The Continuum Limit of Discrete Geometries

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Abstract

In various areas of modern physics and in particular in quantum gravity or foundational space-time physics it is of great importance to be in the possession of a systematic procedure by which a macroscopic or continuum limit can be constructed from a more primordial and basically discrete underlying substratum, which may behave in a quite erratic and irregular way. We develop such a framework within the category of general metric spaces by combining recent work of our own and ingeneous ideas of Gromov et al, developed in pure mathematics. A central role is played by two core concepts. For one, the notion of intrinsic scaling dimension of a (discrete) space or, in mathematical terms, the growth degree of a metric space at infinity, on the other hand, the concept of a metrical distance between general metric spaces and an appropriate scaling limit (called by us a geometric renormalisation group) performed in this metric space of spaces. In doing this we prove a variety of physically interesting results about the nature of this limit process, properties of the limit space as e.g. what preconditions qualify it as a smooth classical space-time and, in particular, its dimension.

Keywords: Gromov-Hausdorff-Distance, Discrete Metric Spaces, Continuum Limit, Geometric Growth at Infinity, Dimension

1 Introduction

In this note we want to report on interesting parallel structures in, on the one hand, a sub-field of current research in quantum gravity or foundational (quantum) space-time physics and, on the other hand, in certain fields of modern mathematics we became aware of only recently (cf. the two beautiful essays by M.Berger, [1], about the work of Gromov in geometric group theory and related fields).

In various areas of modern physics and, in particular, in quantum gravity it is extremely important to develop effective (technical) methods which allow one to perform a suitable continuum limit, starting from discrete and frequently quite irregular structures like e.g. (fluctuating) large and densely netted networks or dynamic graphs.

The underlying physical idea is, that (quantum) space-time on the Planck scale is presumably a very erratic and wildly fluctuating structure which does not resemble very much anything akin to the macroscopic space-time continuum we are accustomed to. One might therefore entertain the idea to model this primordial substratum as a basically discrete dynamic structure of relatively elementary constituents and their interactions and try to reconstruct our continuum concepts by way of performing an appropriate continuum limit, shedding, by the same token, some light on the deep question which discrete concepts are the appropriate counterparts of their continuum cousins.

We note that this problem is virulent in all present approaches to quantum gravity in one form or the other, the most wide spread being string theory and loop quantum gravity (see for example [2],[3] or [4]). There exist however other, less well-known, but nevertheless promising frameworks. While taking an appropriate continuum limit is a desideratum in all the different approaches, the strategies employed are, on the other hand, too different concerning concepts and technical details so that we prefer to concentrate in the following on a particular approach we developed in the recent past and which does not need too many extra assumptions. In contrast to other frameworks which are implicitly or explicitly mainly inspired by ideas, occurring in combinatorial topology and related fields (simplicial complexes to give an example), we rather view primordial space-time as a large dynamic array of interacting elementary degrees of freedom which have the propensity to generate, as an emergent phenomenon, a macroscopic smooth space-time on a coarser level of resolution. It has the further advantage that the mathematical framework, we are going to develop in the following, becomes quite transparent and coherent. Furthermore, it is sufficiently general and flexible so as to be applicable, after slight modifications, to related problems in a wide variety of other contexts.

This change of working philosophy implies that in the following we will mainly work in the metric category of general spaces and not so much in the differential geometric category. We think these metrical aspects are of a more primordial character in physics and are standing between the more pristine topological aspects of a theory and the more advanced analytical properties which are perhaps only very useful (over)idealisations (see also [5]). Our general aim will be the study of large discrete networks assumed to model space-time on a primordial level, where by discrete we do not necessarily mean something like lattices or other countable structures but rather the absence of the usual continuum concepts. We note in particular that it is important to avoid any idea of an ambient space in which our networks are assumed to be embedded. In studying model systems it is of course frequently advantageous to restrict the analysis (mostly for technical convenience) to countable structures. We do not want to comment on the pros and cons of our general working philosophy and refer instead to some of our recent papers (see for example [6],[7],[8]) where more references can be found. We emphasize however that the framework we present in the following is quite self-contained and does not need many prerequisites coming from elsewhere. While being based on a different set of fundamental principles, the work of Borchers and Sen shares some of the aims, pursued by us, for example, reconstructing continuum concepts of space-time from more fundamental notions (see [9],[10]).

Sufficiently interesting and complex examples of discrete geometric structures are large or even infinite graphs. They represent a rich class of models we employed in the description of aspects of the underlying geometric substratum of our networks in the above cited papers and they will also serve as important model geometries in our following analysis.

Just for the record we give some definitions. While in some of our papers (for example in [11]) orientation and direction of edges played a role for doing discrete (functional) analysis, this is, for the time being, not relevant for the following more geometric analysis. Furthermore we deal only with what some graph theorists call *simple graphs.*

Definition 1.1 *A countable labelled (unoriented) graph, $G = (V, E)$, consists of a countable set of vertices (or nodes), $x_i \in V$, and a countable set of edges (or links), $e_{ij} = (x_i, x_j) \in E \subset V \times V$ so that e_{ij} and e_{ji} are identified, put differently, the relation E is assumed to be symmetric.*

Remark: It is important to note that in general we consider abstract graphs, i.e. not assumed to be embedded in an ambient topological space. Both vertices and edges can be considered as representatives of concepts or properties which, in principle, have nothing to do with points and lines in some ambient geometric environment. In our working philosophy the edges frequently represent elementary interactions between certain degrees of freedom or information channels.

For a general orientation see for example [12]. At the moment we introduce only very few concepts from graph theory, making further comments later when needed.

Definition 1.2 *A vertex, x, has vertex degree $v(x) \in \mathbb{N}_0$ or $= \infty$ if it is incident with $v(x)$ edges, put differently, the number of unoriented pairs (x, y), in which x occurs. The degree function $v(x)$ is called locally bounded if $v(x) \in \mathbb{N}_0$. It is called globally bounded if $v(x) \leq v < \infty$ on V. The graph is called regular if $v(x)$ is constant on V.*

Remark: In the functional analysis or operator theory on graphs, graphs with locally or globally bounded vertex degree represent important subclasses.

Whereas this is not really necessary we deal in the following for convenience with *connected* graphs.

Definition 1.3 *A graph is called connected if each pair of nodes, x, y can be connected by a finite edge sequence, γ, starting from x and ending in y. A path is an edge sequence without repetition of vertices with the possible exception of initial and terminal vertex. The number of edges occurring in the path is called the (combinatorial) length, $l(\gamma)$, of the path.*

Observation 1.4 *As these numbers are integer, there exists always a path of minimal length, called a geodesic path. This defines a natural distance concept on graphs (path metric).*

$$d(x,y) := \min_{\gamma}\{l(\gamma), \gamma \text{ connects } x \text{ with } y\} \tag{1}$$

has the properties of a metric, i.e.

$$d(x,x) = 0\ ,\ d(x,y) = 0 \rightarrow x = y\ ,\ d(x,y) = d(y,x)\ ,\ d(x,z) \leq d(x,y) + d(y,z) \tag{2}$$

the last relation being the crucial one.

The proof is more or less obvious.

Remark: With this metric G becomes a complete, discrete metric space. We note however that one can introduce many other types of metrics on graphs. A simple choice is the transition from d to $\lambda \cdot d$, $\lambda \in \mathbb{R}$ (a metric playing a certain role in the following). More intricate types of metrics have for example been studied in [13].

The notion of dimension is an important characteristic in continuum mathematics and physics. In our context it is important to define related concepts on discrete disordered structures which go over in corresponding continuum concepts when performing some continuum limit. On the other hand, in physics it is crucial that these concepts encode certain relevant geometric characteristics of the physical systems under discussion. We formulated such a concept in [14] and showed its usefulness. In [7] we used it among other concepts, to characterize the nature of what we called a *geometric renormalisation process*, i.e. a coarse-graining and rescaling process which, hopefully, allows us to construct a continuum limit, the supposed *fixed point* of the process, from a discrete underlying substratum. The idea has some vague similarities to the *real-space renormalisation group* of statistical mechanics, but it represents, as far as we can see, a technically much more complicated enterprise.

In the following two sections we will mainly study these dimensional concepts and relate them to important concepts being used in *geometric group theory* and related fields in pure mathematics. With applications in physics in our mind, we will mainly be interested in situations where the underlying primordial structure, i.e. in our case the large irregular graphs or networks, has a definite generalized dimension (which, as we will see, is already a particular situation). This holds the more so if this dimension happens to be an integer.

Our aim is, among other things, to provide criteria on this fundamental scale under which these properties do occur. In the remaining sections we will show that what we defined as a geometric renormalisation process has a strong and surprising relation to a relatively recent and intensively studied topic of pure mathematics which is connected with the name of M.Gromov, i.e. the study of the asymptotic limits of general metric spaces, particular subclasses being finitely generated groups and their Cayley graphs and fractal geometries. The combination of these tools and concepts allow us to prove some interesting results about existence and properties of continuum limits of discrete spaces. To sum it up, on the more technical side of the matter, the two topics we are going to study are first, generalized dimension as a characteristic of classes of general metric spaces and second, distances between metric spaces and possible limits of sequences of such spaces.

2 Dimension and Growth on Graphs

The line of thought which led us to the formulation of dimensional concepts on networks and graphs in [14] was essentially motivated by physics. In statistical mechanics, for example, many model systems are placed on a lattice being embedded in an ambient euclidean space, $\mathbb{R}^d$. One observes that this embedding dimension, d, then shows up as a relevant external parameter in many important physical expressions. It governs phase transitions, critical behavior and the decay properties of correlation functions, to mention only a few examples. This restricted context however masks the intrinsic role of something like dimension in these examples.

In a thought experiment we choose to ignore the embedding space and treat the system as an array of interacting degrees of freedom sitting on some labelled graph or discrete set of nodes. The physics has to remain the same. We learn from this that what is really important is the number of nodes or degrees of freedom a given fixed node can have contact with after a certain sequence of consecutive steps on the graph or sequence of elementary interactions.This process defines a natural neighborhood structure, viz. fixes the intrinsic geometry of the system.

We hence make the following series of definition.

Definition 2.1 *Let $B(x,r)$ be the ball of radius r around vertex x on the graph G, viewed as a metric space with respect to the standard graph metric d introduced in the introduction. That is*

$$y \in B(x,r) \; if \, d(x,y) \leq r \tag{3}$$

Denote further by $\partial B(x,k)$ the set of vertices having exactly the distance k from vertex x (which is for the above metric only meaningful for $k \in \mathbb{N}_0$). By $|B(x,r)|$, $|\partial B(x,k)|$ respectively we designate the number of nodes lying in these sets.

We now define *growth function* and *spherical growth function* on G relative to some arbitrary but fixed vertex x. (We use here the notation more common in geometric group theory. In other fields it is also called the distance degree sequence, cf. [15]).

Definition 2.2 *The growth function $\beta(G,x,r)$ is defined by*

$$\beta(G,x,r) = |B(x,r)| \tag{4}$$

Correspondingly we define

$$\partial\beta(G,x,k) := \beta(G,x,k) - \beta(G,x,k-1) \tag{5}$$

With the help of the limiting behavior of β , $\partial\beta$ we introduced two dimensional concepts in [14].

Definition 2.3 *The (upper,lower) internal scaling dimension with respect to the vertex x is given by*

$$\overline{D}_s(x) := \limsup_{r\to\infty}(\ln\beta(x,r)/\ln r) \ , \ \underline{D}_s(x) := \liminf_{r\to\infty}(\ln\beta(x,r)/\ln r) \tag{6}$$

The (upper,lower) connectivity dimension is defined correspondingly as

$$\overline{D}_c(x) := \limsup_{k\to\infty}(\ln\partial\beta(x,k)/\ln k) + 1 \ , \ \underline{D}_k(x) := \liminf_{k\to\infty}(\ln\beta(x,k)/\ln k) + 1 \tag{7}$$

If upper and lower limit coincide, we call it the internal scaling dimension, the connectivity dimension, respectively.

Remark:i) The two notions are not entirely the same in general whereas they coincide for many models (this is quite similar to the many different fractal dimensions).
ii) For regular lattices both yield the expected result, i.e. the embedding dimension. In general however upper and lower limit are different and non-integer. Similarities to fractal dimensions are not accidental. For a more thorough discussion of all these points see [14].
iii) These notions or similar ones have already been introduced earlier (a point we were unaware of at the time of writing [14], see for example [21][22],[23] or [24] and presumably elsewhere) but as far as we can see, such an idea was either only mentioned in passing or its many interesting properties never investigated in greater detail.

It is important (in particular for physics) that these notions display a marked rigidity against all sorts of deformations of the underlying graph and are independent of the reference vertex for locally finite graphs. We mention only two properties in this direction.

Observation 2.4 *i)If the vertex-degree of the graph is locally finite, the numerical values of the above quantities are independent of the reference vertex.*
ii)Insertions of arbitrarily many edges within a k-neigborhood of any vertex do not alter the dimension. Edge deletions, fulfilling a slightly more complicated locality property, do also not change these values (cf. lemma4.10 in [14], the discussion in sect.VII of [7] and theorem 6.8 in [8]). More specifically, edge deletions are called k-local if, in the transition from G to G', only edges are deleted in G so that for the corresponding pairs of nodes, (x,y), it holds that $y \in B_{G'}(x,k)$ with respect to G'.

This geometric concept played an important role in our analysis of ther discrete substructure of continuum space-time and a possible limit behavior towards a continuous (smooth or fractal) macroscopic geometry. It is therefore worth mentioning that similar concepts are playing a crucial role in an important field of pure mathematics, called *geometric group theory* as we learned relatively recently (cf. e.g. [16]). In this latter framework one typically studies *finitely generated* groups and visualize them as so-called *Cayley graphs*. For the record some brief definitions.

Definition 2.5 *The group Γ is finitely generated, if there is a finite subset, $S \subset \Gamma$, such that every $g \in \Gamma$ can be represented as a finite word under group multiplication.*

$$g = s_1 \cdots s_n \tag{8}$$

with $s_i \in S \cup S^{-1} =: S'$, $g^{-1} \in S^{-1}$ if $g \in S$. It is reasonable to exclude the unit element, e from S.

Remark 2.6 *It is important to note that $S \cap S^{-1}$ may happen to be non-empty! There may exist group elements with $g = g^{-1}$. For convenience one may assume $S^{-1} = S$ (i.e. S being inverse closed).*

We define a *(word)metric* on Γ in the following way.

Definition 2.7 *Let g, g' be represented by two words. Then the word metric d_S is given by*

$$d_S(g, g') := \inf_{|w|} |w|(g^{-1} \cdot g') =: l(g^{-1} \cdot g') \tag{9}$$

where $w(g^{-1} \cdot g')$ is a word representation of $g^{-1} \cdot g'$ with elements from S', $|w|$ the length of the word and the infimum is taken, as for graphs, over the length of the different paths, connecting g with g'. Evidently $l(g^{-1} \cdot g')$ is simply the minimal word distance of $g^{-1} \cdot g'$ from e in the group, Γ.

Definition 2.8 *The Cayley graph $C(S, \Gamma)$ has the elements $g \in \Gamma$ as vertices and an (unoriented) edge is drawn between g, g' if $d_S(g, g') = 1$, i.e., $g' = g \cdot s'$, $s' \in S'$.*

Observation 2.9 *With these definitions and with $e \notin S$ the Cayley graph has no elementary loops or multi-edges and its vertex degree is regular ($v(g) = |S \cup S^{-1}|$).*

Remark 2.10 *The notion of Cayley graph may differ from author to author (cf. e.g. [16], [19] or [12]). With $S = S^{-1} = S'$ one may, for example, draw two oppositely oriented edges between nearest neighbor pairs g, g', i.e. $g' = g \cdot s$, $g = g' \cdot s^{-1}$. Such a symmetric directed graph can however be associated with an undirected graph with vertex degree being half as large. In [12] a Cayley graph is a oriented multi-graph. If it happens for some s_1 that $s_1 = s_1^{-1}$, an edge may point for example from e to s_1 and s_1 to e (i.e. both edges coloured with s_1) while for $s_1 \neq s_1^{-1}$ (and $s_1^{-1} \notin S$) g and gs_1 are only connected by one oriented s_1-edge. As these algebraic details are not so important in our context, we deal in the following only with the unoriented Cayley graph defined above.*

We see that finitely generated groups fit nicely into our framework of graph geometry, leading to a particular class of regular graphs. Cayley graphs are however even more special. They represent a subclass of the so-called *vertex transitive* graphs, being examples of *homogeneous spaces*.

Definition 2.11 *A graph is vertex transitive if its automorphism group acts transitively, i.e. for all $x, y \in V$ there exists a graph automorphism, mapping x to y. A graph automorphism is a bijective map*

$$F : (V, E) \to (V, E) \tag{10}$$

which commutes with the incidence relation, i.e. x is linked with y exactly if Fx is linked to Fy.

Observation 2.12 *Cayley graphs are vertex transitive.*

Proof: This follows directly from their definition. Group multiplication from the left yields a subgroup of the automorphism group and acts transitively. □

Remark: Note that multiplication from the right is in general not an isomorphism; it is a more general kind of transformation.

This property is desirable from a physical point of view. It means that the Cayley graph looks alike irrespectively of the reference vertex being selected.

One should note that in general Cayley graphs are not uniquely related to their groups. There usually exist different sets of generators which generate the group. The respective Cayley graphs are hence in general not isomorphic. There exists, however, a much more natural relation between them.

Definition 2.13 *Let F be a map from a metric space, X, to a metric space, Y with metrics d_X, d_Y. It is called a quasi-isometric embedding if the following holds: There exist constants, $\lambda \geq 1, \epsilon \geq 0$, such that*

$$\lambda^{-1} \cdot d_X(x, y) - \epsilon \leq d_Y(F(x), F(y)) \leq \lambda \cdot d_X(x, y) + \epsilon \tag{11}$$

If, furthermore, there exists a constant ϵ' such that for all $y \in Y$ we have $d_Y(y, F(X)) \leq \epsilon'$, that is, $Y \subset U_{\epsilon'}(F(X))$ it is called a quasi-isometry; the spaces are then called quasi-isometric. There is an equivalent definition which shows that the preceding definition is in fact symmetric between X and Y (see for example [16]). That is, there exists a quasi-isometric map G from Y to X with corresponding constants and $d_X(G \circ F(x), x) \leq \rho$ and $d_Y(F \circ G(y), y) \leq \rho$ for some ρ. If $\lambda = 1$ it is called a rough isometry.

Remark: The latter statement can be proved by defining the inverse map, $g : y \to g(y) \in X$, by selecting one of the possibly several vertices, x, such that $g(y) := x$ with $d_Y(f(x), y) \leq \epsilon'$.

We then have

Theorem 2.14 *On a finitely generated group, G, the word metric is unique up to quasi-isometry. That is, two finite sets of generators generate two Cayley graphs, which are quasi-isometric.*

Proof: The proof employs the fact that the generators of the one set can be represented by words of finite length with respect to the other set of generators.□

Quasi-isometry is a very important concept and replaces the dull category of isometric metric spaces (see the following sections). It is in fact a much more natural concept in many respects in this wider context.

In our investigation of the behavior of the dimension functions $\overline{D}_s, \underline{D}_s$ or the growth series $\beta(x,k)$ in our earlier work we found it surprisingly difficult to give general characterizations of sufficiently large classes of graphs with, for example,

$$\overline{D} < \infty \ , \ \overline{D}_s = \underline{D}_s = D_s \ , \ D_s \in \mathbb{N} \quad \text{or} \quad \beta(x,k) \sim k^s \text{ etc.} \tag{12}$$

which, on the other hand, is very important if one wants to get a sufficiently rich overview of continuum limit spaces figuring as possible candidates of space-time continua. Possible relations to fractal geometry were however discussed in section 5.2 of [14].

That it is difficult to give sufficient characterizations of large or infinite graphs, having certain properties was already observed by Erdos and Renyi at the end of the fifties of the last century and led them to invent the field of *random graph* theory (a standard reference being [20]). Furthermore, these problems are related to the so-called *word-problem* in combinatorial group theory, which makes some of the roots of the problems perhaps better understandable. We think, the close connections to the large and florishing field of geometric group theory will be of quite some help in this respect. In the rest of this section we report on some, in our view, useful results being related to such issues. It will however become apparent that our problems, reported above, are no accident. Some of the theorems we will mention and discuss in the sequel are in fact very deep, relatively recent and typically restricted to the highly regular subclass of Cayley graphs or certain generalisations of them.

The two extremes as to dimensional behavior are given by *trees* and *regular lattices* respectively. For a regular infinite tree of uniform vertex degree $v \geq 3$ we have for the growth series relative to an arbitrary vertex x:

$$\beta(0) = 1\,,\ \beta(1) = 1 + v\,,\ \beta(2) = 1 + v(v-1)\,,\ \beta(k) = 1 + \sum_{\nu=0}^{k-1}(v-1)^{\nu} =$$

$$1 + v \cdot \sum_{\mu=1}^{k}(v-1)^{\mu-1} = 1 + v \cdot [1-(v-1)^k]/[1-(v-1)] \sim v^k \tag{13}$$

for large k. This is clearly an exponential growth (for more details concerning this notion see [16]). Our scaling dimension is ∞ but one can define

$$\omega(\Gamma) := \limsup_k (\beta(x,k,\Gamma))^{k^{-1}} \tag{14}$$

as *exponential growth type*.

The other extreme is given by lattices like $\mathbb{Z}^n$ with

$$\beta(x,k,\mathbb{Z}^n) \sim A \cdot k^n \tag{15}$$

and $D(\mathbb{Z}^n) = n$. Such graphs or groups are called being of *polynomial growth.*

Remark: A little remark is in order here. Our graph metric corresponds to the so-called l_1-metric in euclidean space, that is ($x \in \mathbb{Z}^n$)

$$d_{l_1}(x, 0) = \sum_{i=1}^{n} |x_i| \tag{16}$$

Therefore the growth is not exactly the one found for the ordinary euclidean metric. We have for example for $\mathbb{Z}^2$:

$$\beta(k) = 2k^2 + 2k + 1 \leq Ak^2 \tag{17}$$

for some A.

Definition 2.15 *If $\beta(x, k, G) \lesssim k^{\overline{D}}$ for some $\overline{D} \geq 0$, the graph or group is called of polynomial growth. The degree of polynomial growth is then defined by*

$$\overline{D}(G) = \limsup_k \log \beta(k) / \log(k) \tag{18}$$

Strictly speaking, polynomial growth in geometric group theory is usually defined with an integer, d, in the exponent so that $\overline{D} < d$ if it is not an integer.

(Note that $\overline{D}$ is the same as our upper internal scaling dimension we defined before we learned of the existence of the parallel developements in geometric group theory).

It is important to note that quasi-isometric graphs, both having globally bounded vertex degree, have the same growth type.

Theorem 2.16 *Let G_1, G_2 be graphs with vertex degree globally bounded by v and reference vertices x_1, x_2. Let F be a quasi-isometric embedding of G_1 into G_2, then*

$$\beta_1(k, x_1) \leq A \cdot \beta_2(\lambda k + b, x_2) \tag{19}$$

with $b := d_2(x_2, F(x_1)) + \epsilon$. For G_2 having a polynomial growth degree it follows that

$$\overline{D}_1(x_1) \leq \overline{D}_2(x_2)\ ,\ \underline{D}_1(x_1) \leq \underline{D}_2(x_2) \tag{20}$$

If G_1, G_2 are quasi-isometric these inequalities become equalities.

Proof: Conceptually slightly different proofs of parts of the statement can be found in e.g. [14], [7]sect.VII, [36], or [16]. The notions and concepts, used in [14],[7]sect.VII, are slightly different as at the time of writing the papers we were not aware of the existing parallels in geometric group theory. In [7] we proved however a couple of stronger results relating for example the growth function of a graph to the growth function of the corresponding (coarse-grained) *clique-graph.*

With the above defined ball, B, we have

$$F(B(x_1, k)) \subset B(x_2, \lambda k + B) \tag{21}$$

Thus

$$|F(B(x_1,k))| \leq \beta_2(x_2,\lambda k+b) \tag{22}$$

If $F(x)=F(y)$, it follows from the property of quasi-isometric embedding that $d_1(x,y)\leq\lambda\cdot\epsilon$. Due to the globally bounded vertex degree there can be at most A vertices in a ball around x with radius $\lambda\cdot\epsilon$. This can be easily inferred from our estimates on trees by using for example a spanning tree for the subgraph $B(x,\lambda\cdot\epsilon)$. Therefore $B(x_1,k)$ can at most contain $A\cdot|F(B(x_1,k))|$ vertices, yielding the estimate on the two growth functions. If G_1,G_2 are quasi-isometric, such an estimate holds in both directions.

We come now to the estimates on the dimensions. For the lim sup we have the following. By assumption there exists a subsequence, $k_\nu\to\infty$ so that

$$\ln\beta_1(x_1,k_\nu)/\ln k_\nu \to \overline{D}_1(x_1) \tag{23}$$

Due to eqn(19) and the properties of the logarithm the corresponding subsequence with now $A\cdot\beta_2(\lambda k_\nu+b,x_2)$ inserted instead of $\beta_1(x_1,k_\nu)$ is, on the one hand, an upper bound on the previous subsequence and, on the other hand, has to stay below $\overline{D}_2(x_2)$ assymptotically because of the definition of lim sup. This proves the first statement. As to lim inf, take now a subsequence, r'_ν so that

$$\ln\beta_2(x_2,r'_\nu)/\ln r'_\nu \to \underline{D}_2(x_2) \tag{24}$$

We infer from eqn(19) that

$$\beta_1(r_\nu,x_1)\leq A\cdot\beta_2(r'_\nu,x_2) \tag{25}$$

with $r_\nu:=\lambda^{-1}(r'_\nu-B)\to\infty$ for $r'_\nu\to\infty$. As before we conclude that

$$\ln\beta_1(r_\nu,x_1)/\ln r_\nu \leq \underline{D}_2(x_2) \tag{26}$$

assymptotically. This then holds a fortiori for $\underline{D}_1(x_1)$, yielding the second statement. □

From the above we see that we have found a preliminary answer to one point of our programme, i.e. finding classes of graphs or networks having equivalent growth functions and the same dimensions.

Observation 2.17 *Quasi-isometry defines equivalence classes of graphs having the same dimension and equivalent growth functions in the category of graphs with globally bounded vertex degree.*

To get a better feeling to what extent quasi-isometry restricts the structure of graphs, we show that a graph of globally bounded vertex degree can even be roughly isometric to a graph with unbounded vertex degree.

Observation 2.18 *A graph with globally bounded node degree can be roughly isometric to a graph with unbounded node degree.*

Proof: Take for example some regular graph like $\mathbb{Z}^n$. Attach to each node, x, or, more general, to a local neighborhood of such nodes, a subgraph, G_x, such that all its nodes are linked to x, This implies that the distance of two arbitrary nodes in G_x as a subgraph of this new graph, G', have a distance ≤ 2. As the order of G_x can be chosen arbitrary and dependent on x, one easily gets such graphs as mentioned above by this construction. With this G' and the initial graph G we can define the following map

$$F : G' \ni x' \to x \in G \tag{27}$$

where x is the base-vertex in case $x' \in G_x$ or else the identity map. We easily conclude that

$$d(x', y') \leq d(x, y) + 2 \ , \ d(x', y') \geq d(x, y) \tag{28}$$

so F is evidently a rough isometry. □

What remains is to find sufficiently large classes of graph deformations which lead to quasi-isometries. In section 2 we introduced the notion of k-local edge insertions and deletions. We have already shown elsewhere that they do not alter the dimensions of graphs without using explicitly the notion of quasi-isometry. (see the references given there). To complete the discussion we now show that they in fact lead to quasi-isometries.

Proposition 2.19 *Let G' arise from G by a finite sequence of k_ν-local edge insertions or deletions. All these transformations are quasi-isometries as is (because of transitivity) also the resulting operation. A fortiori, they even represent bilipschitz equivalences.*

Definition 2.20 *A map F from the metric space X to the metric space X' is C-lipschitz if $d'(Fx, Fy) \leq C \cdot d(x, y)$. It is a bilipschitz equivalence if it is bijective and such inequalities hold in both directions. This is equivalent to*

$$C^{-1} \cdot d(x, y) \leq d'(Fx, Fy) \leq C \cdot d(x, y) \tag{29}$$

for some positive C.

Observation 2.21 *The usefulness of bilipschitz equivalences is that they also are topological homeomorphisms.*

Proof of the proposition: Let the vertices x, x' denote the same vertices in G, G' respectively. Then $F : x \to x'$ is a bijective map from G onto G' (more specifically, restricted to the respective vertex sets). In the case of k-local edge insertions we have

$$k^{-1} \cdot d(x, y) \leq d'(Fx, Fy) \leq d(x, y) \tag{30}$$

where only the lhs of the estimate is not entirely obvious. Take a minimal path in G' connecting x' and y' having length $d'(x', y')$. It consists of a vertex sequence $(x' = x'_0, x'_1, \ldots, y' = x'_d)$ with consecutive pairs having distance equal to one. In G this sequence corresponds to a sequence $(x = x_0 = x', x_1, \ldots, x_d = y = y')$. Due to the k-locality of edge insertions, two consecutive vertices in the latter sequence can at most be k steps apart, i.e.

$$d(x_i, x_{i+1}) \leq k \tag{31}$$

From this immediately follows that

$$d(x,y) \leq k \cdot d'(x',y') \tag{32}$$

The case of k-local edge deletions is the inverse process if we start from G' and pass over to G by k-local edge insertions. We get

$$d(x,y) \leq d'(x',y') \leq k \cdot d(x,y) \tag{33}$$

That is, by symmetrizing we have the final estimate holding for both insertions and deletions

$$k^{-1} \cdot d(x,y) \leq d'(Fx,Fy) \leq k \cdot d(x,y) \tag{34}$$

This completes the proof. □

3 The Case of Integer Dimension

In [14] we constructed examples of graphs having an arbitrary real number as dimension. If dimension is a stable characteristic of the limit process described in the following sections, such graphs with non-integer dimension are expected to converge to some fractal limit space. Motivated by physical applications, in particular in foundational space-time physics, the case of integer dimension is of course of particular importance. We know that for example $\mathbb{Z}^n$ has integer graph-dimension n. Thus the preceding results guarantee that local deformations of such lattice graphs yield again graphs with integer dimension.

Conclusion 3.1 *Arbitrary k-local deformations of lattice graphs yield again graphs with integer dimension.*

We note that lattices like $\mathbb{Z}^n$ are Cayley graphs of finitely generated abelian groups. The natural question is, are there other classes of Cayley graphs having an integer dimension? Or more generally, are there more general classes of (Cayley) graphs, having polynomial growth? Abelian groups are a particular subclass of so-called *nilpotent* groups. We will not give the definition of nilpotency here, which would need the introduction of some technical machinery (see for example [17]). Anyway, we have the following deep theorem, which was proved in stages by several authors (see [16] p.201).

Theorem 3.2 *(Dixmier, Wolf, Guivarc'h, Bass) Let Γ be a finitely generated nilpotent group. Then Γ is of polynomial growth . More precisely, there exist constants, A, B and an integer d so that*

$$Ak^d \leq \beta(k) \leq Bk^d \tag{35}$$

A stronger result was proved by Bass ([18]).

Theorem 3.3 *If the nilpotent finitely generated subgroup H has finite index in Γ, then H and Γ have the same integer growth degree. In that case one calls the group Γ almost nilpotent.*

This sequence of results culminated in the observation of Gromov:

Theorem 3.4 *(Gromov)A finitely generated group has polynomial growth iff it contains a nilpotent subgroup of finite index (implying that the growth degree is an integer), see [33].*

So we conclude that all k-locally deformed Cayley graphs of almost nilpotent groups have integer dimension, which is an already quite large class.

There exists an important and quite interesting example which is (for various reasons) intensely studied in geometric group theory. It is the so-called *Heisenberg group*, H, (cf. e.g. [16]). Its elements are triangular matrices of the form:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ m & l & 1 \end{pmatrix} \ , \ k, l, m \in \mathbb{Z} \tag{36}$$

with generators

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \ t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \ , \ u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{37}$$

and group laws

$$su = us \ , \ tu = ut \ , \ ts = stu \tag{38}$$

The general element with entries k, l, m is the word $s^k t^l u^m$. So, as a set, H is standing in natural bijection to $\mathbb{Z}^3$, the latter having dimension 3. After some calculations one finds however ([16], p.197)

Theorem 3.5 *For the Heisenberg group we have*

$$Ak^4 \leq \beta(k) \leq Bk^4 \tag{39}$$

i.e., its growth degree (or our internal scaling dimension) is four.

We conjecture that the reason for this surprising behavior derives from one of the three generating relations, when representing H as a Cayley graph. The critical relation is

$$(s^k t^l u^m) \cdot s = s^{(k+1)} t^l u^{(m+l)} \tag{40}$$

(the other two behave smoothly as to the exponents, cf. [16] p.198). That is, applying s to a group element, g, may send it to an element being far away from g (for large l) in a naive metrical ($\mathbb{Z}^3$)-sense.

We note that we studied possibly related phenomena in [7],[8], where we analysed the possibility that space-dimension changes under, what we called, the coarse-graining or renormalisation process. It turned out that only a very peculiar and *non-local* network wiring (with respect to some coarse-grained metric), called by us *critical network states*, allows for such a dimensional reduction under renormalisation. This seems to be again a point where both fields of research are closely related.

We close this section with another surprising and deep result of advanced graph theory, having again certain ramifications into the part of foundational space-time physics we described above. We already remarked that important situations arise when the network under investigation has an asymptotic dimension, D, and, a fortiori, if this number D is integer. We expect that by studying such particular networks one can learn something useful about the reason why our space-time also has an integer dimension at least on a macroscopic scale.

We have seen that Cayley graphs of polynomial growth have an integer growth degree (which is quite a deep result), and that this also holds for k-local deformations of such graphs and for graphs being quasi-isometric to them. A natural class which fulfills a certain kind of *homogeneity* and comprises the Cayley graphs are the so-called *vertex transitive* graphs.

Definition 3.6 *A graph, G, is called vertex transitive if for each pair of vertices, x, y, there exists an element, α_{xy}, in $AUT(G)$ with $\alpha_{xy}(x) = y$.*

The following is quite obvious.

Proposition 3.7 *A vertex transitive graph has equal node degree for all nodes, x. Furthermore, the growth function, $\partial\beta(x,k)$, is independent of x.*

Proof: The first point follows directly from the definition. As to the second property; for each pair of nodes, x, y, there exists by assumption an automorphism, α, with $\alpha(x) = y$ and $\alpha^{-1}(y) = x$. Let z be an element of $\partial B(x,k)$. There exists a minimal path connecting x and z. This path is mapped onto a path of equal length, connecting y and $\alpha(z)$. This path is again minimal. To show that we assume that there exist another, shorter, path connecting y and $\alpha(z)$. It is mapped by α^{-1} onto a path connecting x and z which is shorter than the original minimal path, which is a contradiction. We hence have

$$\alpha(\partial B(x,k)) \subset \partial B(y,k)\ ,\ \ \alpha^{-1}(\partial B(y,k)) \subset \partial B(x,k) \tag{41}$$

and therefore

$$\partial B(y,k) = \alpha \circ \alpha^{-1}(\partial B(y,k)) \subset \alpha(\partial B(x,k)) \subset \partial B(y,k) \tag{42}$$

implying

$$\alpha(\partial B(x,k)) = \partial B(y,k) \tag{43}$$

This proves the proposition. □

We saw that Cayley graphs are a true subclass of the class of vertex transitive graphs, which, on their hand, are the type of networks we would like to call *strongly homogeneous*. As to this latter class we have the following interesting theorem attributed to Sabidussi ([25]) and discussed for example in [19] sect. 3.9.

Theorem 3.8 *Any connected vertex transitive graph is a retract of a Cayley graph, where a subgraph, Y, of a graph, X, is called a retract of X if there exists a graph-homomorphism, f, from X to Y such that the restriction, $f|Y$, of f to Y is the identity map.*

Remark: The construction of the Cayley graph from a vertex transitive graph, X, essentially consists in replacing each vertex, x, by an *independent set* of vertices of size $|G_x|$, with G_x the stabilizer subgroup of x in $AUT(X)$ and connecting these sets in a bipartite way if the original vertices were neighbors in X.

If one can show that these local sets of vertices have finite cardinality, one can, as in the situation of quasi-isometries or local deformations, conclude that the vertex transitive graph has the same dimension as the Cayley graph, so constructed. This was accomplished in an ingeneous analysis by Trofimov ([27]), see also the nice survey by Imrich et al ([26]). An important role in this context is played by the so-called *systems of imprimitivity.* A nice discussion of this concept can e.g. be found in [19] sect.2.5.

The theorem reads:

Theorem 3.9 *If the locally finite connected vertex transitive graph, X, has polynomial growth, the corresponding Cayley graph, described above is the graph of an almost nilpotent group of finite index. Hence its growth degree is an integer and the same holds for the vertex transitive graph X.*

We again conclude that this also holds for all the local deformations of such graphs and graphs being quasi-isometric to them.

4 The Gromov-Hausdorff Distance

In [6] and [7] we developed a technical framework in greater detail which we sketched already in earlier work. That is, we invented a canonical process which allows us (at least in principle) to construct a *continuum limit space* from some underlying highly irregular and erratic primordial substratum, which, we surmise, is underlying our macroscopic space or space-time on the Planck scale level. This represents however a formidable task in any approach to quantum gravity which starts from some microscopic non-smooth space-time. Therefore, several technical points of the procedure could only be incompletely treated.

To tackle this problem on a sufficiently broad scale, i.e. avoiding already from the beginning a too narrow starting point, we chose as our model system, as in the previous sections, large irregular graphs or networks, making the idealisation that the node sets are countably infinite.

Our central idea was it to distill in a sequence of systematic *coarse-graining* or *renormalisation* steps some *large-scale* structure of our networks if there does exist any. We in fact located specific criteria in our microscopic substratum (some *geometric criticallity* or *long-range order*) which are expected to be crucial for the existence of an interesting large-scale limit manifold. We do not intend to repeat a description of this process in greater detail in the following, we only want to make clear what our central goals are.

The process developed by us is reminiscent of the real-space renormalisation process of the statistical mechanics of critical systems. Instead of block-spins and regular lattices we select certain densely entangled subgraphs (called cliques or lumps) in our network and promote them to new nodes in a new (meta-)graph. We draw a (meta-)edge between to cliques if they overlap (to a certain degree), i.e. having a non-void set of common nodes. In this way we construct the so-called *clique-graph*.

One can now choose to forget about the internal structure of these new nodes or average over the inner structure and repeat this process, getting nodes and edges of the next higher level and so on. As in the ordinary renormalisation process the idea is to compare these structures living on the different scales of our graph with each other, study their flow, hoping that they converge to a certain *fixed-structure*. It is evident that this process implements a certain kind of coarse-graining, distilling possibly hidden laqrge-scale characteristics of the graph or network. The main technical tool in our analysis has been the theory of random graphs (see for example [20]).

Performing this task we have to deal with formidable technical problems. Note for example that it is far from clear under what conditions we arrive at a smooth macroscopic manifold or, on the other hand, a chaotic or fractal-like limit point. Furthermore, in order to speak meaningful about concepts like limit or convergence, topological, or even better, metrical concepts have to be introduced or developed. These aspects will be discussed in greater detail in the following.

The idea is to view space or space-time on several scales of resolution at a time, from the very microscopic to the macroscopic regime. In other words, we have to introduce a form of *scaling limit* on graphs or metric spaces in general. It turns out that the performance of such a general rescaling process leads to a whole bunch of deep mathematical questions which have been treated only relatively recently in pure mathematics (see for example [28], [29], [30] or [31]).

What we need in the first place are fruitful notions of distance and convergence in the category of general metric spaces, then leading to further notions like e.g. completeness, compactness etc. in some super-space of spaces. Such concepts have been developed by M.Gromov and other people. What was in fact well-known is the notion of *Hausdorff-distance* of, for example, compact sets lying in some ambient (compact) metric space.

Definition 4.1 *Let X be a metric space, $U_\epsilon(A)$ the ϵ-neigborhood of a subset $A \subset X$. The Hausdorff-distance between $A, B \subset X$ is then given by*

$$d_H(A,B) := \inf\{\epsilon; A \subset U_\epsilon(B), B \subset U_\epsilon(A)\} \tag{44}$$

We have the following lemma

Lemma 4.2 *With X a compact metric space, the closed subsets of X form a compact (i.e. complete) metric space with respect to d_H (see e.g. [29] or [32]).*

In the following it is sometimes useful to make a slight generalisation to pseudo metric spaces as we will encounter situations where spaces or sets have zero Gromov-Hausdorff-distance (for example, the one being a dense subset of the other) while they are not strictly the same. Everything we will prove for metric spaces in the following will also hold for pseudo metric spaces.

Definition 4.3 *A pseudo metric fulfills ther same axioms as a metric with the exception that $d(a,b) = 0 \to a = b$ does not necessarily hold.*

The above distance concept is too narrow to be useful in a more general context. It was considerably generalized by M.Gromov in an important way (see [33]) and later slightly modified by himself and other authors ([28],[29],[34]). What is really beautiful in our view is, that, while it seems to be more abstract, it encodes the really important and crucial aspects of similarity or "nearness" of spaces in a more satisfying way. That is, it measures their structural similarity and not simply the nearness of two structureless sets of points in a space. In general it is a pseudo metric which may even take the value infinity. For compact spaces it is always finite. If one forms equivalence classes of compact spaces under isometries, it becomes a true metric.

The Gromov-Hausdorff distance, d_{GH}, can be formulated in two equivalent ways.

Definition 4.4 *$d_{GH}(X,Y)$ between two metric spaces, X,Y,is defined as the infimum of $d_H^Z(f(X),g(Y))$ over all metric spaces Z and isometric embeddings, f,g, of X,Y into Z.*
Equivalently, one can define d_{GH} by the infimum over $d_H(X,Y)$ in $X \sqcup Y$ equipped with the metric $d_{X\sqcup Y}$ which extends the respective metrics d_X, d_Y in X,Y.

To give a certain impression how typical properties are proved in this context we show that d_{GH} fulfills the triangle inequality (the proof of which is frequently scipped in the literature whereas it is not entirely trivial, see also [36]).

With X,Y,Z metric spaces and (without loss of generality)

$$d_{GH}(X,Y) < \infty \ , \ d_{GH}(Y,Z) < \infty \tag{45}$$

we build the spaces

$$X \sqcup Y \, , \, Y \sqcup Z \, , \, X \sqcup Z \, , \, X \sqcup Y \sqcup Z \tag{46}$$

with $d_{X\sqcup Y}, d_{Y\sqcup Z}$ metrics, extending d_X, d_Y, d_Z. We define the following metric on $X \sqcup Z$, $X \sqcup Y \sqcup Z$, respectively, extending the metrics $d_{X\sqcup Y}, d_{Y\sqcup Z}$.

$$d^*(x,z) := \inf_y (d_{X\sqcup Y}(x,y), d_{Y\sqcup Z}(y,z)) \tag{47}$$

We first show that this defines a metric on $X \sqcup Y \sqcup Z$. The critical property is, as always, the triangle inequality. Inserting the definitions in, for example, the configuration

$$d^*(x,z) + d^*(z,x') \tag{48}$$

and regrouping the terms belonging to $X \sqcup Y, Y \sqcup Z$ respectively one gets

$$\begin{aligned} d^*(x,z) + d^*(z,x') &= \inf_{y,y'} (d(x,y) + d(x',y') + d(y,z) + d(y',z)) \\ &\geq \inf_{y,y'} (d(x,y) + d(x',y') + d(y,y')) \geq d(x,x') \end{aligned} \tag{49}$$

Remark: Note that in general the supremum over metrics is again a metric. This does not hold in general for the infimum.

One now has for the infimum over all admissible metrics:

$$d_{GH}(X,Z) \leq \inf_{d^*}(d_H^*(X,Z)) \leq \inf_{d^*}(d_H^*(X,Y) + d_H^*(Y,Z)) \tag{50}$$

But all the above particular metrics d^* are fixed on $X \sqcup Y, Y \sqcup Z$, being there $d_{X\sqcup Y}, d_{Y\sqcup Z}$ respectively. That is, $d_H^*(X,Y) = d_H^{X\sqcup Y}(X,Y)$ etc. Therefore we can take on the rhs the infimum within the brackets over the respective metrics on both spaces, $X \sqcup Y$ and $Y \sqcup Z$ independently!, getting the final inequality for the GH-distance

$$d_{GH}(X,Z) \leq d_{GH}(X,Y) + d_{GH}(Y,Z) \tag{51}$$

thus proving the statement. $\square$

One has the following lemma.

Lemma 4.5 *Two compact spaces, X, Y, are isometric iff $d_{GH}(X,Y) = 0$. That is, we have a true GH-metric when taking isometry classes ([29], p.73).*

We have the further result

Proposition 4.6 *The space $\mathcal{C}$ of compact metric spaces is complete under d_{GH} (see [34]).*

Remark: It is usually difficult to calculate the GH-distance exactly. In many cases it is however sufficient to find good upper bounds. In convergence questions this is frequently relatively easy.

In the following the concept of an ϵ-net will become a useful tool.

Definition 4.7 *An ϵ-net in a metric space X is a subset, S, which is ϵ-dense in X, i.e. with $x \in X$ it follows that $d(x,S) \leq \epsilon$. Or, stated differently, the union of ϵ-balls with centers in S covers X.*

Of particular importance are finite ϵ-nets.

Definition 4.8 *X is called totally bounded if for all ϵ there exists an ϵ-net which is finite.*

As a consequence we have

Lemma 4.9 *A metric complete space is compact iff it is totally bounded.*

Corollary 4.10 *A compact space has a finite diameter.*

Corollary 4.11 *If X, Y are compact metric spaces, a metric on $X \sqcup Y$ is for example defined by an extension of d_X, d_Y in the form*

$$d_{X\sqcup Y}(x,y) := 1/2 \max\{diam\, X, diam\, Y\} =: 1/2D \tag{52}$$

It follows

$$d_{GH}(X,Y) \leq 1/2D \tag{53}$$

Remark: Usually the crucial and sometimes somewhat tricky point in finding new metrics or extending given metrics is to fulfill the triangle inequality for all! different possible configurations. In the above example this was, somewhat untypically, relatively simple.

To exhibit more clearly that a small GH-distance says something about a certain metric similarity between general metric spaces, we discuss the relation between rough isometry and GH-distance. This is implicitly discussed in example 2 p.491 in [34], the discussion following definition 5.33 in [29] (where GH-distance is introduced via so-called $\epsilon - relations$) or proposition 5.3 in [28]. We state the following result as formulated in [36].

Theorem 4.12 *Two metric spaces have finite GH-distance iff they are roughly isometric.*

Proof: We assume that $d_{GH}(X,Y) < \epsilon$. Then, to each $x \in X$ there exists an y so that $d(x,y) < \epsilon$ in an appropriate metric d on $X \sqcup Y$. We define a map

$$f : x \to f(x) \in Y \tag{54}$$

by selecting one of these elements y. The triangle inequality yields for example for the following configuration

$$d(x,x') \leq d(x,f(x)) + d(f(x),f(x')) + d(f(x'),x') \leq d(f(x),f(x')) + 2\epsilon \tag{55}$$

A corresponding relation holds in the other direction with a map $g : y \to g(y) \in X$. Furthermore, we have $f \circ g(y) \in Y$ with

$$d(y, f \circ g(y)) \leq d(y, g(y)) + d(g(y), f \circ g(y)) \leq 2\epsilon \tag{56}$$

We hence conclude that

$$d_Y(y, f(X)) \leq 2\epsilon \tag{57}$$

for all $y \in Y$. This proves that f, g define rough isometries.

Corollary 4.13 *Furthermore, for the in general non-unique map, $x \to f(x)$ with $d(x,f(x)) < \epsilon$, we have with another choice, $x \to f'(x), d(x,f'(x)) < \epsilon$, that $d(f(x),f'(x)) < 2\epsilon$. Hence all points in Y which are close to x are also close to each other.*

Proof of corollary:

$$d(f(x),f'(x)) \leq d(f(x),x) + d(x,f'(x)) \leq 2\epsilon \tag{58}$$

To prove the other direction in our theorem we introduce, similar to [34] eample 2 or [28] proposition 3.5, a metric d on $X \sqcup Y$, exploiting the rough isometry of X, Y with parameters ϵ, ϵ', i.e.

$$d(x,x') - \epsilon \leq d(f(x),f(x')) \leq d(x,x') + \epsilon \tag{59}$$

implying

$$|d(f(x),f(x')) - d(x,x')| \leq \epsilon \tag{60}$$

and $Y \subset U_{\epsilon'}(f(X))$ with a corresponding relation in the other direction.

We have seen that, with $\rho := \max(\epsilon, \epsilon')$, $f(X)$ is an ρ-net in Y and that, as a consequence of the preceding equation, $\{(x, f(x))\}$ defines a ρ-relation (cf. [29] definition 5.33). Then by

$$d(x,y) := \inf_{x' \in X} (d_X(x,x') + d_Y(f(x'),y) + \rho) \tag{61}$$

a metric is defined on $X \sqcup Y$ extending d_X, d_Y. To illustrate this point we show that the triangle inequality holds for the following configuration

$$\begin{aligned} d(x_1,y) + d(y,x_2) &= \inf_{x',x''} (d(x_1,x') + d(f(x'),y) + d(x_2,x'') + d(f(x''),y) + 2\rho) \\ &\geq \inf_{x',x''} (d(x_1,x') + d(x_2,x'') + d(f(x'),f(x'')) + 2\rho) \end{aligned} \tag{62}$$

From f being a rough isometry we infer

$$d(f(x'),f(x'')) \geq d(x',x'') - \rho \tag{63}$$

and hence

$$\begin{aligned} d(x_1,y) + d(y,x_2) \geq \inf_{x',x''} &(d(x_1,x') + d(x',x'') + d(x'',x_2) + \rho) \\ &\geq d(x_1,x_2) + \rho > d(x_1,x_2) \end{aligned} \tag{64}$$

For $y := f(x)$ we have $d(x, f(x)) \leq \rho$, so X lies in the U_ρ-neigborhood of Y with respect to this metric. On the other hand we have for all y

$$d(x,y) = \inf_{x'} (d(x,x') + d(f(x'),y) + \rho) \leq d(f(x),y) + \rho \tag{65}$$

(by inserting x for x'). Due to the assumptions there exists an x so that $d(f(x),y) \leq \rho$. This proves the theorem. $\square$

Observation 4.14 *Note that the set $\{(x,f(x))\} \cup \{(g(y),y)\}$ define a surjective 2ρ-relation in the sense of [29] definition 5.33, by means of which the GH-distance is defined there.*

Proof: This follows from $d_Y(f \circ g(y), y) \leq \rho$ (cf. the definition of quasi-isometry in section 2) and

$$d_Y(f(x),y) \leq d_Y(f(x), f \circ g(y)) + d_Y(f \circ g(y), y) \leq d_X(x, g(y)) + \rho + \rho \tag{66}$$

and hence

$$|d_X(x,g(y)) - d_Y(f(x),y)| \leq 2\rho \tag{67}$$

We now present the fundamental Gromov-compactness theorem, first for compact spaces, then for more general cases.

Definition 4.15 *We call a family of compact spaces, X_λ, uniformly compact if their diameters are uniformly bounded and if for each $\epsilon > 0$ X_λ is coverable by $N_\epsilon < \infty$ balls of radius ϵ independent of the index λ.*

Theorem 4.16 *(Gromov) A sequence $\{X_i\}$ contains a convergent subsequence in d_{GH} iff $\{X_i\}$ is uniformly compact.*

Proof: see [33],[34] or [29]. Typically an *Arzela-Ascoli-Cantor-diagonal-sequence*-like argument is used in the proof. □

In our framework we are mainly interested in infinite graphs, i.e. non-compact metric spaces being however frequently *proper*.

Definition 4.17 *A metric space, X, is called proper if all its closed balls, $B(x,r)$, are compact.*

We can then extend the above result in the following way. Ordinary GH-convergence works well in the category of compact metric spaces. If the spaces are non-compact, a slightly modified approach is more satisfactory. One problem which may arise is that things in unbounded spaces can "wander away" to infinity. So it is reasonable to pin down the members of the sequence of spaces at certain points, so that they can be better compared. More precisely, we work in the category of *pointed* metric spaces, (X,x), which is, a fortiori pretty normal from the physical point of view as it is like introducing a reference point or a coordinate system.

Definition 4.18 *The sequence of pointed metric spaces, (X_i, x_i), is said to converge to (X,x) in pointed GH-sense if for every $r > 0$ the sequence of closed balls, $B(x_i, r)$, converges to $B(x,r)$ in d_{GH}.*

The Gromov-uniform-compactness theorem now reads:

Theorem 4.19 *If for all r and $\epsilon > 0$ the balls $B(x_i, r)$ of a given sequence (X_i, x_i) are uniformly compact, then a subsequence of spaces converges in pointed GH-sense.*

Remark: There exist various slightly different notions of pointed convergence in the literature. One can for example define pointed GH-distance by admitting only isometries which map the base points onto each other ([28]). Another possibility is to include the distance of the images of the base points in the definition ([35]). The above definition is used in [29].

5 The Scaling-Limit of Infinite Graphs

We now apply the techniques and results, developed in the preceding sections, to our original problem, namely, investigating a particular kind of scaling-limit in our space of metric spaces and apply it to the subclass of locally bounded infinite graphs.

We start with a graph, G, of globally bounded vertex degree, v, and, taking G with the original graph metric, d, as initial metric space, generate a sequence, or, more generally, a directed system of metric spaces, λG, by taking the same graph, G, but now with the scaled metric, λd, defined as

$$\lambda d(x,y) := \lambda \cdot d(x,y) \tag{68}$$

and (usually) taking $\lambda \to 0$. One may, in particular, take subsequences of the kind

$$G_n \ , \ d_n := n^{-1} \cdot d \ , \ n \to \infty \tag{69}$$

or replace n by 2^{-k}.

In a first step we have to provide criteria under which the balls, $B_n(x,r) \subset G_n$, with x a fixed reference vertex in G, are uniformly compact. To this end we have to introduce a new concept.

Definition 5.1 *With (X,d) a metric space, μ a positive Borel measure on X, μ is said to be doubling, if there exists a positive constant, C, being independent of B such that*

$$\mu(2B) \le C \cdot \mu(B) \tag{70}$$

for all balls in X. Note that $2B$ is a ball with the same center as B but twice the radius.

Lemma 5.2 *It easily follows via iteration that*

$$\mu(2^k B) \le C^k \cdot \mu(B) \tag{71}$$

Definition 5.3 *A metric space is called doubling if each ball, B, can be covered by at most C balls with radius half that of B with again C independent of B.*

Proposition 5.4 *If (X,d) has a doubling measure, it is doubling as a metric space.*

Proof: See [28], p.412 (the chapter being written by Semmes).

Now take in the category of metric spaces the notion of *pointed convergence* and apply it to the sequence of spaces $X_n := (X, d_n)$. The ball $B_n(x,r)$ in X_n corresponds as a set to the ball $B(x,nr)$ in X. The sequence of balls $B_n(x,r)$ is uniformly compact if each ball can be covered by at most $N_\epsilon(r)$ ϵ-balls. This means, that all $B(x,nr)$-balls can be covered by at most $N_\epsilon(r)$ balls of radius $n \cdot \epsilon$ in X. This is the case if X carries a doubling measure μ according to the preceding lemma. So what we have to analyse is, under what conditions do graphs cary a doubling counting-measure with

$$\mu(B(x,k)) := |B(x,k)| \tag{72}$$

To this end we have to formulate in the general case (i.e. graphs not necessarily being Cayley or vertex-transitive graphs) a slightly more restricted form of polynomial growth.

Definition 5.5 *We say, a locally finite graph, G, has uniform polynomial growth if there exist constants, $A, B, d > 0$ so that*

$$Ak^d \le \beta(x,k) \le Bk^d \tag{73}$$

for all $k \ge k_0$ and A, B independent of the reference point x.

Remark: It is interesting that the dimension, d, is independent of the reference point in a locally finite graph but to show the doubling property, one needs a stronger result, i.e. the constant A has to stay away from zero if x varies in $V(G)$. This is another weak form of homogeneity of a graph.

If our graph, G, has uniform polynomial growth one has the following estimate for sufficiently large k:

$$\beta(x,2k) \leq B \cdot 2^d \cdot k^d \ , \ \beta(x,k) \geq A \cdot k^d \tag{74}$$

hence

$$\beta(x,2k) \leq B/A \cdot 2^d \cdot \beta(x,k) \tag{75}$$

Conclusion 5.6 *A graph with uniform polynomial growth has a doubling counting measure for sufficiently large k and is hence doubling as a metric space for sufficiently large k. This implies that all balls $B_n(x,r)$ in G_n are uniformly compact.*

The question is now, which graphs have this property of weak homogeneity? It is clear that locally finite vertex-transitive graphs of polynomial growth have this property.

We assume the following. Let G_1 have uniform polynomial growth for sufficiently large $r \geq r_0$, i.e.

$$C_1 \cdot r^d \leq \beta(x,r) \leq C_2 \cdot r^d \tag{76}$$

Assume, furthermore, that the graphs G_1, G_2 have globally bounded vertex degree and that they are quasi-isometric. For convenience we choose all ocurring constants to be symmetric with respect to the quasi-isometric maps, $f : G_1 \rightarrow G_2$, $g : G_2 \rightarrow G_1$ (cf. the definition of quasi-isometry in section 2, we assume also $\epsilon = \epsilon' = \rho$).

We know that $f(G_1)$ is ρ-dense in G_2 and vice versa. Taking a ball, $B(y,r)$, around a vertex $y \in G_2$, there exists by assumption a vertex $x \in G_1$ with $d_2(f(x),y) \leq \rho$. We have shown (see the first part of theorem (2.16)) that

$$\beta_1(x,r) \leq A \cdot \beta_2(y, \lambda r + 2\rho) \tag{77}$$

with A independent of x, y (for globally bounded vertex degree). We hence have

$$\beta_2(y, \lambda r + 2\rho) \geq A^{-1} C_1 r^d \tag{78}$$

Correspondingly we have

$$A\beta_1(x, \lambda r + 2\rho) \geq \beta_2(y,r) \tag{79}$$

Now employing the universal polynomial growth for G_1 we get

$$A^{-1}C_1(\lambda^{-1}(r-2\rho))^d \leq \beta_2(y,r) \leq AC_2(\lambda r + 2\rho) \tag{80}$$

Inserting for r either r (large enough) or $2r$ we arrive at

$$\beta_2(y, 2r) \leq AC_2(2\lambda r + 2\rho)^d \ , \ \beta_2(y, r) \geq A^{-1}C_1(\lambda^{-1}(r - 2\rho))^d \tag{81}$$

and therefore

$$\beta_2(y, 2r)/\beta_2(y, r) \leq A^2 \cdot C_2/C_1 \cdot \lambda^d \cdot \frac{(2\lambda r + 2\rho)^d}{(r - 2\rho)^d} \tag{82}$$

For r sufficiently large (e.g. $2\rho \leq r/2$) we can bound the rhs by a constant from above. We thus have

Theorem 5.7 *Let G_1, G_2 both have globally bounded vertex degree, let G_1 have uniform polynomial growth, and let G_1, G_2 be quasi-isometric. Then also G_2 has uniform polynomial growth. We conclude that graphs with this property have a scaling limit in the sense discussed above.*

We can make use of the theorem in the following way. We know that locally finite vertex transitive graphs of polynomial growth have, by the same token, a uniform polynomial growth. So all graphs of globally bounded node degree, being quasi-isometric to such graphs, qualify as candidates having a continuum limit under scaling. A fortiori we know that they even have an integer dimension. That is, we can dispose of a quite large class of physically interesting examples.

Furthermore, our approach shows that there are certain similarities to *attractors* in dynamical systems. We have in fact a kind of *universality*.

Corollary 5.8 *Let G have the scaling limit X. Let G' have finite GH-distance to G. It follows that G' also converges toward X in d_{GH}.*

Proof: Let $d_{GH}(G, G') \leq a$. The relations between the metrics d and the scaled metrics $\lambda \cdot d$ are bijective. It is hence easy to show that it follows that $d_{GH}(G_n, G'_n) \leq n^{-1} \cdot a$ and therefore

$$d_{GH}(X, G'_n) \leq d_{GH}(X, G_n) + d_{GH}(G_n, G'_n) \tag{83}$$

which proves the statement. $\square$

A last point we want to discuss is the following. From the point of view of physics an analysis of the properties of the continuum limit space, X, as a consequence of certain characteristics of the initial space, G, is extremely important. One can in fact prove quite a lot in this respect. We restrict ourselves, for the sake of brevity, in this paper to showing the following remarkable property.

Let the graph, G, be of uniform polynomial growth (growth degree d). We learned that under this condition its natural counting measure is doubling which implies that G as a metric space is doubling. We know in fact a little bit more. Under the mentioned conditions all balls, $B(x, r)$, in G are totally bounded viz. compact and the number of ϵ-balls, $N_\epsilon(r)$, needed to cover them can be easily estimated with the help of our previous results. It follows from our assumptions that $N_\epsilon(r)$ scales with (ϵ, r) in the form

$$N_\epsilon(r) \sim (r/\epsilon)^d \tag{84}$$

Note that in the following we need this property only for large n, i.e. balls, $B_n(r)$, in G_n which correspond to balls $B(nr)$ in G and, hence, ϵ-balls in G_n corresponding to $n\epsilon$-balls in G.

That is, in order to be allowed to choose ϵ arbitrarily small and still having non-trivial ϵ-balls in the G_n, we can always assume n to be sufficiently large. For convenience we ignore this technical point in the following. The same scaling as above holds for the minimum, $cov_\epsilon(B_r)$, of such $N_\epsilon(r)$. That is, we have

Lemma 5.9 *For sufficiently large n we have*

$$cov_\epsilon(B_r) \sim (r/\epsilon)^d \tag{85}$$

the d coming from the uniform polynomial growth.

We now choose n so large that $d_{GH}(B_X(x,r), B_n(x,r)) \leq \rho$. From our previous results we then have rough isometries f, g between $B_X(x,r), B_n(x,r)$ with $d(f(B_n), y) \leq 2\rho$ for all $y \in B_X(x,r)$ and the same result in the other direction. We select a minimum number, $cov_\epsilon(B_n(x,r))$, of points $x_i \in B_n(x,r)$, so that $B_n(x,r)$ is covered by ϵ-balls centered at x_i. Each $y \in B_X(x,r)$ has distance at most 2ρ to an ϵ-ball centered at some $f(x_i)$. Hence, for all such y there exists a x_i so that

$$d_X(y, f(x_i)) \leq \epsilon + 2\rho \tag{86}$$

The same holds for the opposite direction and the map $g : X \to G_n$. As the number of points x_i was chosen minimal by assumption, we conclude

Proposition 5.10 *We have*

$$cov_{\epsilon+2\rho}(B_X(x,r)) \leq cov_\epsilon(B_n(x,r)) \ , \ cov_{\epsilon+2\rho}(B_n(x,r)) \leq cov_\epsilon(B_X(x,r)) \tag{87}$$

and thus

$$cov_{\epsilon+2\rho}(B_n(x,r)) \leq cov_\epsilon(B_X(x,r)) \leq cov_{\epsilon-2\rho}(B_n(x,r)) \tag{88}$$

or

$$(r/\epsilon + 2\rho)^d \leq cov_\epsilon(B_X(x,r)) \leq (r/\epsilon - 2\rho)^d \tag{89}$$

We can now choose n large enough. This allows us to choose ϵ arbitrarily small. For $\rho \to 0$ and then $\epsilon \to 0$ we get

Theorem 5.11 *Let G be of uniform polynomial growth with growth degree d. Then for the balls, $B_X(x,r)$, in the limit space X it holds*

$$cov_\epsilon(B_X(x,r)) = \lim_{n\to\infty} cov_\epsilon(B_n(x,r)) \tag{90}$$

with the same scaling degree d. Furthermore, all balls in X are totally bounded, hence compact. Taking the infimum over ϵ we call d the covering dimension of X.

As the covering dimension is related to the Hausdorff-measure, this result shows, that also as measure spaces the G_n converge toward a reasonable continuum limit. We note that in this connection a lot more can actually be shown but we choose to stop here.

6 A Brief Outlook

We have shown in the preceding sections that under certain assumptions the existence of a macroscopic continuum limit can be guaranteed when we start from an underlying presumably quite erratic substratum which we modelled as a discrete metric space consisting of elementary degrees of freedom and their elementary interactions or relations. It turned out that the characteristics to be imposed on this primordial network were reasonable from a physical point of view. A particularly important notion was the concept of an intrinsic general dimension of arbitrary discrete spaces. We could show that there even exist criteria such that this dimension takes on integer values.

It is of great importance to learn under what conditions this limit space is a smooth manifold or, on the other hand, a chaotic space of rather fractal type. A particularly important possibility is a space having a superficially smooth structure together with an internal infinitesimal more erratic structure "around" the "classical" points of the base manifold, being kind of a generalized fiber space. In this context a lot more can be investigated as e.g. (functional) analysis, field theory or algebraic structures on the discrete spaces and their respective continuum limits. We indicated how this works in the case of measure theory.

As a last remark, there exists a slightly different approach developed in [31] which we only briefly mentioned. It makes use of the possibility of embedding the general spaces into some $\mathbb{R}^n$ (and relies on several deep theorems). This approach, while perhaps being more useful from a practical point of view, as it is frequently easier to formulate concepts and prove results in $\mathbb{R}^n$, is, on the other hand, slightly less general. So we preferred to begin our analysis within the presumably more general framework. Nevertheless we hope to come back to this latter lines of reasoning in the future.

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