip-normed operator system (Y, L_Y) such that Y is an operator subsystem of a matrix algebra and $\text{dist}_s(X, Y) \leq \epsilon$.

Proof. Let $\epsilon > 0$. Proposition 2.11 implies that

$$\mathcal{D}_1(L_X) = \left(\mathcal{D}_1(L_X) \cap \mathcal{B}_{\text{diam}(X, L_X)}^X \right) + \mathbb{R}1,$$

and thus, since X is nuclear and the set $\mathcal{D}_1(L_X) \cap \mathcal{B}_{\text{diam}(X, L_X)}^X$ is compact (see the second half of the proof of Proposition 2.12), we can find a matrix algebra M_k and u.c.p. maps $\Phi : X \rightarrow M_k$ and $\Gamma : M_k \rightarrow X$ such that $\|(\Gamma \circ \Phi)(x) - x\| \leq \epsilon$ for all $x \in \mathcal{D}_1(L_X)$. Consider the image Y of Φ and the resulting quotient Lip-norm L_Y on Y . Then by Proposition 3.9 we have $\text{dist}_s(X, Y) \leq \epsilon$, yielding the result. \square

By a proof similar to that of Proposition 3.10, we also have the following.

Proposition 3.11. Let (A, L) be a Lip-normed unital exact C^* -algebra. Then for every $\epsilon > 0$ there is a Lip-normed operator system (Y, L_Y) such that Y is an operator subsystem of a matrix algebra and $\text{dist}_s(A, Y) \leq \epsilon$.

A separable C^* -algebra A is said to be a *strong NF algebra* if it is the inductive limit of a generalized inductive system $(A_n, \phi_{n,m})$ with each A_n a finite-dimensional C^* -algebra and each $\phi_{n,m}$ a complete order embedding [1, Defn. 5.2.1] (a *complete order embedding* from a C^* -algebra B to a C^* -algebra A is a c.p. isometry $\Phi : A \rightarrow B$ such that $\Phi^{-1} : \Phi(A) \rightarrow B$ is a c.p. map).

Proposition 3.12. Let (A, L) be a Lip-normed unital strong NF algebra. Then for every $\epsilon > 0$ there is a Lip-normed finite-dimensional C^* -algebra (B, L_B) such that $\text{dist}_s(A, B) \leq \epsilon$.

Proof. Since the set $\mathcal{D}_1(L) \cap \mathcal{B}_{\text{diam}(A, L)}^X$ is compact (see the second half of the proof of Proposition 2.12) and A is strong NF, by [1, Thm. 6.1.1] and the fact that $\mathcal{D}_1(L_X) = (\mathcal{D}_1(L_X) \cap \mathcal{B}_{\text{diam}(X, L_X)}^X) + \mathbb{R}1$ (which follows from Proposition 2.11) we can find a finite-dimensional C^* -algebra B , a unital complete order embedding $\Gamma : B \rightarrow A$, and a (surjective) u.c.p. map $\Phi : A \rightarrow B$ such that $\Phi \circ \Gamma = \text{id}_B$ and $\|(\Gamma \circ \Phi)(a) - a\| \leq \epsilon$ for all $a \in \mathcal{D}_1(L)$. Then L induces a Lip-norm L_B on B via Φ , and $\text{dist}_s(A, B) \leq \epsilon$ by Proposition 3.9. \square

Proposition 3.12 applies for instance to noncommutative tori Lip-normed via the ergodic action of ordinary tori, as described in Example 6.8. In this situation, however, one would hope to be able to approximate by finite-dimensional C^* -algebras Lip-normed via models of the original action, as in the following example.

Example 3.13. In [11] Rieffel shows, in the context of Berezin quantization, that the sphere S^2 is a limit of matrix algebras with respect to quantum Gromov-Hausdorff distance. In fact a more general statement applying to integral coadjoint orbits of

a compact Lie group is established. We will briefly indicate how Rieffel's approach leads to precisely the same estimates for complete distance, adopting the same notation as in [11], to which we refer the reader for more details.

Given a compact group G consider the C^* -algebra B of all bounded operators on a Hilbert space on which G is irreducibly and unitarily represented. Given a rank-one projection $P \in B$ we define for each $T \in B$ the Berezin covariant symbol σ_T with respect to P by $\sigma_T(x) = \tau(T\alpha_x(P))$ where τ is the unnormalized trace on B and α is the action of G on B given by conjugation. Denoting by H the stability subgroup of P for α , we thereby obtain a map σ from B to $A = C(G/H)$ which is unital and positive, and hence u.c.p. since the range is a commutative C^* -algebra. The action α along with a length function ℓ give rise to a Lip-norm L_B on B as in Example 2.6, and similarly the action of G on G/H by left translation combines with ℓ to produce a Lip-norm L_A on A (permitting the value $+\infty$ for convenience). Corollary 2.4 of [11] shows that, for some $\gamma > 0$, there is a bridge between (A, L_A) and (B, L_B) of the form

$$N(f, T) = \gamma^{-1} \|f - \sigma_T\|_\infty.$$

Proposition 1.3 of [11] then shows that $S(A)$ is in the γ -neighbourhood of $S(B)$ under the metric defined by the Lip-norm L on $A \oplus B$ associated to N . But the argument there also applies at the matrix level: given $\varphi \in UCP_n(A)$ we have $\varphi \circ \sigma \in UCP_n(B)$ since σ is u.c.p., and if $L(f, T) \leq 1$ then $\|f - \sigma_T\|_\infty \leq \gamma$ so that

$$\|\varphi(f) - (\varphi \circ \sigma)(T)\| = \|\varphi(f - \sigma_T)\| \leq \|f - \sigma_T\|_\infty \leq \gamma,$$

showing that $UCP_n(A)$ lies in the γ -neighbourhood of $UCP_n(B)$ with respect to the metric $\rho_{L,n}$. Now on the other hand if $\psi \in UCP_n(B)$ then we can consider the adjoint operator $\check{\sigma} : A \rightarrow B$ and take the composition $\psi \circ \check{\sigma}$, which is in $UCP_n(A)$ since $\check{\sigma}$ is unital and positive and hence u.c.p. because its domain is a commutative C^* -algebra. Then if $L(f, T) \leq 1$ we have $\|f - \sigma_T\|_\infty \leq \gamma$ from which we obtain the estimate

$$\begin{aligned} \|(\psi \circ \check{\sigma})(f) - \psi(T)\| &= \|\psi(\check{\sigma}_f - T)\| \\ &\leq \|\check{\sigma}_f - T\| \\ &\leq \|\check{\sigma}_f - \check{\sigma}(\sigma_T)\| + \|\check{\sigma}(\sigma_T) - T\| \\ &\leq \|f - \sigma_T\|_\infty + \|\check{\sigma}(\sigma_T) - T\| \\ &\leq \gamma + \|\check{\sigma}(\sigma_T) - T\| \end{aligned}$$

exactly as in the case $n = 1$ in the discussion following [11, Cor. 2.4]. When G is a compact Lie group Rieffel shows that, by replacing B with the C^* -algebra of bounded operators on the m th tensor power of the original Hilbert space, both the bridge constant γ and the term $\|\check{\sigma}(\sigma_T) - T\|$ can be made arbitrarily small by taking m sufficiently large (see Theorem 3.2 and Sections 3–5 of [11]), yielding an asymptotically vanishing bound on the quantum Gromov-Hausdorff distance as a result of the estimates in the two previous displays for the case $n = 1$. But since these estimates apply equally well for all n we get the same bounds for complete distance.

4. DISTANCE ZERO

This section is aimed at establishing that $\text{dist}_s^n(X, Y) = 0$ (resp. $\text{dist}_s(X, Y) = 0$) is equivalent to the existence of a bi-Lip-isometric unital n -order isomorphism (resp. bi-Lip-isometric unital complete order isomorphism) between X and Y (Theorem 4.10). One direction is straightforward:

Proposition 4.1. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems. If there is a bi-Lip-isometric unital n -order isomorphism between X and Y then $\text{dist}_s^n(X, Y) = 0$. If there is a bi-Lip-isometric unital complete order isomorphism between X and Y then $\text{dist}_s(X, Y) = 0$.

Proof. This follows from Proposition 3.9, taking Φ there to be a bi-Lip-isometric unital n -order isomorphism (resp. bi-Lip-isometric unital complete order isomorphism) from X onto Y and taking Γ to be its inverse. \square

For the converse, it will be convenient to extend our Lip-norms in an adjoint-invariant way (Definition 4.2) and to introduce a collection of matrix semi-norms (Definition 4.4).

Definition 4.2. Let (X, L) be a Lip-normed operator system. We define the semi-norm L^e on X by

$$L^e(x) = \sup \left\{ \frac{|\sigma(x) - \omega(x)|}{\rho_{L,1}(\sigma, \omega)} : \sigma, \omega \in S(X) \text{ and } \sigma \neq \omega \right\}$$

for all $x \in X$ (permitting the value $+\infty$).

Proposition 4.3. The set of self-adjoint elements on which the semi-norm L^e in Definition 4.2 is finite coincides with the domain of L , and on this set $L^e = L$.

Proof. By Proposition 6.1 of [10], $\mathcal{D}(L)$ corresponds to the subspace of affine functions on $S(\mathcal{D}(L))$ which are Lipschitz for ρ_L . This, along with the fact that the restriction map from $S(X)$ to $S(\mathcal{D}(L))$ is a weak* homeomorphism which is isometric for $\rho_{L,1}$ and ρ_L (see the proof of Proposition 2.5), implies the result. \square

Definition 4.4. Let (X, L) be a Lip-normed operator system and $n \in \mathbb{N}$. We define the semi-norm L^n on $M_n \otimes X$ by

$$L^n(x) = \max_{1 \leq i, j \leq n} L^e(x_{ij})$$

for all $x = (x_{ij}) \in M_n(X) \cong M_n \otimes X$.

For the meaning of the notation $\mathcal{D}(\cdot)$ and $\mathcal{D}_\lambda(\cdot)$, as will be applied to the semi-norms L^e and L^n , see Notation 2.2.

Lemma 4.5. Let (X, L) be a Lip-normed operator system. Then $\mathcal{D}(L^e)$ is dense in X and $\mathcal{D}(L^n)$ is dense in $M_n \otimes X$ for all $n \in \mathbb{N}$.

Proof. Since L^e is adjoint-invariant and coincides with L on the dense real subspace $\mathcal{D}(L)$ of X_{sa} by Proposition 4.3, using the decomposition of elements into real and imaginary parts we see that $\mathcal{D}(L^e)$ is dense in X . As a direct consequence $\mathcal{D}(L^n)$ is dense in $M_n \otimes X$ for all $n \in \mathbb{N}$. \square

Lemma 4.6. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems, and let $L \in \mathcal{M}(L_X, L_Y)$ and $n \in \mathbb{N}$. Set $r = \text{dist}_H^{\rho_{L,n}}(UCP_n(X), UCP_n(Y))$. Then for every $\psi \in SCP_n(Y)$ there is a $\varphi \in SCP_n(X)$ such that, for all $(x, y) \in \mathcal{D}(L^e)$,

$$\|\varphi(x) - \psi(y)\| \leq 2n^3 L^e(x, y)r.$$

Proof. To prove the lemma we may assume that $\psi(1)$ has full support in M_n , for otherwise for every $k \in \mathbb{N}$ we can perturb ψ to a convex combination $(1 - k^{-1})\psi + k^{-1}\alpha$ where $\alpha(x) = \omega(x)1_{M_n}$ for some $\omega \in S(Y)$ and all $x \in X$ (in which case the corresponding state σ_ψ on $M_n \otimes Y$ is perturbed to another a state), find a suitable φ_k as in the lemma statement with respect to ψ_k , and then take a point-norm limit point of $\{\varphi_k\}_{k \in \mathbb{N}}$ to obtain the desired ϕ . We can thus consider the map $\psi' \in UCP_n(Y)$ given by

$$\psi'(y) = \psi(1)^{-\frac{1}{2}}\psi(y)\psi(1)^{\frac{1}{2}}.$$

for all $y \in Y$. By assumption we can find a $\varphi' \in UCP_n(X)$ such that

$$\rho_{L,n}(\varphi', \psi') \leq r.$$

Let $\varphi : X \rightarrow M_n$ be the c.p. map given by

$$\varphi(x) = \psi(1)^{\frac{1}{2}}\varphi'(x)\psi(1)^{\frac{1}{2}}$$

for all $x \in X$. Then $\varphi(1) = \psi(1)$, which implies that σ_φ is a state on $M_n \otimes X$, so that $\varphi \in SCP_n(X)$. Since σ_ψ is a state on $M_n \otimes Y$ we must have $\|\psi(1)\| \leq n^3$, and thus if $(x, y) \in \mathcal{D}(L^e)$ then

$$\begin{aligned} \|\varphi(x) - \psi(y)\| &= \|\psi(1)^{\frac{1}{2}}(\varphi'(x) - \psi'(y))\psi(1)^{\frac{1}{2}}\| \\ &\leq \|\psi(1)^{\frac{1}{2}}\| \|\varphi'(x) - \psi'(y)\| \|\psi(1)^{\frac{1}{2}}\| \\ &\leq n^3(\|\varphi'(\text{Re}(x)) - \psi'(\text{Re}(y))\| + \|\varphi'(\text{Im}(x)) - \psi'(\text{Im}(y))\|) \\ &\leq n^3(L(\text{Re}(x), \text{Re}(y)) + L(\text{Im}(x), \text{Im}(y)))\rho_{L,n}(\varphi', \psi') \\ &\leq 2n^3 L^e(x, y)\rho_{L,n}(\varphi', \psi') \\ &\leq 2n^3 L^e(x, y)r \end{aligned}$$

with the second last inequality following from the adjoint invariance of L^e and the fact that $L^e = L$ on $\mathcal{D}(L)$ by Proposition 4.3. \square

Definition 4.7. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems, and let $L \in \mathcal{M}(L_X, L_Y)$ and $n \in \mathbb{N}$. For each $\lambda \geq 0$, $x \in \mathcal{D}(L_X^\lambda)$, and $y \in \mathcal{D}(L_Y^\lambda)$ we set

$$\begin{aligned} \mathcal{N}_{L^n, Y}^\lambda(x) &= \{z \in M_n \otimes Y : (x, z) \in \mathcal{D}_\lambda(L^n)\}, \\ \mathcal{N}_{L^n, X}^\lambda(y) &= \{z \in M_n \otimes X : (z, y) \in \mathcal{D}_\lambda(L^n)\}, \end{aligned}$$

Lemma 4.8. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems, and let $L \in \mathcal{M}(L_X, L_Y)$ and $n \in \mathbb{N}$. Set $r = \text{dist}_H^{\rho_{L,n}}(UCP_n(X), UCP_n(Y))$. If $x \in \mathcal{D}(L_X^\lambda)$ and $\lambda > 2L_X^n(x)$ then

- (i) $\mathcal{N}_{L^n, Y}^\lambda(x)$ is non-empty and closed,

- (ii) the norms of elements of $\mathcal{N}_{L^n, Y}^\lambda(x)$ are bounded by $4(\|x\| + \lambda n^4 r)$, and if x and $y \in \mathcal{N}_{L^n, Y}^\lambda(x)$ are self-adjoint then the norm of y is bounded by $\|x\| + 2\lambda n^4 r$,
- (iii) the norm diameter of $\mathcal{N}_{L^n, Y}^\lambda(x)$ is bounded by $8\lambda n^4 r$,
- (iv) if x' is another element of $\mathcal{D}(L_X^n)$ and λ is also strictly larger than $L_X^n(x')$ then the Hausdorff distance between $\mathcal{N}_{L^n, Y}^\lambda(x)$ and $\mathcal{N}_{L^n, Y}^\lambda(x')$ is bounded by $8\lambda n^4 r + 4\|x - x'\|$, and
- (v) if $x \geq 0$ then there is a (self-adjoint) $y \in \mathcal{N}_{L^n, Y}^\lambda(x)$ with $y \geq -2\lambda n^4 r$.

Statements (i)–(v) also hold for $y \in \mathcal{D}(L_Y^n)$ and $\lambda > 2L_Y^n(y)$ if L_X and $\mathcal{N}_{L^n, Y}^\lambda(x)$ are replaced by L_Y and $\mathcal{N}_{L^n, X}^\lambda(y)$, respectively.

Proof. Since symmetry will take care of the last sentence of the proposition statement, we prove (i)–(v) as written for $x \in \mathcal{D}(L^n)$ and $\lambda > 2L_X^n(x)$. We begin with (i). For $1 \leq i, j \leq n$ we have $\max(L_X(\operatorname{Re}(x_{ij})), L_X(\operatorname{Im}(x_{ij}))) < L^e(x_{ij}) < \lambda/2$. We can thus find $(y_{ij}) \in \mathcal{N}_{L^n, Y}^{\lambda/2}(\operatorname{Re}((x_{ij})))$ and $(z_{ij}) \in \mathcal{N}_{L^n, Y}^{\lambda/2}(\operatorname{Im}((x_{ij})))$ since L_X is the quotient Lip-norm induced by L . Then

$$\begin{aligned} L^n((x_{ij}), (y_{ij}) + i(z_{ij})) &\leq L^n((\operatorname{Re}((x_{ij}))), (y_{ij})) + L^n((\operatorname{Im}((x_{ij}))), (z_{ij})) \\ &< \lambda \end{aligned}$$

so that $\mathcal{N}_{L^n, Y}^{\lambda/2}(x)$ contains $(y_{ij}) + i(z_{ij})$ and is in particular non-empty. That this set is also closed follows from the lower semicontinuity of L^e , which is easily checked.

For (ii), let $y = (y_{ij}) \in \mathcal{N}_{L^n, Y}^\lambda(x)$ and $\psi \in SCP_n(Y)$. By Lemma 4.6 there is a $\varphi \in SCP_n(X)$ such that $\|\varphi(z) - \psi(w)\| \leq 2n^3 L^e(z, w)r$ for all $z, w \in \mathcal{D}(L^e)$. We then have

$$\begin{aligned} |\sigma_\varphi(x) - \sigma_\psi(y)| &= \frac{1}{n} \left| \sum_{i,j} (\varphi(x_{ij})_{ij} - \psi(y_{ij})_{ij}) \right| \\ &\leq \frac{1}{n} \sum_{i,j} |\varphi(x_{ij})_{ij} - \psi(y_{ij})_{ij}| \\ &\leq \frac{1}{n} \sum_{i,j} \|\varphi(x_{ij}) - \psi(y_{ij})\| \\ &\leq 2n^2 \sum_{i,j} L^e(x_{ij}, y_{ij})r \\ &\leq 2n^4 L^n(x, y)r \\ &\leq 2\lambda n^4 r. \end{aligned}$$

It follows that $|\sigma_\psi(y)| \leq 2(\|x\| + \lambda n^4 r)$, and so $|\sigma_\psi(\operatorname{Re}(y))|$ and $|\sigma_\psi(\operatorname{Im}(y))|$ are both bounded by $2(\|x\| + \lambda n^4 r)$, from which we conclude that

$$\|y\| \leq \|\operatorname{Re}(y)\| + \|\operatorname{Im}(y)\| \leq 4(\|x\| + \lambda n^4 r).$$

If x and $y \in \mathcal{N}_{L^n, Y}^\lambda(x)$ are self-adjoint then the above argument shows that the norm of y is in fact bounded by $\|x\| + 2\lambda n^4 r$.

To establish (iii), suppose $y, y' \in M_n \otimes Y$ are such that $(x, y), (x, y') \in \mathcal{D}_\lambda(L^e)$. Let $\psi \in SCP_n(Y)$. Arguing as in the proof of (ii), there exists by Lemma 4.6 a $\varphi \in SCP_n(X)$ such that $\|\varphi(z) - \psi(w)\| \leq n^3 L^e(z, w)r$ for all $z, w \in \mathcal{D}(L^e)$, so that both $|\sigma_\varphi(x) - \sigma_\psi(y)|$ and $|\sigma_\varphi(x) - \sigma_\psi(y')|$ are bounded by $2\lambda n^4 r$, whence $|\sigma_\psi(y - y')| \leq 4\lambda n^4 r$. It follows that $\|y - y'\| \leq 8\lambda n^4 r$, and so we obtain (iii).

For (iv), suppose $y \in \mathcal{N}_{L^n, Y}^\lambda(x)$ and $y' \in \mathcal{N}_{L^n, Y}^\lambda(x')$. Then as in the proof of (ii) we can find a $\varphi \in SCP_n(X)$ such that both $|\sigma_\varphi(x) - \sigma_\psi(y)|$ and $|\sigma_\varphi(x') - \sigma_\psi(y')|$ are bounded by $2\lambda n^4 r$, and the triangle inequality yields

$$\begin{aligned} |\sigma_\psi(y) - \sigma_\psi(y')| &\leq 4\lambda n^4 r + |\sigma_\varphi(x) - \sigma_\varphi(x')| \\ &\leq 4\lambda n^4 r + 2\|x - x'\|. \end{aligned}$$

Hence $\|y - y'\| \leq 8\lambda n^4 r + 4\|x - x'\|$, from which (iv) follows.

Finally, to prove (v) we suppose $x \geq 0$. By part (i) there is a $y = (y_{ij}) \in \mathcal{N}_{L^n, Y}^\lambda(x)$. We then have, for $1 \leq i, j \leq n$,

$$\begin{aligned} L(x_{ij}, (\operatorname{Re}(y))_{ij}) &\leq \frac{1}{2}L^e(x_{ij}, y_{ij}) + \frac{1}{2}L^e(x_{ji}^*, y_{ji}^*) \\ &= \frac{1}{2}L^e(x_{ij}, y_{ij}) + \frac{1}{2}L^e(x_{ji}, y_{ji}) \\ &\leq L^n(x, y) \end{aligned}$$

using the adjoint invariance of L^e , and so $\operatorname{Re}(y)$ is a self-adjoint element of $\mathcal{N}_{L^n, Y}^\lambda(x)$. Suppose now that y is an arbitrary self-adjoint element of $\mathcal{N}_{L^n, Y}^\lambda(x)$. If $\psi \in SCP_n(Y)$ then as in the proof of (ii) there is a $\varphi \in SCP_n(X)$ such that $|\sigma_\varphi(x) - \sigma_\psi(y)| \leq 2\lambda n^4 r$, and thus since $\sigma_\varphi(x) \geq 0$ and y is self-adjoint we infer that $\sigma_\varphi(y) \geq -2\lambda n^4 r$. Hence we conclude that $y \geq -2\lambda n^4 r$. \square

Proposition 4.9. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems, and suppose $\operatorname{dist}_s^n(X, Y) = 0$ for some $n \in \mathbb{N}$. Then there is a unital order isomorphism $\Phi : M_n \otimes X \rightarrow M_n \otimes Y$, and in the case $n = 1$ we may arrange that $\mathcal{D}(L_Y) = \Phi(\mathcal{D}(L_X))$ and $L_Y(\Phi(x)) = L_X(x)$ for all $x \in \mathcal{D}(L_X)$.

Proof. By assumption there is a sequence $\{L_k\}_{k \in \mathbb{N}}$ of Lip-norms in $\mathcal{M}(L_X, L_Y)$ such that $\lim_{k \rightarrow \infty} r_k = 0$ where $r_k = \operatorname{dist}_H^{\rho_{L_k, n}}(UCP_n(X), UCP_n(Y))$. Let $x \in \mathcal{D}(L_X^n)$ and $\lambda > 2L_X^n(x)$. Set $s = 4(\|x\| + 2\lambda n^4)$. In view of Lemma 4.8(ii) we may assume (by removing finitely many of the L_k 's and reindexing the sequence if necessary) that the sets $\mathcal{N}_{L_k, X}^\lambda(x)$ for $k \geq 1$ are all contained in $\mathcal{D}_s(L_Y^n) \cap \mathcal{B}_s^{M_n \otimes Y}$. This latter set is compact, since it is closed by the lower semicontinuity of L_Y^n and for any $t > 0$ the set $\mathcal{D}_t(L_Y) \cap \mathcal{B}_t^Y$ is compact (see the second half of the proof of Proposition 2.12), which implies the compactness of the set $\mathcal{D}_t(L_Y^e) \cap \mathcal{B}_t^Y$ (use the decomposition of elements into real and self-adjoint parts) and hence also the total boundedness of the subset of $M_n(Y) \cong M_n \otimes Y$ of $n \times n$ matrices with entries in $\mathcal{D}_t(L_Y^e) \cap \mathcal{B}_t^Y$. Since by Lemma 4.8(iii) the diameters of $\mathcal{N}_{L_k, X}^\lambda(x)$ converge to zero as $k \rightarrow \infty$, this implies the existence of a subsequence of $\{\mathcal{N}_{L_k, X}^\lambda(x)\}_{k \in \mathbb{N}}$ which converges in Hausdorff distance to some singleton $\{\Phi(x)\}$. This singleton must in fact be the

same for each $\lambda > 2L_X^n(x)$ because for each $k \in \mathbb{N}$ we have

$$\mathcal{N}_{L_k^n, X}^\lambda(x) \subset \mathcal{N}_{L_k^n, X}^{\lambda'}(x)$$

whenever $\lambda \leq \lambda'$. Using a diagonal argument and relabeling indices we may assume that, for all x in a countable dense subset D of $\mathcal{D}(L_X^n)$, the sets $\mathcal{N}_{L_k^n, X}^\lambda(x)$ for $\lambda > 2L_X^n(x)$ converge in Hausdorff distance as $k \rightarrow \infty$ to some singleton $\{\Phi(x)\}$. Then in fact for any $x \in \mathcal{D}(L_X^n)$ and $\lambda > 2L_X^n(x)$ the relabeled sets $\mathcal{N}_{L_k^n, X}^\lambda(x)$ converge as $k \rightarrow \infty$ to some singleton $\{\Phi(x)\}$, since for any $\epsilon > 0$ we can take an $x' \in D$ with $\|x - x'\| \leq \epsilon/16$ and $2L_X^n(x') < \lambda$ (since we may assume that D was chosen so that $D \cap \mathcal{D}_q(L_X^n)$ is dense in $\mathcal{D}_q(L_X^n)$ for all positive rational numbers q) and a $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $\mathcal{N}_{L_k^n, X}^\lambda(x')$ is within Hausdorff distance $\epsilon/2$ of $\{\Phi(x')\}$ and $8\lambda n^4 r_k \leq \epsilon/2$, from which it can be seen using Lemma 4.8(iv) that for all $k \geq k_0$ the set $\mathcal{N}_{L_k^n, X}^\lambda(x)$ lies inside the ball of radius ϵ centred at $\Phi(x')$.

Now if $\mu \in \mathbb{C}$ and $x, x' \in \mathcal{D}(L_X^n)$ then for any $\lambda \geq 0$ it is easily seen that

$$\mathcal{N}_{L_k^n, Y}^{(|\mu|+1)\lambda}(\mu x + x') \supset \{\mu y + y' : y \in \mathcal{N}_{L_k^n, Y}^\lambda(x) \text{ and } y' \in \mathcal{N}_{L_k^n, Y}^\lambda(x')\}.$$

Thus if $\lambda > 2 \max(|\mu|L_X^n(x), L_X^n(x'))$ the sequence of sets $\mathcal{N}_{L_k^n, Y}^\lambda(\mu x + x')$, which we know to converge after the relabeling of the previous paragraph, must converge to $\{\mu\Phi(x) + \Phi(x')\}$ as $k \rightarrow \infty$. Also, for every $x \in \mathcal{D}(L_X^n)$ and $\lambda > L_X^n(x)$ we have $y^* \in \mathcal{N}_{L_k^n, Y}^\lambda(x^*)$ if and only if $y \in \mathcal{N}_{L_k^n, Y}^\lambda(x)$, so that $\{\Phi(x^*)\}$ is equal to the limit of $\{y^* : y \in \mathcal{N}_{L_k^n, Y}^\lambda(x)\}$, which by the continuity of the involution must be $\{(\Phi(x))^*\}$. Hence we have defined a *-linear map $\Phi : \mathcal{D}(L_X^n) \rightarrow M_n \otimes Y$. Note also that Φ is unital since $1_Y \in \mathcal{N}_{L_k^n, Y}^\lambda(1_X)$ for all $\lambda > 0$. We furthermore have by Lemma 4.8(ii) that the norm of Φ is bounded by 4 on D , and thus, since D is dense in $\mathcal{D}(L_X^n)$ which in turn is dense in $M_n \otimes X$ by Lemma 4.5, Φ extends uniquely to a bounded *-linear map from $M_n \otimes X$ to $M_n \otimes Y$, which we will again denote by Φ .

Now by another diagonal argument and index relabeling we may assume that $\mathcal{N}_{L_k^n, X}^\lambda(y)$ converges in Hausdorff distance as $k \rightarrow \infty$ to a singleton $\{\Gamma(y)\}$ for all y in a countable dense subset of $\mathcal{D}(L_Y^n)$ which contains $\Phi(D)$. We thus obtain, as above, a bounded unital *-linear map $\Gamma : \mathcal{D}(L_Y^n) \rightarrow \mathcal{D}(L_X^n)$. We will show that Φ and Γ are mutual inverses. Suppose $x \in \mathcal{D}(L)$ and $\lambda > 2 \max(L_X^n(x), L_Y^n(\Phi(x)))$. For each $k \in \mathbb{N}$ choose $x'_k \in \mathcal{N}_{L_k^n, X}^\lambda(\Phi(x))$ and $y_k \in \mathcal{N}_{L_k^n, Y}^\lambda(x)$. Let $\epsilon > 0$ and $\varphi \in SCP_n(Y)$. Pick $k_0 \in \mathbb{N}$ large enough so that, for all $k \geq k_0$, $2\lambda n^4 r_k \leq \epsilon$ and $\|y_k - \Phi(x)\| \leq \epsilon$. Then as in the proof of Lemma 4.8(i) for any $k \geq k_0$ we can find a $\psi \in SCP_n(X)$ such that

$$|\sigma_\varphi(x) - \sigma_\psi(y_k)| \leq 2n^4 L^\epsilon(x, y_k) r_k \leq 2n^4 \lambda r_k \leq \epsilon$$

and similarly $|\sigma_\varphi(x'_k) - \sigma_\psi(\Phi(x))| \leq \epsilon$, whence by the triangle inequality

$$\begin{aligned} |\sigma_\varphi(x) - \sigma_\varphi(x'_k)| &\leq 2\epsilon + |\sigma_\psi(y_k) - \sigma_\psi(\Phi(x))| \\ &\leq 3\epsilon. \end{aligned}$$

Therefore $\|x - x'_k\| \leq 6\epsilon$, and so we have $\lim_{k \rightarrow \infty} x'_k = x$. Hence

$$\Gamma(\Phi(x)) = \Gamma\left(\lim_{k \rightarrow \infty} y_k\right) = \lim_{k \rightarrow \infty} \Gamma(y_k) = \lim_{k \rightarrow \infty} x'_k = x.$$

By a similar argument $\Phi(\Gamma(y)) = y$ for all $y \in \mathcal{D}(L_Y^n)$, and hence by continuity we conclude that Φ and Γ are mutual inverses.

Next we show that Φ and Γ are positive. If $x \in \mathcal{D}(L_X^n)$, $x \geq 0$, and $\lambda > 2L_X^n(x)$, then Lemma 4.8(v) yields, for all $k \in \mathbb{N}$, a $y_k \in \mathcal{N}_{L_k^n, Y}^\lambda(x)$ with $y_k \geq -2\lambda n^4 r_k$. Then y is the limit as $k \rightarrow \infty$ of the positive elements $y_k + 2\lambda n^4 r_k$ and hence y itself is positive. Thus Φ is positive, and by a symmetric argument so is Γ . Hence Φ is a unital order isomorphism.

It remains to show that if $n = 1$ then Φ is isometric with respect to L_X and L_Y , that is, $\mathcal{D}(L_Y) = \Phi(\mathcal{D}(L_X))$ and $L_Y(\Phi(x)) = L_X(x)$ for all $x \in \mathcal{D}(L_X)$. Let $x \in \mathcal{D}(L_X)$ and $\lambda > \max(1, 2L_X(x))$. Suppose $\sigma, \sigma' \in S(Y)$, and let $\epsilon > 0$. Choose $k \in \mathbb{N}$ large enough so that we can find $\omega, \omega' \in S(X)$ with $\lambda \rho_{L_k, 1}(\sigma, \omega) \leq \epsilon$ and $\lambda \rho_{L_k, 1}(\sigma', \omega') \leq \epsilon$, as well as a $y_k \in \mathcal{N}_{L_k^1, X}^\lambda(x)$ with $\|\Phi(x) - y_k\| \leq \epsilon$. Then

$$\begin{aligned} |\sigma(\Phi(x)) - \omega(x)| &\leq |\sigma(\Phi(x)) - \sigma(y_k)| + |\sigma(y_k) - \omega(x)| \\ &\leq \epsilon + \lambda \rho_{L_k, 1}(\sigma, \omega) \\ &\leq 2\epsilon \end{aligned}$$

and similarly $|\sigma'(\Phi(x)) - \omega'(x)| \leq 2\epsilon$. Thus, since

$$\begin{aligned} \rho_{L_Y, 1}(\omega, \omega') &\leq \rho_{L_k, 1}(\omega, \sigma) + \rho_{L_X, 1}(\sigma, \sigma') + \rho_{L_k, 1}(\sigma', \omega') \\ &\leq \rho_{L_Y, 1}(\sigma, \sigma') + 2\epsilon, \end{aligned}$$

we have

$$\begin{aligned} |\sigma(\Phi(x)) - \sigma'(\Phi(x))| &\leq |\sigma(\Phi(x)) - \omega(x)| + |\omega(x) - \omega'(x)| + |\omega'(x) - \sigma'(\Phi(x))| \\ &\leq 4\epsilon + \rho_{L_X, 1}(\omega, \omega') L_X(x) \\ &\leq 2\epsilon(2 + L_X(x)) + \rho_{L_Y, 1}(\sigma, \sigma') L_X(x). \end{aligned}$$

Dividing by $\rho_{L_Y, 1}(\sigma, \sigma')$ and letting $\epsilon \rightarrow 0^+$, we conclude that $L_Y(\Phi(x)) \leq L_X(x)$. Since the above argument also applies to Γ we must in fact have $\mathcal{D}(L_Y) = \Phi(\mathcal{D}(L_X))$ and $L_Y(\Phi(x)) = L_X(x)$ for all $x \in \mathcal{D}(L_X)$. \square

Theorem 4.10. Let (X, L_X) and (Y, L_Y) be Lip-normed operator systems.

- (i) If $n \in \mathbb{N}$ then $\text{dist}_s^n(X, Y) = 0$ if and only if there is a bi-Lip-isometric unital n -order isomorphism between X and Y .
- (ii) We have $\text{dist}_s(X, Y) = 0$ if and only if there is a bi-Lip-isometric unital complete order isomorphism between X and Y .

Proof. Proposition 4.1 takes care of the “if” in each part, and so we need only worry about the “only if” direction. Suppose $\text{dist}_s^n(X, Y) = 0$ for some $n \in \mathbb{N}$. Then by Lemma 3.3 we have $\text{dist}_s^m(X, Y) = 0$ for each $m = 1, \dots, n$. Applying the proof of Proposition 4.9 successively for each $m = 1, \dots, n$ so that we can apply a diagonal argument across these values of m , we can find, for each $m = 1, \dots, n$, a unital order isomorphism $\Phi_m : M_m \otimes X \rightarrow M_m \otimes Y$ such that, for each $x \in M_m \otimes X$ and

$\lambda > 2L_X^m(x)$, the singleton $\{\Phi(x)\}$ is the limit of $\mathcal{N}_{L_k^m, Y}^\lambda(x)$ as $k \rightarrow \infty$ with respect to Hausdorff distance for a sequence of Lip-norms $L_k \in \mathcal{M}(L_A, L_B)$. By Proposition 4.9 we may also arrange that $\mathcal{D}(L_Y) = \Phi(\mathcal{D}(L_X))$ and $L_Y(\Phi_1(x)) = L_X(x)$ for all $x \in \mathcal{D}(L_X)$. We will show that $\Phi_m = \text{id}_m \otimes \Phi_1$ for each $m = 2, \dots, n$. Suppose then that $x \in \mathcal{D}(L_X)$. For each $k \in \mathbb{N}$ choose $y_k \in \mathcal{N}_{L_k^1, Y}^\lambda(x)$. If e_{ij} is a standard matrix unit in M_m then $L_k^m(e_{ij} \otimes x, e_{ij} \otimes y_k) = L_k^e(x, y_k)$ so that $e_{ij} \otimes y_k \in \mathcal{N}_{L_k^m, Y}^\lambda(e_{ij} \otimes x)$. Thus $\mathcal{N}_{L_k^m, Y}^\lambda(e_{ij} \otimes x)$ must converge to the singleton containing

$$\lim_{k \rightarrow \infty} e_{ij} \otimes y_k = e_{ij} \otimes \lim_{k \rightarrow \infty} y_k = (\text{id}_m \otimes \Phi_1)(e_{ij} \otimes x),$$

whence $\Phi_m(e_{ij} \otimes x) = (\text{id}_m \otimes \Phi_1)(e_{ij} \otimes x)$. Since by Lemma 4.5 the span of elements of the form $e_{ij} \otimes x$ with $x \in \mathcal{D}(L_X^e)$ is dense in $M_m \otimes X$, we conclude that $\Phi_m = \text{id}_m \otimes \Phi_1$, so that Φ_1 is a bi-Lip-isometric n -order isomorphism. We thus obtain (i).

For (ii) we can use essentially the same proof (note that $\text{dist}_s(X, Y) = 0$ implies $\text{dist}_s^n(X, Y) = 0$ for all $n \in \mathbb{N}$), with the diagonal arguments now extended across all $n \in \mathbb{N}$. \square

Corollary 4.11. Let A and B be unital C^* -algebras with Lip-norms L_A and L_B , respectively.

- (i) We have $\text{dist}_s^1(A, B) = 0$ if and only if there is a bi-Lip-isometric unital order isomorphism between A and B .
- (ii) If $n \geq 2$ then $\text{dist}_s^n(A, B) = 0$ if and only if there is a bi-Lip-isometric $*$ -isomorphism between A and B .
- (iii) We have $\text{dist}_s(A, B) = 0$ if and only if there is a bi-Lip-isometric $*$ -isomorphism between A and B .

Proof. The corollary is an immediate consequence of Theorem 4.10 and the fact that a unital 2-order isomorphism between A and B is automatically a $*$ -isomorphism [2]. \square

We remark that a unital order isomorphism between unital C^* -algebras need not be a $*$ -isomorphism. For instance, a unital C^* -algebra A is always unitaly order isomorphic to its opposite algebra A^{op} , but these need not be $*$ -isomorphic, as the examples in [8] demonstrate.

5. f -LEIBNIZ COMPLETE DISTANCE AND CONVERGENCE

Let $(\mathcal{R}, \text{dist}_s)$ be the metric space, under complete distance, of equivalence classes of Lip-normed operator systems with respect to bi-Lip-isometric unital complete order isomorphism. For economy we will simply refer to the elements of \mathcal{R} as Lip-normed operator systems. We have not been able to determine whether the metric space $(\mathcal{R}, \text{dist}_s)$ is complete (cf. Theorem 12.11 of [10]). However, we will establish in this section a kind of relative completeness within the metric subspace of Lip-normed unital C^* -algebras which asserts the convergence, with respect to complete distance, of sequences which are Cauchy with respect to a larger distance (“ f -Leibniz complete distance”) the Lip-norms in whose definition are required to satisfy a type of weak Leibniz property (“ f -Leibniz”) which introduces some control with respect

to products. Let $f : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$ be a continuous function. Given a Lip-normed unital C^* -algebra (A, L) , the Lip-norm L is said to be *f-Leibniz* if it satisfies the *f-Leibniz property*

$$\max(L(\operatorname{Re}(xy)), L(\operatorname{Im}(xy))) \leq f(L(x), L(y), \|y\|, \|x\|)$$

for all $x, y \in \mathcal{D}(L)$. If L is the restriction of an adjoint-invariant semi-norm L' on A which is finite on a dense $*$ -subalgebra and satisfies the usual Leibniz rule

$$L'(xy) \leq L'(x)\|y\| + \|x\|L'(y)$$

for all $x, y \in \mathcal{D}(L)$, then L is *f-Leibniz* for the function $f(a, b, c, d) = ac + bd$ and we simply say that L is *Leibniz*. The Lip-normed unital C^* -algebras of Example 2.6 are Leibniz, as are those obtained from Lipschitz semi-norms on functions over a compact metric space. We denote by \mathcal{R}_{alg} the subset of \mathcal{R} consisting of Lip-normed unital C^* -algebras, and for (A, L_A) and (B, L_B) in \mathcal{R}_{alg} we define the *f-Leibniz complete distance* $\operatorname{dist}_{s,f}(A, B)$ in the same way that the complete distance is defined (Definition 3.2) except that the infimum is now taken over the *f-Leibniz* Lip-norms in $\mathcal{M}(L_A, L_B)$ (if no such *f-Leibniz* Lip-norm exists we set $\operatorname{dist}_{s,f}(A, B) = \infty$). Note that $\operatorname{dist}_{s,f}$ might not satisfy the triangle inequality without further hypotheses on f , but that will not be of consequence for our application here, and we can still speak of Cauchy sequences with respect to $\operatorname{dist}_{s,f}$ in the obvious sense. It can be seen that the estimates in Example 3.13 for complete distance also apply to *f-Leibniz* complete distance for suitable f (although f may depend on the matrix algebra), and if N is a bridge between two Leibniz Lip-normed C^* -algebras of the form that appears in Proposition 3.8 then the resulting Lip-norm on the direct sum is Leibniz (see Section 6 for examples of the use of bridges like those in Proposition 3.8).

I would like to thank Narutaka Ozawa for suggesting the idea behind the proof of the following lemma. Given a sequence $\{A_k\}_{k \in \mathbb{N}}$ of C^* -algebras we denote by $\prod A_k$ the C^* -algebra of bounded sequences with the supremum norm and by $\bigoplus A_k$ the C^* -subalgebra of sequences converging to zero.

Lemma 5.1. Let $\{A_k\}_{k \in \mathbb{N}}$ be a sequence of unital C^* -algebras and X a separable operator subsystem of $\prod A_k / \bigoplus A_k$, and let $n \in \mathbb{N}$. Suppose that, for each $z \in M_n \otimes X$, at least one lift (and hence every lift) $\sum e_{ij} \otimes (z_k^{ij})_k \in M_n \otimes (\prod A_k)$ of z with respect to the quotient $(M_n \otimes (\prod A_k)) / (M_n \otimes (\bigoplus A_k)) \cong M_n \otimes (\prod A_k / \bigoplus A_k)$ satisfies

$$\lim_{k \rightarrow \infty} \left\| \sum e_{ij} \otimes z_k^{ij} \right\| = \|z\|.$$

Then for every $\varphi \in UCP_n(X)$ there are $\varphi_k \in UCP_n(A_k)$ for $k \in \mathbb{N}$ such that for all $(x_k)_k + \bigoplus A_k \in X$ we have

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \rightarrow \infty} \varphi_k(x_k).$$

Proof. First we consider an arbitrary finite-dimensional operator subsystem Y of X and show that the conclusion of the lemma holds with respect to elements of Y . Letting $\pi : \prod A_k \rightarrow \prod A_k / \bigoplus A_k$ be the quotient map, there exists, by elementary linear algebra, a unital linear map $x \mapsto \alpha(x) = (\alpha(x)_k)_k$ from X to $\prod A_k$ such that $\pi \circ \alpha = \operatorname{id}_X$. We may assume that α is Hermitian for otherwise we can replace it

with its real part $(\alpha + \alpha^*)/2$. Since Y is finite-dimensional the unit ball of $M_n \otimes Y$ is compact, and so by our assumption on lifts of elements of $M_n \otimes X$ we can find a sequence $\delta_1, \delta_2, \dots$ of positive real numbers with $\lim_{k \rightarrow \infty} \delta_k = 0$ such that, for all $\sum e_{ij} \otimes x_{ij}$ in the unit ball of $M_n \otimes Y$ and $k \in \mathbb{N}$,

$$\left\| \sum e_{ij} \otimes x_{ij} \right\| - \delta_k < \left\| \sum e_{ij} \otimes \alpha(x_{ij})_k \right\| < \left\| \sum e_{ij} \otimes x_{ij} \right\| + \delta_k.$$

This implies in particular that for each sufficiently large $k \in \mathbb{N}$ the map $\pi_k \circ \alpha$ is injective on Y , where $\pi_k : \prod A_k \rightarrow A_k$ is the projection map. Let $\varphi \in UCP_n(Y)$. For each sufficiently large $k \in \mathbb{N}$ we can define the linear map $\psi_k : (\pi_k \circ \alpha)(Y) \rightarrow M_n$ by

$$\psi_k(a) = \varphi((\pi_k \circ \alpha)^{-1}(a))$$

for all $a \in (\pi_k \circ \alpha)(Y)$. Then ψ_k is unital and Hermitian, and $\|\text{id}_n \otimes \psi_k\| \leq (1 - \delta_k)^{-1}$. By [14, Thm. 2.10] the completely bounded norm $\|\psi_k\|_{cb}$ is equal to $\|\text{id}_n \otimes \psi_k\|$ and hence is at most $(1 - \delta_k)^{-1}$. By the Wittstock extension theorem (see [17]) there is an extension of ψ_k to A_k with the same completely bounded norm. We denote this extension also by ψ_k . By the Wittstock decomposition theorem (see [17]) there exist completely positive maps ψ_k^+ and ψ_k^- from A_k to M_n such that $\psi_k = \psi_k^+ - \psi_k^-$ and $\|\psi_k\|_{cb} \geq \|\psi_k^+ + \psi_k^-\|$. We then have

$$\psi_k^+(1) = \psi_k(1) + \psi_k^-(1) = 1 + \psi_k^-(1) \geq 1$$

and

$$\|\psi_k^+(1)\| = \|\psi_k^+\| \leq \|\psi_k^+ + \psi_k^-\| \leq \|\psi_k\|_{cb} \leq (1 - \delta_k)^{-1}.$$

Also,

$$\|\psi_k^+ - \psi_k\| = \|\psi_k^+(1) - \psi_k(1)\| = \|\psi_k^+(1) - 1\| \leq (1 - \delta_k)^{-1} - 1.$$

Since $\psi_k^+(1) > 0$ we can define the u.c.p. map $\varphi_k : A_k \rightarrow M_n$ by

$$\varphi_k(a) = \psi_k^+(1)^{-\frac{1}{2}} \psi_k^+(a) \psi_k^+(1)^{-\frac{1}{2}}$$

for all $a \in A_k$. Then

$$\begin{aligned} \|\psi_k - \varphi_k\| &\leq \|\psi_k - \psi_k^+\| + \|\psi_k^+ - \varphi_k\| \\ &\leq ((1 - \delta_k)^{-1} - 1) + \|1 - \psi_k^+(1)^{-\frac{1}{2}}\| \|\psi_k^+\| (1 + \|\psi_k^+(1)^{-\frac{1}{2}}\|) \\ &\leq ((1 - \delta_k)^{-1} - 1) + 2(1 - (1 - \delta_k)^{\frac{1}{2}})(1 - \delta_k)^{-1}, \end{aligned}$$

and this last expression tends to zero as $k \rightarrow \infty$. It follows that, for all $(x_k)_k + \bigoplus A_k \in Y$,

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \rightarrow \infty} \psi_k(x_k) = \lim_{k \rightarrow \infty} \varphi_k(x_k).$$

Now suppose that $X_1 \subset X_2 \subset \dots$ is an increasing sequence of finite-dimensional operator subsystems of X with union dense in X . Let $\varphi \in UCP_n(X)$. Then for each $j \in \mathbb{N}$ there exists by the first paragraph u.c.p. maps φ_k on A_k for sufficiently large k (and hence for all k) such that

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \rightarrow \infty} \varphi_k(x_k)$$

for all $(x_k)_k + \bigoplus A_k \in X_j$. By applying a diagonal argument over $j \in \mathbb{N}$ we can assume that the equality in the above display holds for all $(x_k)_k + \bigoplus A_k \in \bigcup_{j \in \mathbb{N}} X_j$.

A straightforward approximation argument then shows that this equality in fact holds for all $(x_k)_k + \bigoplus A_k \in X$, completing the proof. \square

Lemma 5.2. Let (Z, L_Z) be a Lip-normed operator system, X and Y operator systems, $\Phi : Z \rightarrow X$ and $\Gamma : Z \rightarrow Y$ u.c.p. maps with $\Phi(\mathcal{D}(L_Z))$ and $\Gamma(\mathcal{D}(L_Z))$ dense in X_{sa} and Y_{sa} , respectively, and L_X and L_Y the quotient Lip-norms induced via Φ and Γ , respectively. Then

$$\text{dist}_s(X, Y) \leq \sup_{n \in \mathbb{N}} \text{dist}_H^{\rho_{L_Z, n}}(UCP_n(X), UCP_n(Y)),$$

with $UCP_n(X)$ and $UCP_n(Y)$ considered as subsets of $UCP_n(Z)$.

Proof. Set $r = \sup_{n \in \mathbb{N}} \text{dist}_H^{\rho_{L_Z, n}}(UCP_n(X), UCP_n(Y))$ (as can be seen from the proof of Proposition 2.9, this supremum is bounded by $\text{diam}(Z, L_Z)$). As in [10, Example 5.6] for any $\gamma > 0$ we can construct a bridge between two copies of (Z, L_Z) by setting $N(z, z') = \gamma^{-1} \|z - z'\|$. Let M be the Lip-norm

$$M(z, z') = \max(L_Z(z), L_Z(z'), N(z, z'))$$

on $\mathcal{D}(L_Z) \oplus \mathcal{D}(L_Z)$ and L the quotient Lip-norm induced by M via the u.c.p. map $(z, z') \mapsto (\Phi(z), \Gamma(z'))$. Then $L \in \mathcal{M}(L_X, L_Y)$. Denote the projections of $Z \oplus Z$ onto the first and second direct summand by π_1 and π_2 , respectively.

Now suppose $\varphi \in UCP_n(X)$. Then by assumption for some $\psi \in UCP_n(Y)$ we have $\rho_{L_Z, n}(\varphi \circ \Phi, \psi \circ \Gamma) \leq r$. Also, if $(z, z') \in \mathcal{D}_1(M)$ then $\|z - z'\| \leq \gamma$ so that $\|(\psi \circ \Gamma)(z) - (\psi \circ \Gamma)(z')\| \leq \gamma$ and hence $\rho_{M, n}(\psi \circ \Gamma \circ \pi_1, \psi \circ \Gamma \circ \pi_2) \leq \gamma$ (where to avoid confusion we have included the compositions with the projection maps π_1 and π_2 , contrary to our usual practice). Thus, since $\rho_{L, n}$ is the restriction of $\rho_{M, n}$ via the identification arising from the quotient map, we have by the triangle inequality

$$\begin{aligned} \rho_{L, n}(\varphi, \psi) &\leq \rho_{M, n}(\varphi \circ \Phi, \psi \circ \Gamma) + \rho_{M, n}(\psi \circ \Gamma \circ \pi_1, \psi \circ \Gamma \circ \pi_2) \\ &\leq r + \gamma. \end{aligned}$$

Hence $\text{dist}_s(X, Y) \leq r + \gamma$, which yields the result since γ was arbitrary. \square

In the proof of the following theorem, we will abbreviate expressions of the form $\text{dist}_H^{\rho_{L, n}}(UCP_n(X), UCP_n(Y))$ to $\rho_{L, n}(UCP_n(X), UCP_n(Y))$ to reduce the number of subscripts, and whenever we have a quotient Lip-norm then we will identify the state space of the quotient operator system with a subset of the state space of the original operator system under the induced isometry (Proposition 2.13) as is our usual practice in the case of projections onto direct summands.

Theorem 5.3. Let $\{(A_k, L_k)\}$ be a sequence in \mathcal{R}_{alg} which is Cauchy with respect to f -Leibniz complete distance for a given continuous $f : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$. Then $\{(A_k, L_k)\}$ converges in \mathcal{R}_{alg} with respect to complete distance.

Proof. To show that $\{(A_k, L_k)\}$ converges it suffices to show the convergence of a subsequence, and so we may assume that $\text{dist}_{s, f}(A_k, A_{k+1}) < 2^{-k}$ for all $k \in \mathbb{N}$. Then there exist f -Leibniz Lip-norms $L_{k, k+1} \in \mathcal{M}(L_k, L_{k+1})$ with

$$\rho_{L_{k, k+1}, n}(UCP_n(A_k), UCP_n(A_{k+1})) < 2^{-k}$$

for all $n, k \in \mathbb{N}$. Let Z be the set of sequences $(x_k)_k$ with $x_k \in \mathcal{D}(L_k)$ such that, for some $\lambda > L_1(x_1)$, $x_{k+1} \in \mathcal{N}_{L_{k,k+1}, A_{k+1}}^\lambda(x_k)$ for all $k \in \mathbb{N}$ (see Definition 4.7). We will show that Z is a subset of the direct product $\prod A_k$. Let J_k be the semi-norm on $\prod_{j=1}^k \mathcal{D}(L_j)$ given by

$$J_k((x_j)_j) = \sup_{1 \leq j \leq k-1} L_{j,j+1}(x_j, x_{j+1}).$$

By [10, Lemma 12.2] J_k is a Lip-norm. Denote by Q_k the quotient Lip-norm on $\mathcal{D}(L_1) \oplus \mathcal{D}(L_k)$ induced by J_k via the projection map. Then

$$\rho_{Q_k,1}(S(A_1), S(A_k)) \leq 2^{-1} + 2^{-2} + \dots + 2^{-k} < 1.$$

Suppose $(x_k)_k \in Z$, and let $\lambda > L_1(x_1)$ be such that $x_{k+1} \in \mathcal{N}_{L_{k,k+1}, X_{k+1}}^\lambda(x_k)$ for all $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, x_k is an element of $\mathcal{N}_{Q_k, A_k}^\lambda(x_1)$ and hence by Lemma 4.8 has norm bounded by $\|x_1\| + 2\lambda\rho_{Q_k,1}(S(A_1), S(A_k)) \leq \|x_1\| + 2\lambda$. Therefore $(x_k)_k$ is a bounded sequence and so belongs to $\prod A_k$, as we wished to show.

We define the semi-norm L_Z on Z by

$$L_Z((x_k)_k) = \sup_{k \in \mathbb{N}} L_{k,k+1}(x_k, x_{k+1})$$

(which is finite by the definition of Z). Theorem 12.9 of [10] then shows that L_Z is a Lip-norm on Z . Now since the elements of Z are bounded sequences $(x_k)_k$ with $L_{k,k+1}(x_k, x_{k+1})$ uniformly bounded over k , it follows that by the f -Leibniz property that if $(x_k)_k, (y_k)_k \in Z$ then $\operatorname{Re}((x_k y_k)_k), \operatorname{Im}((x_k y_k)_k) \in Z$ whence $(x_k y_k)_k \in Z + iZ$. Thus $Z + iZ$ is closed under multiplication, and so the operator system B obtained by taking the closure of $Z + iZ$ in $\prod A_k$ must in fact be a C^* -algebra.

Let A be the C^* -subalgebra of $\prod A_k / \bigoplus A_k$ which is the image of B under the quotient map $\pi : \prod A_k \rightarrow \prod A_k / \bigoplus A_k$, and let L be the quotient Lip-norm on A induced by L_Z . Then (A, L) is a Lip-normed unital C^* -algebra. Our goal now is to show that $\{(A_k, L_k)\}$ converges to (A, L) with respect to complete distance. By Lemma 5.2 it suffices to show that, for all $n \in \mathbb{N}$, $UCP_n(A)$ coincides with the Hausdorff limit $H_n \subset UCP_n(B)$ of $\{UCP_n(A_k)\}_{k \in \mathbb{N}}$, which exists due to the completeness, in the Hausdorff metric, of the set of closed subspaces of the compact set $UCP_n(B)$. Note that for each $k' \in \mathbb{N}$ the image of Z under the projection onto $\mathcal{D}(L_{k'})$ is surjective (any element of $\mathcal{D}(L_{k'})$ can be recursively extended to a sequence in $Z \subset \prod A_k$ using the fact that L_k and L_{k+1} are quotients of $L_{k,k+1}$ for every $k \in \mathbb{N}$) so that we may indeed view each $UCP_n(A_k)$ as subset of $UCP_n(B)$. Note also that the convergence of $\{UCP_n(A_k)\}_{k \in \mathbb{N}}$ to H_n is uniform over n because the Cauchy condition is uniform over n by assumption.

If $\{\varphi_k\}_{k \in \mathbb{N}}$ is a sequence such that $\varphi_k \in UCP_n(A_k)$ and $\{\varphi_k \circ \pi_k\}_{k \in \mathbb{N}}$ is point-norm convergent (necessarily to an element of H_n), then setting

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \rightarrow \infty} \varphi_k(x_k)$$

for $(x_k)_k + \bigoplus A_k \in X$ we obtain a map $\varphi : X \rightarrow M_n$. This map is u.c.p. in view of the identification $M_n \otimes (\prod A_k / \bigoplus A_k) \cong (M_n \otimes (\prod A_k)) / (M_n \otimes (\bigoplus A_k))$ and the

fact that positive elements in quotients lift to positive elements. We thus see that $H_n \subset UCP_n(A)$.

It remains to show that $H_n \supset UCP_n(A)$. With a view to applying Lemma 5.1, we will show that every $x \in M_n \otimes A$ has a lift $(x_k)_k \in M_n \otimes \prod A_k$ satisfying $\lim_{k \rightarrow \infty} \|x_k\| = \|x\|$. Notice first that if $(z_k)_k \in M_n \otimes Z$ then for some λ not depending on j we have $|\|z_j\| - \|z_{j+1}\|| \leq 2^{-j+1}n^4\lambda$ by Lemma 4.8(ii) (since each z_j is self-adjoint), so that $\{\|z_k\|\}_{k \in \mathbb{N}}$ is a Cauchy sequence and hence $\|\pi((z_k)_k)\| = \lim_{k \rightarrow \infty} \|z_k\|$. Now suppose $x \in M_n \otimes A$ and let $(x_k)_k$ be a lift of x to $M_n \otimes B$. Then $(x_k^*x_k)_k \in M_n \otimes B$, and so there exists a $(y_k)_k \in M_n \otimes Z$ such that $\|x_k^*x_k - y_k\| < \varepsilon$ for all $k \in \mathbb{N}$, and from above we have $\|\pi((y_k)_k)\| = \lim_{k \rightarrow \infty} \|y_k\|$. Let $\varepsilon > 0$, and choose $k_0 \in \mathbb{N}$ such that, for all $j, k \geq k_0$, $|\|y_j\| - \|y_k\|| < \varepsilon$. Then, for all $j, k \geq k_0$,

$$\begin{aligned} |\|x_j\|^2 - \|x_k\|^2| &= |\|x_j^*x_j\| - \|x_k^*x_k\|| \\ &\leq |\|x_j^*x_j\| - \|y_j\|| + |\|y_j\| - \|y_k\|| + |\|y_k\| - \|x_k^*x_k\|| \\ &< 3\varepsilon. \end{aligned}$$

It follows that $\{\|x_k\|^2\}_{k \in \mathbb{N}}$ is a Cauchy sequence and hence converges. Thus $\lim_{k \rightarrow \infty} \|x_k\|$ exists, and it must equal $\|x\|$. We can therefore apply Lemma 5.1, so that given $\varphi \in UCP_n(A)$ there exist $\varphi_k \in UCP_n(A_k)$ for $k \in \mathbb{N}$ such that for all $(x_k)_k + \bigoplus A_k \in A$ we have

$$\varphi((x_k)_k + \bigoplus A_k) = \lim_{k \rightarrow \infty} \varphi_k(x_k),$$

whence $H_n \supset UCP_n(A)$. Thus H_n and $UCP_n(A)$ coincide, completing the proof. \square

Using the arguments of this section we might hope to show that the metric space $(\mathcal{R}, \text{dist}_s)$ is complete. However, without the sharp control on the norms of non-self-adjoint elements that the f -Leibniz property provides in Theorem 5.3, we would not be able to apply Lemma 5.1.

6. TOTAL BOUNDEDNESS

We will establish a version of Theorem 13.5 in [10] (“the quantum Gromov compactness theorem”) for complete distance using approximation by Lip-normed operator subsystems of matrix algebras. As before $(\mathcal{R}, \text{dist}_s)$ is the metric space of equivalence classes of Lip-normed operator systems with respect to bi-Lip-isometric unital complete order isomorphism.

Notation 6.1. For a Lip-normed operator system (X, L) and $\varepsilon > 0$ we denote by $\text{Afn}_L(\varepsilon)$ the smallest integer k such that there is a Lip-normed operator system (Y, L_Y) with Y an operator subsystem of the matrix algebra M_k and $\text{dist}_s(X, Y) \leq \varepsilon$. If no such integer k exists we write $\text{Afn}_L(\varepsilon) = \infty$. We denote by \mathcal{R}_{fa} the subset of \mathcal{R} consisting of Lip-normed operator systems (X, L) for which $\text{Afn}_L(\varepsilon)$ is finite for all $\varepsilon > 0$.

We remark that every Lip-normed nuclear operator system and Lip-normed unital exact C^* -algebra is contained in \mathcal{R}_{fa} by Propositions 3.10 and 3.11, respectively. Note also that \mathcal{R}_{fa} is a closed subset of \mathcal{R} under the complete distance topology.

Lemma 6.2. Let X be a finite-dimensional operator system and $C \geq 0$. Then the set $\mathcal{C} = \{(X, L) \in \mathcal{R} : \text{diam}(X, L) \leq C\}$ is totally bounded.

Proof. Proposition 13.13 and the proof of Proposition 13.14 in [10] show that, given $\epsilon > 0$, there is a finite subset $\mathcal{S} \subset \mathcal{C}$ such that for every $(X, L) \in \mathcal{C}$ there is a $(X, L') \in \mathcal{S}$ and a bridge N between (X, L) and (X, L') of the form $N(x, y) = \epsilon^{-1}\|x - y\|$. Let M be the Lip-norm in $\mathcal{M}(L, L')$ given by

$$M(x, y) = \max(L(x), L'(y), N(x, y)).$$

Now if $\phi \in UCP_n(X)$ and $(x, y) \in \mathcal{D}_1(M)$ then $\|x - y\| \leq \epsilon$ so that $\|\phi(x) - \phi(y)\| \leq \epsilon$, and so by Proposition 2.10 we have $\rho_{M,n}(\phi \circ \pi_1, \phi \circ \pi_2) \leq \epsilon$, where π_1 and π_2 are the projections of $X \oplus X$ onto the first and second direct summands, respectively. Hence $\text{dist}_s((X, L), (X, L')) \leq \epsilon$, from which we conclude that \mathcal{C} is totally bounded. \square

Theorem 6.3. Let \mathcal{C} be a subset of \mathcal{R}_{fa} . Then \mathcal{C} is totally bounded if and only if

- (i) there is an $M > 0$ such that the diameter of every element of \mathcal{C} is bounded by M , and
- (ii) there is a function $F : (0, \infty) \rightarrow (0, \infty)$ with $\text{Afn}_L(\epsilon) \leq F(\epsilon)$ for all $(X, L) \in \mathcal{C}$.

Proof. For the “only if” direction, suppose that \mathcal{C} is a totally bounded subset of \mathcal{R}_{fa} . If there did not exist an $M > 0$ bounding the complete diameter of every element of \mathcal{C} , then we could find a sequence $\{(X_k, L_k)\}_{k \in \mathbb{N}}$ such that $\text{diam}(X_{k+1}, L_{k+1}) \geq \text{diam}(X_k, L_k) + 1$ for every $k \in \mathbb{N}$, in which case $\text{dist}_s(X_k, X_{k'}) \geq 1$ for all $k, k' \in \mathbb{N}$, contradicting total boundedness. To verify condition (ii), we can find a finite $(\epsilon/2)$ -dense subset \mathcal{G} of \mathcal{C} and set

$$G(\epsilon) = \max\{\text{Afn}_L(\epsilon/2) : (X, L) \in \mathcal{G}\}.$$

Then by the triangle inequality $\text{Afn}_L(\epsilon) \leq G(\epsilon)$ for any $(X, L) \in \mathcal{C}$.

To prove the converse, suppose that conditions (i) and (ii) hold. By (ii) we see that it is sufficient to prove, for $k \geq j \geq 1$ and $M > 0$, the total boundedness of the collection of Lip-normed operator systems (X, L) where X is an operator subsystem of the matrix algebra M_k with X_{sa} of real linear dimension j (in which case we will say that X has *Hermitian dimension* j) and $\text{diam}(X, L) \leq M$. Since the closed unit ball of the self-adjoint part of M_k is compact, the set of closed unit balls of the self-adjoint parts of operator subsystems of M_k of Hermitian dimension j is totally bounded in the Hausdorff metric. Also, by Lemma 6.2, for every $M > 0$ and operator subsystem X of M_k of Hermitian dimension j the set of Lip-normed operator systems (X, L) with $\text{diam}(X, L) \leq M$ is totally bounded. Thus we need only show that, for every $\epsilon > 0$ and $M > 0$, if X and Y are operator subsystems of M_k of Hermitian dimension j the closed unit balls of the self-adjoint parts of which are within Hausdorff distance $(4k)^{-1}\epsilon \min(M^{-1}, 1)$, and L_X is a Lip-norm on X with $\text{diam}(X, L_X) \leq M$, then there is a Lip-norm L_Y on Y with $\text{dist}_s((Y, L_Y), (X, L_X)) \leq \epsilon$ and $\text{diam}(Y, L_Y) \leq M + \epsilon$. So let X and Y be such operator systems and L_X such a Lip-norm on X for given $\epsilon > 0$ and $M > 0$. We may assume that $\epsilon < 1/2$. Set $\delta = (4k)^{-1}\epsilon \min(M^{-1}, 1)$. By [1, Lemma 3.2.3] there is a (real linear) projection P

from $(M_k)_{\text{sa}}$ onto Y_{sa} of norm $\leq k$. The restriction Q of P to X_{sa} is a bijection, for if $x \in X_{\text{sa}}$ with $\|x\| = 1$ then we can find a $y \in Y_{\text{sa}}$ with $\|y - x\| < \delta \leq \epsilon/k$ so that

$$\|Q(x)\| \geq \|y\| - \|Q(x - y)\| \geq 1 - \epsilon > \frac{1}{2},$$

yielding injectivity and hence also bijectivity since X_{sa} and Y_{sa} are of equal finite dimension. The above display also shows that the norm of Q^{-1} is bounded by 2. We next define a semi-norm L_X on $Q^{-1}(Y)$ by $L_X(x) = L_Y(Q(x))$ (note that $\mathcal{D}(L_Y)$ is equal to Y_{sa} by finite-dimensionality). Since the restriction of Q to X is bijective and L_Y is a Lip-norm we must have $L_X(x) = 0$ if and only if $x \in \mathbb{R}1$. Thus L_X is a Lip-norm in view of the finite-dimensionality of X_{sa} , and (X, L_X) is a Lip-normed operator system since $\mathcal{D}(L_X) = X_{\text{sa}}$ and $\mathcal{D}_1(L_X)$ is closed in X_{sa} by the bijectivity and continuity, respectively, of Q .

On $\mathcal{D}(L_X) \oplus \mathcal{D}(L_Y)$ we define the semi-norm N by $N(x, y) = \epsilon^{-1}\|x - y\|$. We will argue that N is a bridge. For this it suffices to show that, for all $x \in \mathcal{D}(L_X)$,

$$N(x, Q(x)) \leq L_Y(Q(x)),$$

for then in condition (ii) of Definition 3.5 given $x \in \mathcal{D}(L_X)$ we can take $Q(x)$, and given $y \in \mathcal{D}(L_Y)$ we can take $Q^{-1}(y)$. So let $x \in \mathcal{D}(L_X)$. Then we can find a $y \in Y_{\text{sa}}$ such that $\|x - y\| \leq \delta\|x\|$, so that

$$\|x - Q(x)\| \leq \|x - y\| + \|Q(x - y)\| \leq (1 + k)\|x - y\| \leq 2k\delta\|x\| \leq 4k\delta\|Q(x)\|.$$

Applying this estimate with x replaced by $x - \lambda 1$ where λ is the infimum of the spectrum of $Q(x)$, we have

$$\begin{aligned} \|x - Q(x)\| &= \|x - \lambda 1 - Q(x - \lambda 1)\| \leq 4k\delta\|Q(x - \lambda 1)\| \\ &= 4k\delta\|Q(x) - \lambda 1\| \\ &\leq 4kM\delta L_Y(Q(x) - \lambda 1) \\ &= 4kM\delta L_Y(Q(x)) \\ &\leq \epsilon L_Y(Q(x)) \end{aligned}$$

with Proposition 2.11 yielding the second inequality in the string. We thus conclude that N is a bridge. Let L be the Lip-norm in $\mathcal{M}(L_X, L_Y)$ given by

$$L(x, y) = \max(L_X(x), L_Y(y), N(x, y)).$$

It remains to show that $\text{dist}_H^{\rho_{L,n}}(UCP_n(X), UCP_n(Y)) \leq \epsilon$ for all $n \in \mathbb{N}$, for then $\text{dist}_s(X, Y) \leq \epsilon$ and hence also $\text{diam}(X, L_X) \leq \text{diam}(Y, L_Y) + \epsilon \leq M + \epsilon$. Let $\varphi \in UCP_n(X)$. By Arveson's extension theorem we can extend φ to a u.c.p. map $\varphi' : M_k \rightarrow M_n$. We then have

$$\begin{aligned} \rho_{L,n}(\varphi, \varphi'|_Y) &= \sup\{|\varphi'(x) - \varphi'(y)| : (x, y) \in \mathcal{D}_1(L)\} \\ &\leq \sup\{|\varphi'(x) - \varphi'(y)| : (x, y) \in X \oplus Y \text{ and } \|x - y\| \leq \epsilon\} \\ &\leq \epsilon. \end{aligned}$$

Similarly, if $\varphi \in UCP_n(Y)$ then extending it by Arveson's theorem to a u.c.p. map $\varphi' : M_k \rightarrow M_n$ we have $\rho_{L,n}(\varphi, \varphi'|_Y) \leq \epsilon$. Thus $\text{dist}_H^{\rho_{L,n}}(UCP_n(X), UCP_n(Y)) \leq \epsilon$, as desired. \square

An immediate consequence of Theorem 6.3 is the separability of \mathcal{R}_{fa} .

Corollary 6.4. The metric space \mathcal{R}_{fa} is separable.

Question 6.5. Given $n > 1$ and $M > 0$, is the set of all n -dimensional Lip-normed operator systems of diameter at most M totally bounded and/or separable?

We may think of $\log \text{Afn}_L(\epsilon)$ as an analogue of Kolmogorov ϵ -entropy. From the computational viewpoint, however, the value of this quantity seems to be limited by the apparent difficulty in establishing lower bounds. Using local approximation we will next define a quantity $\text{Rcp}_L(\epsilon)$ which is more amenable to obtaining estimates than $\text{Afn}_L(\epsilon)$ and provides a ready means for obtaining upper bounds for $\text{Afn}_L(\epsilon)$ (see Proposition 6.7) with a view to the application of Theorem 6.3, as we will illustrate in the case of noncommutative tori in Example 6.8. It can also be shown (by suitably adjusting the proof of [5, Prop. 3.9] for instance) that by taking $\limsup_{\epsilon \rightarrow 0^+} \log \text{Rcp}_L(\epsilon) / \log(\epsilon^{-1})$ we obtain a generalization of the Kolmogorov dimension of a compact metric space, whose utility depends on our ability to estimate $\log \text{Rcp}_L(\epsilon)$ from below. We will not be concerned here with obtaining lower bounds for $\log \text{Rcp}_L(\epsilon)$, but we point out that this can often be done by using the Hilbert space geometry implicit in the given operator system or C^* -algebra as in [5].

For a Lip-normed nuclear operator system (X, L) and $\epsilon > 0$ we denote by $CPA_L(\epsilon)$ the collection of triples (α, β, B) where B is a finite-dimensional C^* -algebra and $\alpha : X \rightarrow B$ and $\beta : B \rightarrow X$ are u.c.p. maps with $\|(\beta \circ \alpha)(x) - x\| < \epsilon$ for all $x \in \mathcal{D}_1(L)$. This collection is non-empty by Proposition 2.11. Conversely, if (X, L) is any Lip-normed operator system (X, L) and $CPA_L(\epsilon)$ is non-empty for each $\epsilon > 0$, then X is nuclear owing to the density of $\mathcal{D}(L)$ in X_{sa} .

Definition 6.6. Let (X, L) be a Lip-normed nuclear operator system. For $\epsilon > 0$ we set

$$\text{Rcp}_L(\epsilon) = \inf\{\text{rank}(B) : (\alpha, \beta, B) \in CPA_L(\epsilon)\}$$

where rank refers to the cardinality of a maximal set of pairwise orthogonal minimal projections.

Proposition 6.7. If (X, L) is a Lip-normed nuclear operator system and $\epsilon > 0$ then $\text{Rcp}_L(\epsilon) \geq \text{Afn}_L(\epsilon)$.

Proof. Let $\epsilon > 0$. Then there is a triple $(\alpha, \beta, B) \in CPA_L(\epsilon)$ with $\text{rank}(B) = \text{Rcp}_L(\epsilon)$. Set $Y = \alpha(X)$, and let L_Y be the Lip-norm on Y induced by L via α . Then $\text{dist}_s(X, Y) \leq \epsilon$ by Proposition 3.9, and since B unittally embeds into a matrix algebra of the same rank we obtain $\text{Afn}_L(\epsilon) \leq \text{Rcp}_L(\epsilon)$. \square

Example 6.8 (Noncommutative tori). Let $\rho : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{T}$ be an antisymmetric bicharacter and for $1 \leq i, j \leq k$ set

$$\rho_{ij} = \rho(e_i, e_j)$$

with $\{e_1, \dots, e_d\}$ the standard basis for \mathbb{Z}^d . We call the universal C^* -algebra A_ρ generated by unitaries u_1, \dots, u_d satisfying

$$u_j u_i = \rho_{ij} u_i u_j$$

a *noncommutative d -torus*. Given a noncommutative d -torus A_ρ with generators u_1, \dots, u_d there is an ergodic action $\gamma : \mathbb{T}^d \cong (\mathbb{R}/\mathbb{Z})^d \rightarrow \text{Aut}(A_\rho)$ determined on the generators by

$$\gamma_{(t_1, \dots, t_d)}(u_j) = e^{2\pi i t_j} u_j$$

(see [6]). Let ℓ be a length function on \mathbb{T}^d (for instance, we could take the distance to 0 with respect to the metric induced from the Euclidean metric on \mathbb{R}^d). By Example 2.6 we then obtain a Lip-norm L arising from the action γ and length function ℓ . Let τ be the tracial state on A_ρ defined by

$$\tau(a) = \int_{\mathbb{T}^d} \gamma_{(t_1, \dots, t_d)}(a) d(t_1, \dots, t_d)$$

for all $a \in A_\rho$, where $d(t_1, \dots, t_d)$ is normalized Haar measure.

Let $\mathcal{A}(d, \ell)$ be the subset of \mathcal{R} consisting of all noncommutative d -tori Lip-normed as above with respect to the length function ℓ . This is in fact a subset of \mathcal{R}_{fa} by Proposition 3.10, since noncommutative tori are nuclear. We will show using Theorem 6.3 that $\mathcal{A}(d, \ell)$ is totally bounded.

For $(n_1, \dots, n_d) \in \mathbb{N}^d$ we denote by $R(n_1, \dots, n_d)$ the set of points (k_1, \dots, k_d) in \mathbb{Z}^d such that $|k_i| \leq n_i$ for $i = 1, \dots, d$. For $a \in A_\rho$, we define for every $(n_1, \dots, n_d) \in \mathbb{N}^d$ the partial Fourier sum

$$s_{(n_1, \dots, n_d)}(a) = \sum_{(k_1, \dots, k_d) \in R(n_1, \dots, n_d)} \tau(a u_d^{-k_d} \dots u_1^{-k_1}) u_1^{k_1} \dots u_d^{k_d}$$

and for each $n \in \mathbb{N}$ the Cesàro mean

$$\sigma_n(a) = (n+1)^{-d} \sum_{(n_1, \dots, n_d) \in R(n, n, \dots, n)} s_{(n_1, \dots, n_d)}(a).$$

As in classical Fourier analysis (see for example [4]) it can be shown that if K_n is the Fejér kernel

$$K_n(t) = \sum_{k=-n}^n \left(1 - \frac{|k|}{n+1}\right) e^{2\pi i k t} = \frac{1}{n+1} \left(\frac{\sin((n+1)t/2)}{\sin(t/2)}\right)^2$$

then for all $a \in A_\rho$ and $n \in \mathbb{N}$ we have

$$\|a - \sigma_n(a)\| \leq \sum_{k=1}^d \int_{\mathbb{T}} \|a - \gamma_{r_k(t)}(a)\| K_n(t) dt$$

where $r_k(t)$ denotes the d -tuple which is t at the k th coordinate and 0 elsewhere, and dt is normalized Haar measure (see for example the proof of [16, Thm. 22]). It follows that if $a \in \mathcal{D}_1(L)$ then

$$\|a - \sigma_n(a)\| \leq \sum_{k=1}^d \int_{\mathbb{T}} \ell(r_k(t)) K_n(t) dt.$$

In particular the Cesàro means of elements of $\mathcal{D}_1(L)$ converge at a rate which does not depend on ρ , a fact which we will use below.

Now by [10, Lemma 9.4] there is a constant $M > 0$ such that $\text{diam}(A_\rho, L) \leq M$ for all $(A_\rho, L) \in \mathcal{A}(d, \ell)$. Hence to obtain the total boundedness of $\mathcal{A}(d, \ell)$ we need only check condition (ii) in Theorem 6.3. Let $\epsilon > 0$. If B is a finite-dimensional C^* -algebra and $\alpha : A_\rho \rightarrow B$ and $\beta : B \rightarrow A_\rho$ are u.c.p. maps with $\|(\beta \circ \alpha)(x) - x\| < \epsilon$ for all $x \in \mathcal{D}_1(L) \cap \mathcal{B}_M^{A_\rho}$, then it is readily seen that Proposition 2.11 implies that the triple (α, β, B) lies in $CPA_L(\epsilon)$. From the previous paragraph there is an $n \in \mathbb{N}$ which does not depend on ρ such that each element of $\mathcal{D}_1(L) \cap \mathcal{B}_M^{A_\rho}$ is within ϵ of its n th Cesàro mean, which is a linear combination of elements in $\{u_1^{k_1} \cdots u_d^{k_d} : |k_i| \leq n\}$ with coefficients bounded in modulus by M (since the operation of taking a Cesàro mean decreases the moduli of Fourier coefficients, which are bounded by the norm of the given element). Thus in view of Proposition 6.7 it suffices to show the existence of a finite-dimensional C^* -algebra B and u.c.p. maps $\alpha : A_\rho \rightarrow B$ and $\beta : B \rightarrow A_\rho$ such that $\|(\beta \circ \alpha)(x) - x\| < \epsilon$ for all $x \in \{u_1^{k_1} \cdots u_d^{k_d} : |k_i| \leq n\}$ with the rank of B not depending on ρ , and this is a consequence of [15, Lemma 5.1].

Hanfeng Li has informed me that he can show that the map from the space of antisymmetric bicharacters on \mathbb{Z}^d to $\mathcal{A}(d, \ell)$ determined by $\rho \mapsto A_\rho$ is continuous (as Rieffel showed for quantum Gromov-Hausdorff distance in [10, Thm. 9.2]). In fact, given any field of strongly continuous ergodic actions of a compact group on a continuous field of unital C^* -algebras over a compact metric space X , at any point of X the continuity of complete distance is equivalent to the local constancy (or, equivalently, the lower semicontinuity) of the function on X which records the multiplicity of the action in the fibre algebras. This is a result of the fact that Li (unpublished notes) has worked out a general version of Rieffel's result on coadjoint orbits as described in Example 3.13.

REFERENCES

- [1] B. Blackadar and E. Kirchberg. Generalized inductive limits of finite-dimensional C^* -algebras. *Math. Ann.* **307** (1997), 343–380.
- [2] M.-D. Choi. A Schwarz inequality for positive linear maps on C^* -algebras. *Ill. J. Math.* **18** (1974), 565–574.
- [3] E. G. Effros and Z.-J. Ruan. *Operator Spaces*. London Mathematical Society Monographs, New Series **23**, Clarendon, New York, 2000.
- [4] Y. Katznelson. *An Introduction to Harmonic Analysis*, Second Edition. New York, Dover Publications, 1976.
- [5] D. Kerr. Dimension and dynamical entropy for metrized C^* -algebras. *Commun. Math. Phys.* **232** (2003), 501–534.
- [6] D. Olesen, G. K. Pedersen, and M. Takesaki. Ergodic actions of compact Abelian groups. *J. Operator Theory* **3** (1980), 237–269.
- [7] V. I. Paulsen. *Completely bounded maps and dilations*. Pitman Research Notes in Mathematics **146**, Longman, New York, 1986.
- [8] N. C. Phillips. Continuous-trace C^* -algebras not isomorphic to their opposite algebras. *Internat. J. Math.* **12** (2001), 263–275.
- [9] M. A. Rieffel. Group C^* -algebras as compact quantum metric spaces. arXiv:math.OA/0205195, 2002.
- [10] M. A. Rieffel. Gromov-Hausdorff distance for quantum metric spaces. arXiv:math.OA/0011063 v2, 2001.

- [11] M. A. Rieffel. Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance. arXiv:math.OA/0108005, 2001.
- [12] M. A. Rieffel. Metrics on state spaces. *Doc. Math.* **4** (1999), 559–600.
- [13] M. A. Rieffel. Metrics on states from actions of compact groups. *Doc. Math.* **3** (1998), 215–229.
- [14] R. R. Smith. Completely bounded maps between C^* -algebras. *J. London Math. Soc.* **27** (1983), 157–166.
- [15] D. V. Voiculescu. Dynamical approximation entropies and topological entropy in operator algebras. *Commun. Math. Phys.* **170** (1995), 249–281.
- [16] N. Weaver. Lipschitz algebras and derivations of von Neumann algebras. *J. Funct. Anal.* **139** (1996), 261–300.
- [17] S. Wassermann. *Exact C^* -algebras and Related Topics*. Lecture Notes Series **19**, GARC, Seoul National University, 1993.

DAVID KERR, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA “LA SAPIENZA,” P.LE ALDO MORO, 2, 00185 ROME, ITALY

E-mail address: `kerr@mat.uniroma1.it`