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Attacks on the Faithfulness of the Burau Representation of the Braid Group B_4

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Abstract

The faithfulness of the Burau representation of the 4-strand braid group, B_4, remains an open question. In this work, there are two main results. First, we specialize the indeterminate t to a complex number on the unit circle, and we find a necessary condition for a word of B_4 to belong to the kernel of the representation. Second, by using a simple algorithm, we will be able to exclude a family of words in the generators from belonging to the kernel of the reduced Burau representation.

Keywords: braid group, Burau representation, faithful

1. Introduction

Magnus and Peluso (1969) showed that the Burau representation is faithful for $n \leq 3$. Moody (1991) showed that it is not faithful for $n \geq 9$; this result was improved to $n \geq 6$ by Long and Paton (1992). The non-faithfulness for $n = 5$ was shown by Bigelow (1999). The question of whether or not the Burau representation for $n = 4$ is faithful is still open.

In our work, we attack the question of faithfulness of the Burau representation of B_4. In section 3, we specialize the indeterminate t to a complex number $e^{i\alpha}$, where $\alpha \in \mathbb{R}$. Then we show that if $\frac{\alpha}{\pi} \notin \mathbb{Q}$ and $4\epsilon + 3m \neq 0$, then the word $b^{\epsilon_1} a^{m_1} b^{\epsilon_2} a^{m_2} ....b^{\epsilon_n} a^{m_n}$ does not belong to the kernel of the representation . Here $\epsilon_i = 0, 1$ or 2, $m_i \in \mathbb{Z}$, $\epsilon = \sum_{i=1}^{n} \epsilon_i$ and $m = \sum_{i=1}^{n} m_i$, for $1 \leq i \leq n$. In section 4, we let $a = \sigma_1\sigma_2\sigma_3$ and $b = a\sigma_1$, where $\sigma_1, \sigma_2, \sigma_3$ are generators of B_4. Then we find the general form of the words a^n and b^n and we prove that they are not in the kernel of the representation for any non-zero natural number n. In section 5, we introduce a simple algorithm which computes all words of the form $a^i b^j$ and $a^i b^j a^k$ for integers i, j and k. We then conclude that there is no word of such forms in the kernel of the representation.

2. Preliminaries

Definition 1. *(Artin, 1965) The braid group, B_n, is an abstract group generated by $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ with the following relations*

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \textit{ for all } i, j = 1, ..., n-1 \textit{ with } |i-j| \geq 2,$$

and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \textit{ for } i = 1, ..., n-2.$$

Definition 2. *(Burau, 1936) The reduced Burau representation of B_n is defined by*

$$\alpha_n\colon B_n \to GL(n-1, \mathbb{Z}[t, t^{-1}])$$

$$\sigma_1 = \begin{pmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}, \sigma_{n-1} = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & -t \end{pmatrix},$$

$$\sigma_i = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-(i+2)} \end{pmatrix}, \textit{ where } 2 \leq i \leq n-2.$$

In particular, setting $n = 4$, we have

$$\sigma_1 = \begin{pmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & -t \end{pmatrix},$$

where t is an indeterminate.

Let $a = \sigma_1\sigma_2\sigma_3$ and $b = a\sigma_1$. Then we get

$$a = \begin{pmatrix} 0 & 0 & -t \\ t & 0 & -t \\ 0 & t & -t \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 0 & -t \\ -t^2 & t & -t \\ 0 & t & -t \end{pmatrix}.$$

Since the determinants $|\sigma_1| = |\sigma_2| = |\sigma_3| = -t$, it follows that $|a| = -t^3$ and $|b| = t^4$.

Theorem 1. *(Holtzma, 2008) Let B_4 be the braid group of order 4 then*

1. $B_4 =< a, b|a^4 = b^3, a^2bab = baba^2 >$,
2. $Z(B_4) =< a^4 >$.

Using Theorem 1, we can show that the elements of B_4 are either of the form $b^{\epsilon_1}a^{m_1}b^{\epsilon_2}a^{m_2}.......b^{\epsilon_n}a^{m_n}$ or of the form obtained by permuting a^{m_i} and b^{ϵ_i}. Here $\epsilon_i = 0, 1, 2$ and $m_i \in \mathbb{Z}$ $(i = 1, ..., n)$.

3. Necessary Condition for Elements in the Kernel of the Reduced Burau Representation

Let t be a non-zero complex number on the unit circle, $t = e^{i\alpha}$, where α is a non-zero real number.

Theorem 2. *Given a non zero integer n and a word u in B_4, $\epsilon_i = 0, 1$ or 2, $m_i \in \mathbb{Z}$, $\epsilon = \sum_{i=1}^{n} \epsilon_i$ and $m = \sum_{i=1}^{n} m_i$. Suppose that $\frac{\alpha}{\pi} \notin \mathbb{Q}$. If u is a non empty word of the form $b^{\epsilon_1}a^{m_1}b^{\epsilon_2}a^{m_2}.......b^{\epsilon_n}a^{m_n}$ such that $4\epsilon + 3m \neq 0$, then u does not belong to the kernel of the representation.*

Proof. $|u| = |b|^\epsilon |a|^m = 1$. So $(-1)^m t^{4\epsilon+3m} = 1$. Then we have 2 cases.

(i) If m is even then $e^{i\alpha(4\epsilon+3m)} = 1$ and so $(4\epsilon+3m)\alpha = 2k\pi$, where $k \in \mathbb{Z}$. This implies that $\frac{\alpha}{\pi} \in \mathbb{Q}$, which is a contradiction.

(ii) If m is odd then $e^{i\alpha(4\epsilon+3m)} = -1$ and so $(4\epsilon + 3m)\alpha = (2k + 1)\pi$, where $k \in \mathbb{Z}$. This implies that $\frac{\alpha}{\pi} \in \mathbb{Q}$, which is a contradiction.

Likewise for a word obtained from u by permuting b^{ϵ_i} and a^{m_i}. □

Corollary 1. *For a word u of the form $b^{\epsilon_1}a^{m_1}b^{\epsilon_2}a^{m_2}.......b^{\epsilon_n}a^{m_n}$ to belong to the kernel of the representation, m has to belong to $4\mathbb{Z}$ and ϵ has to belong to $3\mathbb{Z}$.*

4. The Words a^n and b^n

In this section, we find the general form of the words a^n and b^n, for any integer n. Denote by I_3 the identity matrix. We recall, from section 2, that $a = \sigma_1\sigma_2\sigma_3$ and $b = a\sigma_1$. It is easy to see that $b^{-1} = b^2a^{-4}$ and $b^{-2} = ba^{-4}$.

Proposition 1. *Consider the matrix* $J = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$. *For any $k \in \mathbb{N}$, we have*

1. $a^{4k} = t^{4k}I_3$, $\quad a^{4k+1} = t^{4k+1}J$,
   $a^{4k+2} = t^{4k+2}J^2$, $\quad a^{4k+3} = t^{4k+3}J^3$,
2. $a^{-4k} = t^{-4k}I_3$, $\quad a^{-(4k+1)} = t^{-(4k+1)}J^3$,
   $a^{-(4k+2)} = t^{-(4k+2)}J^2$, $\quad a^{-(4k+3)} = t^{-(4k+3)}J$.

Proof. We prove this proposition using mathematical induction principle. For $k = 0$ and $k = 1$, direct computations give us that $J^4 = 1$,

$$a^0 = I_3, a^1 = tJ, a^2 = t^2J^2, a^3 = t^3J^3,$$

$$a^4 = t^4I_3, a^5 = t^5J, a^6 = t^6J^2, a^7 = t^7J^3,$$

$$a^{-1} = t^{-1}J^3, a^{-2} = t^{-2}J^2, a^{-3} = t^{-3}J, a^{-4} = t^{-4}I_3,$$

and

$$a^{-5} = t^{-5}J^3, a^{-6} = t^{-6}J^2, a^{-7} = t^{-7}J.$$

Suppose that the proposition is true for all integers less than or equal to k. We then show it is still true for $k + 1$. We show (1):

$$a^{4(k+1)} = a^{4k}a^4 = t^{4k}I_3 = t^{4(k+1)}I_3$$

$$a^{4(k+1)+1} = a^{4k}a^5 = t^{4k}I_3t^5J = t^{4(k+1)+1}J$$

$$a^{4(k+1)+2} = a^{4k}a^6 = t^{4k}I_3t^6J^2 = t^{4(k+1)+2}J^2$$

$$a^{4(k+1)+3} = a^{4k}a^7 = t^{4k}I_3t^7J^3 = t^{4(k+1)+3}J^3$$

We show (2):

$$a^{-4(k+1)} = a^{-(4k)}a^{-4} = t^{-(4k)}I_3t^{-4}I_3 = t^{-4(k+1)}I_3$$

$$a^{-(4(k+1)+1)} = a^{-4k}a^{-5} = t^{-4k}I_3t^{-5}J^3 = t^{-(4(k+1)+1)}J^3$$

$$a^{-(4(k+1)+2)} = a^{-4k}a^{-6} = t^{-4k}I_3t^{-6}J^2 = t^{-(4(k+1)+2)}J^2$$

$$a^{-(4(k+1)+3)} = a^{-4k}a^{-7} = t^{-4k}I_3t^{-7}J = t^{-(4(k+1)+3)}J$$

Therefore the proposition is true for all $k \in \mathbb{N}$. □

Proposition 2. *For all $k \in \mathbb{N}$, we have*

1. $b^{3k} = t^{4k}I_3, \quad b^{3k+1} = t^{4k}b, \quad b^{3k+2} = t^{4k}b^2,$
2. $b^{-3k} = t^{-4k}I_3, \quad b^{-(3k+1)} = t^{-4(k+1)}b^2, \quad b^{-(3k+2)} = t^{-4(k+1)}b.$

Proof. We prove the proposition using mathematical induction principle. Direct computations show

$$b^0 = I_3, b^1 = b, b^2 = b^2$$

$$b^3 = t^4I_3, b^4 = t^4b, b^5 = t^4b^2$$

$$b^{-1} = t^{-4}b^2, b^{-2} = t^{-4}b, b^{-3} = t^{-4}I_3$$

$$b^{-4} = (t^{-4})^2b^2, b^{-5} = (t^{-4})^2b$$

Suppose that it is true for all integers less than or equal to k. We now show it is still true for $k + 1$. We show (1).

$$b^{3(k+1)} = b^{3k}b^3 = t^{4k}I_3t^4I_3 = t^{4(k+1)}I_3$$

$$b^{3(k+1)+1} = b^{3k}.b^4 = t^{4k}I_3.t^4b = t^{4(k+1)}b$$

$$b^{3(k+1)+2} = b^{3k}.b^5 = t^{4k}I_3.t^4b^2 = t^{4(k+1)}b^2$$

As for (2):

$$b^{-3(k+1)} = b^{-3k}b^{-3} = t^{-4k}I_3t^{-4}I_3 = t^{-4(k+1)}I_3$$

$$b^{-(3(k+1)+1)} = b^{-3k}.b^{-4} = t^{-4k}I_3.t^{-8}b^2 = t^{-4((k+1)+1)}b^2$$

$$b^{-(3(k+1)+2)} = b^{-3k}.b^{-5} = t^{-4k}I_3.t^{-8}b = t^{-4((k+1)+1)}b \qquad \square$$

Lemma 1. *For any non-zero integer n, a^n and b^n do not belong to the kernel of the reduced burau representation $B_4 \to GL(3, \mathbb{Z}[t, t^{-1}])$.*

Proof. Consider the matrix $J = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$. Direct computations show that

$$J^2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \text{ and } J^3 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Let $n \in \mathbb{Z}$, so n has one of the forms

$$\pm 4k, \pm(4k+1), \pm(4k+2) \text{ and } \pm(4k+3) \text{ , where } k \in \mathbb{Z}.$$

This implies, by Proposition 1, that a^n has to be one of the following words:

$$t^{4k}I_3,\ t^{4k+1}J,\ t^{4k+2}J^2,\ t^{4k+3}J^3\ t^{-4k}I_3,\ t^{-(4k+1)}J^3,\ t^{-(4k+2)}J^2,\ t^{-(4k+3)}J.$$

We denote by J^k_{ii} the diagonal entry of the matrix J^k, which lies in the ith row and in the ith column ($1 \le i \le 3, 1 \le k \le 3$). Since J_{11}, J^2_{11} and J^3_{22} are zeros, it follows that a^n is not the empty word for any integer n.

On the other hand, n is written in either one of the following forms:

$$\pm 3k, \pm(3k+1), \pm(3k+2), \text{ where } k \in \mathbb{Z}$$

This implies, by Proposition 2, that b^n has to be one of the following words:

$$t^{4k}I_3,\ t^{4k}b,\ t^{4k}b^2,\ t^{-4k}I_3,\ t^{-4(k+1)}b^2,\ t^{-4(k+1)}b$$

Direct computations show that

$$b = \begin{pmatrix} 0 & 0 & -t \\ -t^2 & t & -t \\ 0 & t & -t \end{pmatrix} \text{ and } b^2 = \begin{pmatrix} 0 & -t^2 & t^2 \\ -t^3 & 0 & t^3 \\ -t^3 & 0 & 0 \end{pmatrix}.$$

The 1-1 entries in both of the matrices b and b^2 are equal to zeros , it follows that b^n is not the empty word for any integer n. $\square$

5. Words of The Form a^ib^j And $a^ib^ja^k$

In this section, we use Mathematica to excute a program that computes words of the form a^ib^j and $a^ib^ja^n$ for all non zero integers i, j and n. In order to excute this program, we consider the following notations:

$c_1 = a^{4k_1}$, $c_2 = a^{4k_2+1}$, $c_3 = a^{4k_3+2}$, $c_4 = a^{4k_4+3}$,
$c_5 = a^{-4k_5}$, $c_6 = a^{-(4k_6+1)}$, $c_7 = a^{-(4k_7+2)}$, $c_8 = a^{-(4k_8+3)}$,
$z_1 = a^{4x_1}$, $z_2 = a^{4x_2+1}$, $z_3 = a^{4x_3+2}$, $z_4 = a^{4x_4+3}$,
$z_5 = a^{-4x_5}$, $z_6 = a^{-(4x_6+1)}$, $z_7 = a^{-(4x_7+2)}$, $z_8 = a^{-(4x_8+3)}$,
$d_1 = b^{3s_1}$, $d_2 = b^{3s_2+1}$, $d_3 = b^{3s_3+2}$, $d_4 = b^{-3s_4}$,
$d_5 = b^{-(3s_5+1)}$, $d_6 = b^{-(3s_6+2)}$,

$L = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$, $S = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ and $R = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\}$.

Then we excute the following codes:

Algorithm1: For $a^i b^j$
For $m = 1, m < 9, m + +,$
$\quad$ For $n = 1, n < 7, n + +,$
Print [[L[[m]].S[[n]], c, m, d, n]]

and

Algorithm2: For $a^i b^j a^k$
For $m = 1, m < 9, m + +,$
$\quad$ For $n = 1, n < 7, n + +,$
$\quad\quad$ For $r = 1, r < 9, r + +,$
Print [[L[[m]].S[[n]].R[[r]], c, m, d, n, c, r]]

These codes compute all words of the form $a^i b^j$ and $a^i b^j a^k$ for all non zero integers i, j and k. Among these words, the ones that might possibly belong to the kernel of the representation are those which have the form $t^\alpha I_3$. More precisely, these words are

c_1d_4, c_5d_1, c_5d_4, $c_1d_1z_5, c_1d_4z_1, c_1d_4z_5, c_2d_1z_6, c_2d_4z_4, c_2d_4z_6, c_3d_1z_7, c_3d_4z_3,$
$c_3d_4z_7, c_4d_1z_8, c_4d_4z_2, c_4d_4z_8, c_5d_1z_1, c_5d_1z_5, c_5d_4z_1, c_5d_4z_5, c_6d_1z_2, c_6d_1z_8, c_6d_4z_2,$
$c_6d_4z_8, c_7d_1z_7, c_7d_4z_3, c_7d_4z_7, c_8d_1z_4, c_8d_1z_6, c_8d_4z_4, c_8d_4z_6.$

It is clear that each of these words is the empty word. For example, $c_1d_1z_5 = t^\alpha I_3$, where $\alpha = 4k_1 + 4s_1 - 4x_5$. If $\alpha = 0$, then $x_5 = k_1 + s_1$.

Since $a^4 \in Z(B_4)$ and $a^4 = b^3$, it follows that $c_1d_1z_5 = a^{4k_1} b^{3s_1} a^{-4x_5} = a^{4k_1} b^{3s_1} a^{-4(k_1+s_1)} = 1$.

Therefore we get the following theorem.

Theorem 3. *For integers i, j, k, there are no words of the form $a^i b^j a^k$ which lies in the kernel of the Burau representation.*

References

Artin, E. (1965). *The Collected Papers of Emil Artin,* Addison-Wesley Publishing Company, Inc. http://dx.doi.org/10.1007/978-1-4612-5717-2

Burau, W. (1936). *Uber Zopfgruppen und gleichsinnig verdrillte Verkettungen,* Abh. Math. Sem. Hamburg, 11, 179-186. http://dx.doi.org/10.1007/bf02940722

Bigelow, S. (1999). The Burau representation of the braid group B_n is not faithful for n = 5. *Geometry and Topology, 3,* 397-404. http: //dx.doi.org/10.2140/gt.1999.3.397

Holtzman, C. (2008). *Sous groupes de petit indice des groupes de tresses et systeme de reecriture,* Doctoral thesis, institut de mathematiques de bourgogne.

Long, D ., & Paton, M. (1992). The Burau representation of the braid group B_n is not faithful for n ≥ 6. *Topology, 32,* 439-447. http://dx.doi.org/10.1016/0040-9383(93)90030-y

Magnus, W. & Peluso, A. (1969). On a theorem of V. I. Arnold. *Comm. Pure Appl. Math, 22,* 683-692. http://dx.doi.org/10.1002/cpa.3160220508

Moody, J. (1991). The Burau representation of the braid group B_n is not faithful for large n. *Bull. Amer. Math.Soc., 25,* 379-384. http://dx.doi.org/10.1090/S0273-0979-1991-16080-5