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CENTER FOR NUMERICAL ANALYSIS

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1. Introduction.

The notion of univariate smoothing splines was introduced in the polynomial case by I. J. Schoenberg [12]. An important abstract generalization was later given by M. Atteia [2] [3]. P. M. Anselone and P. J. Laurent have studied a construction method for these functions. [1], [6]. Smoothing splines for the case of two variables have been considered by G. M. Nielson in [8].

The purpose of this note is to generalize spline smoothing functions in analogy with the recent generalization introduced by A. Sard for spline interpolating functions [10].

The proof of the existence and unicity of such a function will be analogous to the proof given by M. Atteia [3] in the case of smoothing spline functions for functionals.

Using Sard's method, we shall establish the property of best approximation of the spline functions. Finally, we improve the error estimate using an idea of J. Meinguet [7] and A. Sard [11].

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A final practical example will be given to illustrate the utility of smoothing splines for operators.

2. Existence and unicity.

Let X, Z, $Z^1,\ldots,Z^m$ be Hilbert spaces, and let G be a continuous linear operator on X into Z. We wish to approximate Gx, $x \in X$, using m experimental observations $h_i \in Z^i$ which represent F^ix, F^i being a continuous linear operator on x into Z^i, $1 \leq i \leq m$. The elements $h_i \in Z^i$, given experimentally, are approximations of the elements F^ix.

Let U be a continuous, linear and surjective operator from X onto a Hilbert space Y.

Denote by Ω the Cartesian product of the spaces $Z^1,\ldots,Z^m$. Define an operator V on X into Ω as follows:

$$Vx = [F^1x, F^2x, \ldots, F^mx] \, . \tag{1}$$

Introducing the Cartesian product $Z^o = Y \times \Omega$, we define the operator L from X into Z^o by

$$Lx = [Ux, Vx] \, . \tag{2}$$

For each $h = [h_1, h_2,\ldots,h_m] \in \Omega$, we put $h^o = [0, h] \in Z^o$, where 0 denotes the origin of Y.

Following the ideas expressed in [2], [3], [6], we furnish Z^o with the quadratic norm

$$||[y_o, z_1, z_2, \ldots, z_m]||^2_{Z^o} = ||y||^2_Y + \rho \sum_{i=1}^{m} ||z_i||^2_{Z^i}, \tag{3}$$

where ρ is a positive parameter at our disposal.

We call a "smoothing spline for operators" any element $s \in X$ which minimizes the functional

$$\Phi(x) = ||Ux||^2_Y + \rho ||Vx-h||^2_\Omega \quad . \tag{4}$$

Then we have by definition

$$||Ls - h^o||^2_{Z^o} = \operatorname*{Min}_{x \in X} ||Lx-h^o||^2_{Z^o} \; . \tag{5}$$

We define $N = \{x \in X: F^i x = 0,\ 1 \leq i \leq m\}$ and denote by M the orthogonal complement of N. We introduce the following hypotheses:

1^o. $\ker U \subset M$, $\ker U \cap N = 0$;

2^o. The ranges of F^i, $1 \leq i \leq m$, are closed;

3^o. There exists a constant $B < \infty$, such that

$$||x||^2 \leq B^2[||Ux||^2 + \rho \sum_{i=1}^{m} ||F^i x||^2], \quad x \in X \; . \tag{6}$$

Then the operator L has the following properties:

a) L is bounded and hence continuous. Indeed, taking account of the norm introduced in Z^o one has

$$||Lx||^2_{Z^o} \leq \{||U||^2_Y + \rho \sum_{i=1}^{m} ||F^i||^2_{Z^i}\} ||x||^2_X \; . \tag{7}$$

b) L maps X bijectively onto $V^{o} = LX$. This is an immediate consequence of 1^{o}.

c) LX is a closed linear subspace of Z^{o}. This follows from the hypothesis 2^{o} and from the assumption that U is surjective.

d) L^{-1} is continuous. (The hypothesis 3^{o} assures the existence and the continuity of L^{-1}).

The element $h^{o} \epsilon Z^{o}$ being given, we denote by p the orthogonal projection of h^{o} onto the subspace LX. The orthogonality is relative to the scalar product induced by the norm (3). The projection exists uniquely. Then we have

$$Ls = p \quad \text{and} \quad s = L^{-1}p \tag{8}$$

and we can infer the existence and unicity of the smoothing spline defined above.

3. <u>The property of best approximation. Estimation of the error.</u>

Suppose that one wishes to approximate Gx, $x \epsilon X$ by means of a linear combination of the form

$$Gx \approx \sum_{i=1}^{m} E^{i}h_{i}, \tag{9}$$

where E^{i} is a continuous linear map from Z^{i} into Z. We will call $\sum_{i=1}^{m} E^{i}h_{i}$ an <u>admissible</u> approximation if the E^{i} are chosen in such a manner that the operator

$$R = G - \sum_{i=1}^{m} E^{i}F^{i} \tag{10}$$

admits a representation of the form

$$R = QU, \tag{11}$$

where Q is an linear continuous operator on Y into X.

The necessary and sufficient condition for existence of the representation (11) is the inclusion Ker U $\subset$ Ker R (Sard's quotient theorem [9]).

Denote by $\mathcal{A}$ the set of admissible approximations. From (9), (10), (11) we infer the existence of a continuous operator Q_1 from Z^o into Z such that

$$R_1x = Gx - \sum_{i=1}^{m} E^i h_i = Q_1(Lx - h^o). \tag{12}$$

We will say that $A_o x = \sum_{i=1}^{m} E_o{}^i h_i \in A$ is an <u>optimal</u> approximation of Gx, if the operators E_o^i, $i = \overline{1,m}$ are such that $||Q_1||$ is minimal.

<u>Theorem</u>. The operator $A_o x = Gs$ (s being the smoothing spline defined previously) minimizes the norm $||Q_1||$ among all the admissible operators, and we have the following estimation.

$$||R_1^o x||^2 \leq K^2||Lx - h^o||^2 \tag{13}$$

where

$$K = \inf_{A \in \mathcal{A}} ||Q_1|| = \inf_{A \in \mathcal{A}} \sup_{\substack{x \in X \\ ||Lx-h^o||=1}} ||R_1x|| = \sup_{\substack{x \in X \\ ||Lx-h^o||=1}} ||R_1^o x|| = ||Q_1^o|| . \tag{14}$$

<u>Proof</u>: Put

$$R_1^o x = Gx - A^o x = Gx - Gs = Q_1^o(Lx - h^o) \tag{15}$$

Then we see that

$$R_1^o x = R_1^o (x - s) . \tag{16}$$

Introduce on X the norm

$$||x||_{\mathcal{X}} = ||Ux||_Y^2 + \rho \sum_{i=1}^{m} ||F^i x - h^i||^2_{Z^i} \tag{17}$$

and designate by $\mathcal{X}$, the space X endowed with this norm. We have the following inequality

$$||R_1^o x||^2 \leq K^2 ||x||^2_{\mathcal{X}} \tag{18}$$

or

$$||R_1^o x||^2 = ||R_1^o (x - s)||^2 \leq K^2 ||x - s||^2_{)} \tag{19}$$

where

$$K = \sup_{0 \neq x \varepsilon X} \frac{||R_1^o x||}{||x||_{\mathcal{X}}} = \sup_{\substack{x,s \varepsilon X \\ x-s \neq 0}} \frac{||R_1^o (x-s)||}{||x-s||_{\mathcal{X}}} . \tag{20}$$

Since (17) can be written

$$||x||_{\mathcal{X}} = ||Lx - h^o||_{Z^o} , \tag{21}$$

we see that

$$K = \sup_{\substack{x \,\varepsilon\, X \\ 0 \neq Lx-h^o \varepsilon Z^o}} \frac{||R_1^o x||}{||Lx-h^o||_{Z^o}} , \tag{22}$$

from which we deduce (18).

Returning to formulas (20) and (21), we see that

$$K = \sup_{\substack{x,s \,\varepsilon\, X \\ x-s \neq 0}} \frac{||R_1^o(x-s)||}{||x-s||_{\mathcal{X}}} = \sup_{\substack{s,x \,\varepsilon\, X \\ 0 \neq L(x-s) \varepsilon\, Z^o}} \frac{||R_1^o(x-s)||}{||Lx-h^o-(Ls-h^o)||_{Z^o}} =$$

$$= \sup_{\substack{x,s \,\varepsilon\, X \\ 0 \neq L(x-s) \varepsilon Z^o}} \frac{||R_1^o(x-s)||}{||L(x-s)||_{Z^o}} = ||Q_1^o \mid V^o|| \leq ||Q_1^o|| \qquad (23)$$

By $||Q_1^o \mid V^o||$ on signifies the norm of the operator Q_1^o restricted to the subspace LX.

Starting now from the definition of the norm, we prove the inverse inequality

$$||Q_1^o|| \leq \sup_{\substack{x \,\varepsilon\, X \\ 0 \neq L(x-s) \varepsilon Z^o}} \frac{||R_1^o x||}{||Lx-h^o||} = \sup_{\substack{x \,\varepsilon\, \mathcal{X} \\ 0 \neq x-s \varepsilon\, \mathcal{X} \\ 0 \neq L(x-s) \varepsilon Z^o}} \left[\frac{||R_1^o(x-s)||}{||x-s||} \frac{||Lx-h^o-(Ls-h^o)||_{Z^o}}{||Lx-h^o||_{Z^o}}\right] . \qquad (24)$$

But since

$$||Lx - h^o - (Ls - h^o)||_{Z^o} \leq ||Lx - h^o||_{Z^o}$$

we deduce that

$$||Q_1^o|| \leq \sup_{\substack{x \,\in\, \mathcal{X} \\ 0 \neq x-s \varepsilon\, \mathcal{X}}} \frac{||R_1^o(x-s)||}{||x-s||} = K. \qquad (25)$$

From (22) and (25) we conclude that $K = ||Q_1^o||$. Finally for all admissible A we have

$$||Q_1|| = \sup_{\substack{x \in \mathcal{X} \\ Lx-h^o \neq 0}} \frac{||R_1 x||}{||Lx-h^o||_{Z^o}} \geq \sup_{\substack{x \in \mathcal{X} \\ Lx-h^o \neq 0}} \frac{||R_1^o x||}{||Lx-h^o||_{Z^o}} = \sup_{\substack{x \in \mathcal{X} \\ L(x-s) \neq 0}} \frac{||R_1^o (x-s)||}{||L(x-s)||_{Z^o}} \qquad (26)$$

whence

$$K = ||Q_1^o|| = \inf_{A \in \mathcal{A}} ||Q_1|| \,. \qquad (27)$$

We can seek to improve the estimation of the error given in (13) by following the idea exploited by Meinguet [7] for the case of functionals (see also for example [4], [5]).

For this we introduce the set D X

$$D = \{x \in X: ||Lx - h^o||^2_{Z^o} \leq d^2\} \qquad (28)$$

and

$$GD = \{Gx;\ x \in D\}. \qquad (29)$$

The set GD is closed, convex and has center Gs, s being the smoothing spline studied above. Let Ls be the projection of h^o on LX. Then we have the following estimation

$$\|R_1^o x\|^2 \le \|Q_1^o\|^2[\|Lx - h^o\|^2_{Z^o} - \|Ls - h^o\|^2] \le \|Q_1^o\|^2[d^2 - \|Ls - h^o\|^2_{Z^o}]. \quad (30)$$

Indeed from (14) and (23) we deduce

$$\|R_1^o x\|^2 = \|R_1^o(x-s)\|^2 \le \|Q_1^o\|^2 \|Lx - h^o - (Ls - h^o)\|^2_{Z^o} \quad ,$$

and because of the equalities

$$\|Lx - h^o - (Ls - h^o)\|^2_{Z^o} = \|Lx - h^o\|^2_{Z^o} - \langle \underset{\varepsilon\, V}{Ls - h^o}, \underset{\varepsilon\, V}{L(x-s)} \rangle_{Z^o} -$$

$$- \langle Lx - h^o, Ls - h^o \rangle_{Z^o} = \|Lx - h^o\|^2_{Z^o} - \|Ls - h^o\|^2_{Z^o} \quad ,$$

there results finally the estimation (30).

4. <u>Example</u>.

We shall describe a category of problems to which the smoothing splines for operators can be applied.

Suppose that we have a partial differential equation with boundary conditions. With the help of an analogue computer or otherwise one can often obtain some approximations of the solution at several particular points

$$x = x_i, \qquad u(x_i, y) = u_k(y), \qquad 1 \le i \le N_1 .$$

Denote by $u(x,y)$ the exact solution of the boundary value problem. Similarly, for particular values of y, $y = y_k$, we can obtain

$$u(x, y_k) = u_k(x), \qquad 1 \le k \le N_2 .$$

Suppose that we wish to compute approximately $u(x_i, y_j)$ or $\frac{\partial u(x_i, y_j)}{\partial x}$ using the given experimental information.

Consider as G the operator

$$G: u(x,y) \rightarrow u(x_i, y_j)$$

or

$$G: u(x,y) \rightarrow \frac{\partial u(x_i, y_j)}{\partial x} .$$

Choosing in a convenient way the spaces X, Y, Z, Z^i, and fixing the operator U related to the given practical problem (see the example given by Sard for interpolating spline [10]), we can obtain in principle an approximation of $u(x_i, y_j)$, or $\frac{\partial u(x_i y_j)}{\partial x}$, which is optimal in the sense made precise above.

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