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Extremal Solutions by Monotone Iterative Technique for Hybrid Fractional Differential Equations

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Abstract This paper highlights the mathematical model of biological experiments, that have an effect on our lives. We suggest a mathematical model involving fractional differential operator, kind of hybrid iterative fractional differential equations. Our technique is based on monotonous iterative in the nonlinear analysis. The monotonous sequences described extremal solutions converging for hybrid monotonous fractional iterative differential equations. We apply the monotonous iterative method under appropriate conditions to prove the existence of extreme solutions. The tool relies on the Dhage fixed point Theorem. This theorem is required in biological studies in which increasing or decreasing know freshly split bacterial and could control.

Keywords: *fractional differential equation, fractional differential operator, fractional calculus, monotonous sequences, extreme solution*

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1. Introduction

A class of mathematical models based on differential equations plays a big role in all areas of life, such as physics, engineering, agriculture, and medicine, etc. In this research, we use biological mathematical models in the studying of the growth and reproduction of bacteria or decay. It is well known that the bacteria are tiny bodies, generally made of a cell that does not have chlorophyll. But for the virus, it is the little things of life on Earth. Bacteria are likely to multiply very quickly under favorable conditions, the formation of settlements of millions or even billions of organisms in such a tiny space like a drop of water. Any bacteria have a particular temperature rank that can survive. For a particular rather bacteria, this range can be too high, too low, or someplace in between, although it is still a narrow range [1].

Results extension and predictions of mathematical models must be considered for further understanding of the different processes. These results are essential in more fields like science and engineering. An example is the application of differential equations for growth and bacterial cell division [2]. Humans have learned to exploit bacteria and other microbe beneficial uses, such as genetically engineered human insulin. Right now it is more convenient to insert the human insulin gene in bacteria and let produced in large industrial enzymes. It is therefore important achieve bacterial growth [3].

A class of fractional differential equations manipulates a big role in almost all sciences such as engineering, medicine, economics, social, linguistic and physics. This area is the more general field of mathematical analysis. It includes, fractional order that is, the value of $\iota < \alpha < \iota + 1$. This field is not new, but older age ranges between 200 and 300 years, but now has become the subject of studies and concerns because it gives effective results and more general, and therefore, we find a lot of scholars working in this field ([4,5]). The significance of differential equations was that polls hybrid entails a number of dynamic systems treated as special cases ([6,7]). Dhage and Lakshmikantham [8], Dhag and Jadhav [9], showed some of the main results of the hybrid linear differential equations of the first order and disturbances of second type. [10]. A good mathematical model for growth bacteria is described by iterative differential equation. Ibrahim [11] established the existence of a class of fractional iterative differential equation (Cauchy type) utilizing the non-expansive operator technique. This class is generated by the authors [12].

This paper deals with the mathematical model of biological experiments, that have an affect on our lives. We impose a mathematical model involving fractional differential operator, type hybrid iterative fractional differential equations. Our method is based on monotonous iterative in the nonlinear analysis. The monotonous sequences described extremal solutions converging for hybrid monotonous fractional iterative differential equations. We employ the monotonous iterative method under appropriate conditions to show that the existence of extreme solutions. The tool relies on the Dhage fixed point Theorem. This theorem is required in biological studies in which increasing or decreasing know freshly split bacterial and could control.

2. Preliminaries

Recall the following preliminaries:

Definition 2.1 The definition of fractional (arbitrary) order derivative function $\psi(t)$ of order $\alpha > 0$ for *Riemann-Liouville* is

$$D_s^{\alpha}\psi(s) = \frac{1}{\Gamma(\iota-\alpha)}\left[\frac{d}{ds}\right]^{\iota}\int_0^s \frac{\psi(\beta)}{(s-\beta)^{\alpha-\iota+1}}d\beta \qquad (1)$$
$$(\iota-1) \le \alpha < \iota,$$

in which ι is a whole number and α is a real number.

Definition 2.2 In which the function ψ is defined such as $\psi:[0,\infty[\to \mathfrak{R},$ fractional (arbitrary) order derivative function (ψ) of order $\alpha > 0$ for the *Caputo* is

$$D_s^{\alpha}\psi(s) = \frac{1}{\Gamma(\iota-\alpha)}\int_0^s \frac{\psi^{(\iota)}(\beta)}{(s-\beta)^{\alpha-\iota+1}}d\beta \qquad (2)$$
$$(\iota-1) \le \alpha < \iota,$$

in which ι is a whole number and α is a real number.

Definition 2.3 The fractional (arbitrary) integral of order $\alpha > 0$ for Riemann-Liouville is defined by the formula

$$I_s^{\alpha}\psi(s) = \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\psi(\beta)}{(s-\beta)^{1-\alpha}}d\beta. \qquad (3)$$

Based on the Riemann-Liouville differential operator, we impose the following useful definitions:

Definition 2.4 We said that $\iota \in C(I,\mathfrak{R})$ is a function which is a lower solution for the equation introduced on I if

1. $s \mapsto \left(\iota(s)-\psi\left((s,\iota(s)),\iota(\iota(s))\right)\right),$ is continuous, and
2. $D^{\alpha}\left[\iota(s)-\psi\left(s,v(s),v(v(s))\right)\right] \ge \aleph\left((s,\iota(s)),\iota(\iota(s))\right),$ $s \in I, \iota(s_0) \ge v_0.$

Definition 2.5 We said that $\tau \in C(I,\mathbb{R})$ is a function which is an upper solution for the equation introduced on I if

1. $s \mapsto \left(\tau(s)-\psi\left((s,\tau(s)),\tau(\tau(s))\right)\right),$ is continuous, and
2. $D^{\alpha}\left[\tau(s)-\psi\left(s,v(s),v(v(s))\right)\right] \le \aleph\left((s,\tau(s)),\tau(\tau(s))\right),$ $s \in I, \tau(s_0) \le v_0.$

Definition 2.6 [14] Let $\alpha > 0.$ The function E_{α} introduced by

$$E_{\alpha}u = \sum_{i=0}^{\infty}\frac{u^i}{\Gamma(i\alpha+1)} \qquad (4)$$

provided that the series converging be called the Mittag-Leffer function of order $\alpha.$

Remark 2.1 The exponential function corresponding to $\alpha = 1.$ Figure 1 shows the Mittag Leffer function for different values of α . More general class of functions follows:

Definition 2.7 Let $\alpha_1, \alpha_2 > 0.$ The function E_{α_1,α_2} introduced by

$$E_{\alpha_1,\alpha_2}u = \sum_{i=0}^{\infty}\frac{u^i}{\Gamma(i\alpha_1+\alpha_2)} \qquad (5)$$

provided that the series converges is called Mittag- Leffer function two parameters with parameters α_1 and α_2.

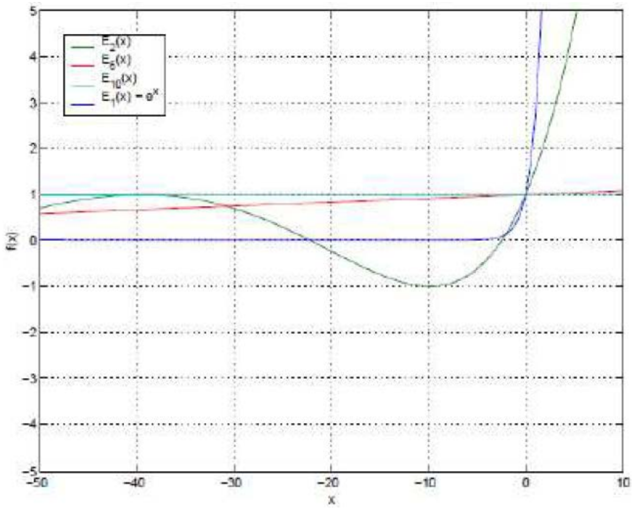

Figure 1. The Mittag-Leffler Function for different $\alpha, \beta = 1$

Remark 2.2 Clearly the Mittag-Leffer functions a parameter can be defined in terms of their counterparts in two parameters using the relationship in $E_{\alpha}u = E_{\alpha,1}u.$

Cases. [15] Some special cases of the Mittag-Leffer function $E_{\alpha}u$ as follow:

1. $E_0(u) = \frac{1}{1-u}, |u| < 1,$
2. $E_1(u) = e^u,$
3. $E_1(Mu) = e^{(Mu)}, M \in C,$
4. $E_2(u) = \cos h\sqrt{u}, u \in C,$
5. $E_2(-u^2) = \cos u, u \in C.$

Definition 2.8 Assume the closed period bounded interval $I = [s_0, s_0 + a]$ in $\mathfrak{R}$ ($\mathfrak{R}$ the real line), for some $s_0 \in \mathfrak{R},$ $a \in \mathfrak{R}.$ The initial value problem of fractional iterative hybrid differential equations (*FIHDE*) can be formulated as

$$\begin{aligned} &D^{\alpha}\left[v(s)-\psi\left(s,v(s),v(v(s))\right)\right] \\ &= \aleph\left((s,v(s)),\tau(v(s))\right), s \in I \end{aligned} \qquad (6)$$

with $v(s_0) = v_0,$ where $\psi, \aleph : I \times \mathfrak{R} \to \mathfrak{R}$ are continuous. A solution $v \in C(I,\mathfrak{R})$ of the *FIHDE* (6) can be defined by

1. $s \mapsto v - \psi(s,v,v(v))$ is a function which is continuous $\forall v \in \mathfrak{R},$ and
2. v contented the equations in (6). In which $C(I,\mathfrak{R})$ space is of real-valued continuous functions defined on I.

Also, our definition of the fractional iterative of hybrid equation integrated *FIHIE* as following:

$$\begin{aligned} v(t) &= \left[v_0 - \psi\left(s_0, v_0, v(v_0)\right)\right] + \psi\left(s, v(s), v(v(s))\right) \\ &+ \int_0^s \aleph\left(\beta, v(\beta), v(v(\beta))\right)\frac{(s-\beta)^{\alpha-1}}{\Gamma(\alpha)}d\beta, \\ &\left(s \in I, v(s_0) = v_0\right). \end{aligned} \qquad (7)$$

Note that the fractional hybrid differential equations can be found in [16].

2.1. Assumptions

In the following assumptions relating to function ψ is very important in the studying of Eq(6).

(a0) The function $v \mapsto (v - \psi(s_0, v, v(v)))$ is injective in $\mathfrak{R}$.

(b0) $\aleph$ is a bounded real-valued function on $I \times \mathfrak{R}$.

(a1) The function $v \mapsto (v - \psi(s, v, v(v)))$ is increasing in $\mathfrak{R}$ for all $s \in I$.

(a2) There is a constant $\ell > 0$ so that

$$\left|\psi(s, v, v(v)) - \psi(s, z, z(z))\right| \le \frac{\ell|v - z|}{M + |v - z|}, M > 0,$$

$\forall s \in I,\ v, z \in \mathfrak{R}$ and $\ell \le M$.

(b1) There is a constant $\kappa > 0$ o that $|\aleph(s, v, v(v))| \le \kappa \forall s \in I$ and $\forall v \in \mathfrak{R}$.

(b2) The function $v \mapsto \aleph(s, v, v(v))$ is nondecreasing in $\mathfrak{R}$, $\forall s \in I$.

(b3) There is a real number $M > 0$ so that

$$\aleph(s, v, v(v)) - \aleph(s, z, z(v)) \ge \frac{-M}{1 + T^{\alpha}}\left[(v - \psi(s, v, v(v))) - (z - \psi(s, z, z(v)))\right]$$

$\forall s \in I$, and $v, z \in \mathfrak{R}$ with $v \ge z$.

(b4) There is a constant $\kappa > 0$ so that

$$\left|\tilde{\aleph}(s, v, v(v))\right| = \left|\aleph(s, v, v(v)) + \frac{M}{1 + T^{\alpha}}(v - \psi(s, v, v(v)))\right| \le \kappa$$

for each $s \in I$ and $v \in \mathfrak{R}$.

3. Main Results

In this section, our purpose is to discuss the technique of monotone iterative to *FIHDE*(6) some under appropriate conditions are charged solutions proving the existence of extreme.

Lemma 3.1 [17] Let $0 < \alpha < 1$ and $z \in \ell^{-1}(0, T)$.

1. The equality $D^{\alpha} I^{\alpha} z(s) = z(s)$ achieves
2. The equality

$$D^{\alpha} I^{\alpha} z(s) = z(s) - \frac{I^{1-\alpha} z(s_0)}{\Gamma(\alpha)}(s - s_0)^{\alpha - 1}$$

exerted on nearly throughout in *I*.

Lemma 3.2 Suppose the assumptions $(a0)$-$(b0)$ are achieved. Thereafter for any $\xi \in C(I, \mathfrak{R})$ and $0 < \alpha < 1$, $v \in C(I, \mathfrak{R})$ is function which is a solution of the *FIHDE*

$$D^{\alpha}\left[v(s) - \psi(s, v(s), v(v(s)))\right] = \xi(s), s \in I \qquad (8)$$

and

$$v(s_0) = v_0 \qquad (9)$$

if and only if *v* must be the solution of the fractional iterative of hybrid equation integrated *FIHIE*

$$v(t) = \left[v_0 - \psi(s_0, v_0, v_0(v_0))\right] + \psi(s, v(s), v(v(s))) + \int_0^s \xi(\beta)\frac{(s - \beta)^{\alpha - 1}}{\Gamma(\alpha)} d\beta, s \in I. \qquad (10)$$

Proof. Firstly we suppose that *v* is a solution for the Cauchy problem (6) defined on *I*. Since the Riemann-Liouville fractional integral I^{α} operator is a monotonous, then operating Eq.(6) by I^{α}. In view of Lemma 3.1, we obtain

$$\begin{aligned} & I^{\alpha} D^{\alpha}\left[v(s) - \psi(s, v(s), v(v(s)))\right] \\ & = v(s) - \psi(s, v(s), v(v(s))) \\ & - \frac{I^{1-\alpha}\left[v(s) - \psi(s, v(s), v(v(s)))\right]\Big|_{s = s_0}}{\Gamma(\alpha)}(s - s_0)^{\alpha - 1} \\ & = I^{\alpha}\xi(s). \end{aligned}$$

followed by (9), we obtain

$$\begin{aligned} & \left[v(s) - \psi(s, v(s), v(v(s)))\right] \\ & = v(s) - \psi(s, v(s), v(v(s))) \\ & - \frac{I^{1-\alpha}\left[v(s) - \psi(s, v(s), v(v(s)))\right]\Big|_{s = s_0}}{\Gamma(\alpha)}(s - s_0)^{\alpha - 1} \\ & = s_0 - \psi(s_0, v_0, v(v_0)) + I^{\alpha}\xi(s), \end{aligned}$$

namely

$$\begin{aligned} v(s) &= \left[v_0 - \psi(s_0, v_0, v(v_0))\right] \\ &+ \psi(s, v(s), v(v(s))) + I^{\alpha}\xi(s) \\ &= \left[v_0 - \psi(s_0, v_0, v(v_0))\right] + \psi(s, v(s), v(v(s))) \\ &+ \int_0^s \xi(\beta)\frac{(s - \beta)^{\alpha - 1}}{\Gamma(\alpha)} d\beta. \end{aligned}$$

Therefore, (10) is satisfied.

On the contrary, suppose that the function *v* fulfills the Eq(10) in *I*. Thus the application D^{α} on either side of (10), (8) holds. Again, substituting in (10) by $s = s_0$ income

$$v(s_0) - \psi(s_0, v(s_0), v(v_0(s))) = v_0 - \psi(s_0, v_0, v(v_0))$$

the map $v \to v - (s, v, v(v))$ is increasing in $\mathfrak{R}$. $\forall s \in I$, the map $v \to v - (s, v, v(v))$ is injective in $\mathfrak{R}$, and $v(s_0) = v_0$. This completes the proof.

Theorem 3.1 Let ι_0 and τ_0 be the lower and upper solutions respectively fulfilling $\iota_0(s) \le \tau_0(s)$ on *I* and let the assumptions (a1)-(a2) and (b3)-(b4) achieved. Then there exist monotone sequences $\{\iota_t\}$ and $\{\tau_t\}$ so that $\iota_t \to \iota$ and $\tau_t \to \tau$ regularly on *I*, in which ι is minimal and τ is maximal solutions for the *FIHDE*(6) on *I* and

$$\iota_0 \le \iota_1 \le \iota_2 \le \ldots \le \iota_t \le \tau_t \le \ldots \le \tau_1 \le \tau_0. \tag{11}$$

Proof. For any $\mu \in C(I,\mathfrak{R})$ with $\iota_0(s) \le \mu(s) \le \tau_0(s)$ on *I*, deem fractional iterative hybrid differential equation,

$$\begin{aligned} & D^{\alpha}\left[v(s)-\psi\left(s,v(s),v(v(s))\right)\right] \\ & =\aleph\left(s,\mu(s),\mu(\mu(s))\right) \\ & -\frac{M}{1-T^{\alpha}}\begin{bmatrix} v(s)-\psi\left(s,v(s),v(v(s))\right) \\ -\mu(s)-\psi\left(s,\mu(s),\mu(\mu(s))\right) \end{bmatrix} \\ & v(s_0)=v_0, \iota_0 \le v_0 \le \tau_0(s), \forall s \in I. \end{aligned} \tag{12}$$

Presently the Eq(12) is tantamount the issue

$$\begin{aligned} & D^{\alpha}\left[v(s)-\psi\left(s,v(s),v(v(s))\right)\right] \\ & +\frac{M}{1-T^{\alpha}}\left[v(s)-\psi\left(s,v(s),v(v(s))\right)\right] \\ & =\tilde{\aleph}\left(s,\mu(s),\mu(\mu(s))\right), s \in I \\ & v(s_0)=v_0. \end{aligned}$$

Integrating factor using the equation above can be formed

$$\begin{aligned} & D^{\alpha}\left[E_1(Ms)\left(v(s)-\psi\left(s,v(s),v(v(s))\right)\right)\right] \\ & =E_1(Ms)\tilde{\aleph}\left(s,\mu(s),\mu(\mu(s))\right), s \in I \\ & v(s_0)=v_0 \end{aligned}$$

where $E_1(Ms)=e^{Ms}$ is called Mittag-Leffer function see to some special cases above and by Lemma 3.2 the iterative differential equation up is equivalent to hybrid *FIHIE*(7)

$$\begin{aligned} & v(s)=\psi\left(s,v(s),v(v(s))\right) \\ & +E_1\left(-M(s-s_0)\left(\psi\left(s_0,v_0,v(v_0)\right)\right)\right) \\ & +\int_0^s E_1\left(-M(s-s_0)\right)\tilde{\aleph}\left(s,\mu(\beta),\mu(\mu(\beta))\right)d\beta, s \in I. \end{aligned}$$

By hypothesis (b4), there is one solution $v \in C(I,\mathfrak{R})$ of the *FIHDE*(6) defined in *I* due to the principle of Banach contraction.

Define the map *W* on $[\iota_0,\tau_0]$ by $W\mu = v$. Such a mapping will utilize for defining sequences $\{\iota_t\}$ and $\{\tau_t\}$. Let us presently show that

1. $\iota_0 \le W\iota_0$ and $\tau_0 \le W\tau_0$,
2. *W* is monotonous operator in this sector

$$[\iota_0,\tau_0]=\{v \in C(I,R) \mid \iota_0(s) \le v(s) \le \tau_0(s), s \in I\}.$$

To show (1), set $W\iota_0=\iota_1$, in which ι_1 be a unique solution of the Eq(12) on *I* with $\mu=\iota_0$. Indicate Θ as follows:

$$\begin{aligned} & \Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right) \\ & =\left(\iota_1(s)-\psi\left(s,\iota_1(s),\iota(\iota_1(s))\right)\right) \\ & -\left(\iota_0(s)-\psi\left(s,\iota_0(s),\iota(\iota_0(s))\right)\right) \end{aligned} \tag{13}$$

for $\Theta \in C(I,\mathfrak{R})$. Then $\Theta(s_0)-\psi\left(s_0,\Theta(s_0),\Theta(\Theta(s_0))\right) \ge 0$, and

$$\begin{aligned} & D^{\alpha}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right)\right] \\ & =D^{\alpha}\left[\left(\iota_1(s)-\psi\left(s,\iota_1(s),\iota(\iota_1(s))\right)\right)\right] \\ & -D^{\alpha}\left[\left(\iota_0(s)-\psi\left(s,\iota_0(s),\iota(\iota_0(s))\right)\right)\right] \\ & \ge\aleph\left(s,\iota_0(s),\iota(\iota_0(s))\right) \\ & -\frac{M}{1+T^{\alpha}}\begin{bmatrix} \left(\iota_1(s)-\psi\left(s,\iota_1(s),\iota(\iota_1(s))\right)\right) \\ -\left(\iota_0(s)-\psi\left(s,\iota_0(s),\iota(\iota_0(s))\right)\right) \end{bmatrix} \\ & -\aleph\left(s,\iota_0(s),\iota(\iota_0(s))\right) \\ & =-\frac{M}{1+T^{\alpha}}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right)\right]. \end{aligned} \tag{14}$$

This proves that

$$\begin{aligned} & \Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right) \\ & \ge E_1(-Ms)\left[\Theta(s_0)-\psi\left(s,\Theta(s_0),\Theta(\Theta(s_0))\right)\right] \ge 0, \forall s \in I \end{aligned}$$

and than from (14), we get

$$\begin{aligned} & \left(\iota_1(s)-\psi\left(s,\iota_1(s),\iota(\iota_1(s))\right)\right) \\ & \ge\left(\iota_0(s)-\psi\left(s,\iota_0(s),\iota(\iota_0(s))\right)\right) \end{aligned}$$

$\forall s \in I$. Since assumption (a1) achieves, $\iota_0(s) \le \iota_1(s) \forall s \in I$, or, equivalently, $\iota_0 \le W\iota_0$. Similarly, we can show that $\tau_0 \le W\tau_0$.

To show (2), let $\mu_1,\mu_2 \in [\iota_0,\tau_0]$ be such that $\mu_1 \le \mu_2$ on *I* . Therefore, we obtain

$$\begin{aligned} & \aleph\left(s,\mu_1(s),\mu(\mu_1(s))\right)-\aleph\left(s,\mu_2(s),\mu(\mu_2(s))\right) \\ & \ge\frac{M}{1+T^{\alpha}}\begin{bmatrix} \left(\mu_1(s)-\psi\left(s,\mu_1(s),\mu(\mu_1(s))\right)\right) \\ -\left(\mu_2(s)-\psi\left(s,\mu_2(s),\mu(\mu_2(s))\right)\right) \end{bmatrix} \end{aligned} \tag{15}$$

for all $s \in I$.

Assume that $v_1=W\mu_1$ and $v_2=W\mu_2$ and set

$$\begin{aligned} & \Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right) \\ & =\left(v_2(s)-\psi\left(s,v_2(s),v(v_2(s))\right)\right) \\ & -\left(v_1(s)-\psi\left(s,v_1(s),\mu(v_1(s))\right)\right) \end{aligned}$$

for certain $\Theta \in C(I,\mathfrak{R})$. Then,

$$\Theta(s_0)-\psi\left(s_0,\Theta(s_0),\Theta(\Theta(s_0))\right) \ge 0$$

and

$$\begin{aligned}
&D^{\alpha}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right)\right]\\
&=D^{\alpha}\left[\left(v_2(s)-\psi\left(s,v_2(s),v\left(v_2(s)\right)\right)\right)\right]\\
&-D^{\alpha}\left[\left(v_1(s)-\psi\left(s,v_1(s),\mu\left(v_1(s)\right)\right)\right)\right]\\
&=\aleph\left(s,\mu_1(s),\mu\left(\mu_1(s)\right)\right)\\
&-\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(v_2(s)-\psi\left(s,v_2(s),v\left(v_2(s)\right)\right)\right)\\-\left(\mu_2(s)-\psi\left(s,\mu_2(s),\mu\left(\mu_2(s)\right)\right)\right)\end{bmatrix}\\
&-\aleph\left(s,\mu_2(s),\mu\left(\mu_2(s)\right)\right)\\
&-\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(v_1(s)-\psi\left(s,v_1(s),\mu\left(v_1(s)\right)\right)\right)\\-\left(\mu_1(s)-\psi\left(s,\mu_1(s),\mu\left(\mu_1(s)\right)\right)\right)\end{bmatrix}\\
&\geq-\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(\mu_2(s)-\psi\left(s,\mu_2(s),\mu\left(\mu_2(s)\right)\right)\right)\\-\left(\mu_1(s)-\psi\left(s,\mu_1(s),\mu\left(\mu_1(s)\right)\right)\right)\end{bmatrix}\\
&-\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(\mu_2(s)-\psi\left(s,\mu_2(s),\mu\left(\mu_2(s)\right)\right)\right)\\-\left(\mu_1(s)-\psi\left(s,\mu_1(s),\mu\left(\mu_1(s)\right)\right)\right)\end{bmatrix} \qquad (16)\\
&+\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(v_1(s)-\psi\left(s,v_1(s),\mu\left(v_1(s)\right)\right)\right)\\-\left(\mu_1(s)-\psi\left(s,\mu_1(s),\mu\left(\mu_1(s)\right)\right)\right)\end{bmatrix}\\
&=-\frac{M}{1+T^{\alpha}}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right)\right].
\end{aligned}$$

As above, the foregoing inequality yields that $v_2 \geq v_1$ on I which implicates that $W\mu_2 \geq W\mu_1$ prove (2). Presently we defined both sequences $\{\iota_t\}$ and $\{\tau_t\}$ by

$$\iota_t = W\iota_{t-1},\ and,\ \tau_t = W\tau_{t-1}$$

for $t = 1, 2, \ldots$. The monotony of the operator W implies that

$$\iota_0 \leq \iota_1 \leq \ldots \leq \iota_t \leq \tau_t \leq \tau_{t-1} \leq \ldots \leq \tau_1 \leq \tau_0.$$

It is plain to demonstrate that the sequences $\{\iota_t\}$ and $\{\tau_t\}$ are regularly bounded and equi-continuous on *I:* Clearly, $\{\iota_t\}$ and $\{\tau_t\}$ are the solutions in from

$$\begin{cases}D^{\alpha}\left[\iota_t(s)-\psi\left(s,\iota_t(s),\iota_t\left(\iota_t(s)\right)\right)\right]\\=\aleph\left(s,\iota_t(s),\iota_t\left(\iota_t(s)\right)\right)\\-\dfrac{M}{1+T^{\alpha}}\left[\iota_t(s)-\psi\left(s,\iota_t(s),\iota_t\left(\iota_t(s)\right)\right)\right]\\\qquad\iota_t(s_0)=v_0\end{cases} \qquad (17)$$

and

$$\begin{cases}D^{\alpha}\left[\tau_t(s)-\psi\left(s,\tau_t(s),\tau_t\left(\tau_t(s)\right)\right)\right]\\=\aleph\left(s,\tau_{t-1}(s),\tau\left(\tau_{t-1}(s)\right)\right)\\-\dfrac{M}{1+T^{\alpha}}\begin{bmatrix}\tau_t(s)-\psi\left(s,\tau_t(s),\tau\left(\tau_t(s)\right)\right)\\-\tau_{t-1}(s)-\psi\left(s,\tau_{t-1}(s),\tau\left(\tau_{t-1}(s)\right)\right)\end{bmatrix}\\\qquad\tau_t(s_0)=v_0\end{cases} \qquad (18)$$

To demonstrate that ι and τ are solution of extremes for *FIHDE*(6) on *I*, we should check that if v is any another solution for *FIHDE*(6), so that $\iota_0(s) \leq v_0(s) \leq \tau_0(s)$, $s \in I$ then

$$\iota_0(s) \leq \iota(s) \leq v(s) \leq \tau(s) \leq \tau_0(s), s \in I.$$

Presume that for some $t \in N$, $\iota_t \leq v \leq \tau_t$ on *I* and set

$$\begin{aligned}
&\Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right)\\
&=\left(v(s)-\psi\left(s,v(s),v\left(v(s)\right)\right)\right)\\
&-\left(\iota_{t+1}(s)-\psi\left(s,\iota_{t+1}(s),\iota\left(\iota_{t+1}(s)\right)\right)\right)
\end{aligned}$$

for some

$$\Theta \in C(I,\mathfrak{R}) \Rightarrow \Theta(s_0)-\psi\left(s_0,\Theta(s_0),\Theta\left(\Theta(s_0)\right)\right)=0,$$

and

$$\begin{aligned}
&D^{\alpha}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right)\right]\\
&=D^{\alpha}\left[v(s)-\psi\left(s,v(s),v\left(v(s)\right)\right)\right]\\
&-D^{\alpha}\left[\iota_{t+1}(s)-\psi\left(s,\iota_{t+1}(s),\iota\left(\iota_{t+1}(s)\right)\right)\right]\\
&=\aleph\left(s,v(s),v\left(v(s)\right)\right)-\aleph\left(s,\iota(s),\iota\left(\iota_t(s)\right)\right)\\
&-\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(\iota_{t+1}(s)-\psi\left(s,\iota_{t+1}(s),\iota\left(\iota_{t+1}(s)\right)\right)\right)\\-\left(\iota_t(s)-\psi\left(s,\iota_t(s),\iota\left(\iota_t(s)\right)\right)\right)\end{bmatrix} \qquad (19)\\
&\geq-\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(v(s)-\psi\left(s,v(s),v\left(v(s)\right)\right)\right)\\-\left(\iota_t(s)-\psi\left(s,\iota_t(s),\iota\left(\iota_t(s)\right)\right)\right)\end{bmatrix}\\
&+\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(\iota_{t+1}(s)-\psi\left(s,\iota_{t+1}(s),\iota\left(\iota_{t+1}(s)\right)\right)\right)\\-\left(\iota_t(s)-\psi\left(s,\iota_t(s),\iota\left(\iota_t(s)\right)\right)\right)\end{bmatrix}\\
&=-\frac{M}{1+T^{\alpha}}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta(\Theta(s))\right)\right].
\end{aligned}$$

This yields that $\iota_{t+1}(s) \leq v(s)$ for the whole $s \in I$. In the same way, it proves that $v(s) \leq \tau_{t+1}(s)$ for the whole $s \in I$. Since $\iota_0 \leq v \leq \tau_0$ on I , we have, by induction precept which $\iota_t \leq v \leq \tau_t$ on I for all t, $t = 0,1,2,\ldots$. Taking the limit as $t \to \infty$, we conclude which $\iota \leq v \leq \tau$ on *I*. Thus ι and τ are straightly the minimal and maximal solutions for the *FIHDE*(6) on *I*.

Corollary 3.1 Let ι_0 and τ_0 are straightly the solutions of lower and upper for the *FIHDE*(6) on *I* fulfilling $\iota_0(s) \leq \tau_0(s)$ on *I* and that all conditions of Theorem 3.1 are fulfilled assumptions (*b*4) substituted for (*b*2). And the *FIHDE*(6) provides extreme solutions on *I*.

Corollary 3.1 comes from Theorem 3.1 by substituting the constant *M* assumptions (*b*3) with *M* = 0. Therewith, we discuss the case when $\aleph(s,v,v(v))$ is non-increasing in nearly v throughout for $s \in I$. Let ι_0 and τ_0 are straightly the solutions of lower and upper for the

FIHDE(6) on *I*. Now consider the two sequences ι_t and τ_t iterations definite in the following way:

$$D^{\alpha}\left[\left(\iota_{t+1}(s)-\psi\left(s,\iota_{t+1}(s),\iota\left(\iota_{t+1}(s)\right)\right)\right)\right] = \aleph\left(s,\iota(s),\iota\left(\iota_t(s)\right)\right), s\in I, \iota_{t+1}(s_0)=\iota_0; \quad (20)$$

and

$$D^{\alpha}\left[\left(\tau_{t+1}(s)-\psi\left(s,\tau_{t+1}(s),\tau\left(\tau_{t+1}(s)\right)\right)\right)\right] = \aleph\left(s,\tau(s),\tau\left(\tau_t(s)\right)\right), s\in I, \tau_{t+1}(s_0)=\tau_0; \quad (21)$$

for $t=0,1,2,3,\dots$

Next we shown that every sequences of ι_t and τ_t having two alternating sequences converging uniformly and monotonously with the solutions of extremal for *FIHDE*(6) on *I*. We need the following result, which can be found in [18]

Lemma 3.3 Let $\iota,\tau\in C(I,\mathfrak{R})$ be lower and upper solutions of *FIHDE*(6) satisfying $\iota(s)\le\tau(s)$, $s\in I$ and let the assumptions (*a*1)-(*a*2) and (*b*1) achieved. Then, there is a solution *v*(*s*) of (6), in the closed set $\overline{\mho}$, satisfying

$$\iota(s)\le v(s)\le\tau(s), s\in I.$$

Theorem 3.2 Let the assumptions (*a*1)-(*a*2), (*b*1) and (*b*4) achieved. Then either,

1. $\{\iota_t\}$ iterates presented by Eq(20) and a unique solution *v* of *FIHDE*(6) introduced in *I* fulfill:

$$\iota_0\le\iota_2\le\iota_4\le\dots\le\iota_{2t}\le v(s)\le\iota_{2t+1}\le\dots\le\iota_5\le\iota_3\le\iota_1 \quad (22)$$

for every $s\in I$, if $\iota_0(s)\le\iota_{2t}(s)$, $s\in I$. Moreover, the sequence $\{\iota_{2t}\}$ and $\{\iota_{2t+1}\}$ and monotonically converge uniformly toward $\iota_\Diamond$ and $\iota^\Diamond$ fulfilling $\iota_\Diamond\le v\le\iota^\Diamond$ for all $s\in I$; or

2. $\{t_t\}$ iterates given by Eq(21) and a unique solution *v* of *FIHDE*(6) posed in *I* fulfill:

$$\tau_0\le\tau_2\le\tau_4\le\dots\le\tau_{2t}\le v(s)\le\tau_{2t+1}\le\dots\le\tau_5\le\tau_3\le\tau_1 \quad (23)$$

for every $s\in I$, if $\tau_0(s)\le\tau_{2t}(s)$, $s\in I$. In addition, the sequence $\{\tau_{2t}\}$ and $\{\tau_{2t+1}\}$ converge uniformly monotonically toward $\tau_\Diamond$ and $\tau^\Diamond$ fulfilling $\tau_\Diamond\le v\le\tau^\Diamond$ for all $s\in I$.

Actually as extreme solutions are unique, $\iota_\Diamond=\tau_\Diamond=\iota$ and $\iota^\Diamond=\tau^\Diamond=\tau^\Diamond$ on *I* fulfilling

$$\iota(s)\le v(s)\le\tau(s), s\in I.$$

Proof. By Lemma 3.3, there exist a lower solution ι_0, an upper solution τ_0 and a solutio *v* for the *FIHDE*(6), such that

$$\iota_0(s)\le v(s)\le\tau_0(s), s\in I.$$

We will just demonstrate that case (1), since the demonstrate of case(2) followers with similar arguments. Presume $\iota_0\le\iota_2$ on *I*. First we demonstrate that

$$\iota_0(s)\le\iota_2(s)\le v(s)\le\iota_3(s)\le\iota_1(s), s\in I. \quad (24)$$

Set Θ as follows:

$$\Theta(s)-\psi\left(s,\Theta(s),\Theta\left(\Theta(s)\right)\right) = \left(\iota_1(s)-\psi\left(s,\iota_1(s),\iota\left(\iota_1(s)\right)\right)\right) - \left(\iota_0(s)\psi\left(s,\iota_0(s),\iota\left(\iota_0(s)\right)\right)\right) \quad (25)$$

for $s\in I$. Next,

$$\begin{aligned} &D^{\alpha}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta\left(\Theta(s)\right)\right)\right] \\ &= D^{\alpha}\left[\iota_1(s)-\psi\left(s,\iota_1(s),\iota\left(\iota_1(s)\right)\right)\right] \\ &-D^{\alpha}\left[\iota_0(s)\psi\left(s,\iota_0(s),\iota\left(\iota_0(s)\right)\right)\right] \\ &\ge\aleph\left(\psi\left(s_0,\iota_0(s),\iota\left(\iota_0(s)\right)\right)\right) \\ &-\aleph\left(\psi\left(s_0,\iota_0(s),\iota\left(\iota_0(s)\right)\right)\right) \\ &=0 \end{aligned}$$

and $\Theta(s_0)=0$. Hence,

$$\iota_1(s)-\psi\left(s,\iota_1(s),\iota\left(\iota_1(s)\right)\right)\ge\iota_0(s)\psi\left(s,\iota_0(s),\iota\left(\iota_0(s)\right)\right)$$

$\forall_s\in I$. In view of the assumption (a1), we obtain $\iota_1(s)\ge\iota_0(s)$ on *I*. Let

$$\Theta(s)-\psi\left(s,\Theta(s),\Theta\left(\Theta(s)\right)\right) = \left(v(s)-\psi\left(s,v(s),v\left(v(s)\right)\right)\right) \times\left(\iota_1(s)-\psi\left(s,\iota_1(s),v\left(\iota_1(s)\right)\right)\right) \quad (26)$$

for $s\in I$. Consequently, we have

$$\begin{aligned} &D^{\alpha}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta\left(\Theta(s)\right)\right)\right] \\ &= D^{\alpha}\left[\left(v(s)-\psi\left(s,v(s),v\left(v(s)\right)\right)\right)\right] \\ &-D^{\alpha}\left[\left(\iota_1(s)-\psi\left(s,\iota_1(s),v\left(\iota_1(s)\right)\right)\right)\right] \\ &=\aleph\left(s_0,v(s),v\left(v(s)\right)\right)-\aleph\left(s_0,\iota_0(s),\iota\left(\iota_0(s)\right)\right) \\ &\le 0 \end{aligned}$$

and $\Theta(s_0)=0$. Hence,

$$v(s)-\psi\left(s,v(s),v\left(v(s)\right)\right)\le\iota_1(s)-\psi\left(s,\iota_1(s),v\left(\iota_1(s)\right)\right)$$

$\forall_s\in I$. Since assumption (a1) achieved, one has $v(s)\le\iota_3(s)$ on *I*. By employing similar procedures, we may prove respectively that

$$\iota_2(s)\le v(s),\ \iota_3(s)\le\iota_1(s),\ \textit{and}\ v(s)\le\iota_3(s),\ s\in I.$$

In order to demonstrate Eq(22), the induction principle is applied, i.e suppose that Eq(22) is true for some *t* and demonstrate that it is true for (*t* + 1). Let

$$\Theta(s)-\psi\left(s,\Theta(s),\Theta\left(\Theta(s)\right)\right)$$
$$=\left(\iota_{2t+2}(s)-\psi\left(s,\iota_{2t+2}(s),v\left(\iota_{2t+2}(s)\right)\right)\right)$$
$$-\left(\iota_{2t+1}(s)-\psi\left(s,\iota_{2t+1}(s),v\left(\iota_{2t+1}(s)\right)\right)\right).$$

Next, using the monotonicity of $\aleph$, which

$$D^{\alpha}\left[\Theta(s)-\psi\left(s,\Theta(s),\Theta\left(\Theta(s)\right)\right)\right]$$
$$=D^{\alpha}\left[\left(\iota_{2t+2}(s)-\psi\left(s,\iota_{2t+2}(s),v\left(\iota_{2t+2}(s)\right)\right)\right)\right]$$
$$-D^{\alpha}\left[\left(\iota_{2t+1}(s)-\psi\left(s,\iota_{2t+1}(s),v\left(\iota_{2t+1}(s)\right)\right)\right)\right]$$
$$=\aleph\left(s,\iota_{2t+2}(s),\iota\left(\iota_{2t+2}(s)\right)\right)-\aleph\left(s,\iota_{2t+1}(s),\iota\left(\iota_{2t+1}(s)\right)\right)$$
$$\le 0.$$

and $\Theta(0)=0$. This proves $\Theta(s)\le 0$ and thus $\iota_{2t+2}(s)\le\iota_{2t+1}(s)$. The repetition of arguments similar we can obtain

$$\iota_0\le\iota_2\le\iota_4\le\ldots\le\iota_{2t}\le\iota_{2t+2}$$
$$\le v\le\iota_{2t+3}\le\iota_{2t=1}\le\ldots\le\iota_5\le\iota_3\le\iota_1$$

on *I*. Since Eq(22) is right for *t* = 1, it is following by principle of induction which Eq(22) achieved for all *t*. Obviously, that the sequences $\{\iota_{2t}\}$, $\{\iota_{2t+1}\}$ are equicontinuous and uniformly bounded; thus in virtue of the Arzela-Ascoli Theorem, they are uniformly converges and monotonously to $\iota_{\Diamond}(s)$, $\iota^{\Diamond}(s)$ respectively and that $\iota_{\Diamond}(s)\le v(s)\le\iota^{\Diamond}(s)$ on *I*. Case (2) yields with similar arguments. This completes the proof.

Corollary 3.2 Let the hypothesis (*a*1) - (*a*2), (*b*1) and (*b*2) satisfied. Moreover, let

$$\aleph\left(s,\iota_1(s),\iota\left(\iota_1(s)\right)\right)-\aleph\left(s,\iota_2(s),\iota\left(\iota_2(s)\right)\right)$$
$$\ge-\frac{M}{1+T^{\alpha}}\begin{bmatrix}\left(\iota_1(s)-\psi\left(s,\iota_1(s),\iota\left(\iota_1(s)\right)\right)\right)\\ \left(\iota_2(s)-\psi\left(s,\iota_2(s),\iota\left(\iota_2(s)\right)\right)\right)\end{bmatrix}$$

$\forall_s\in I$, where $\iota_1(s)\ge\iota_2(s)$ on *I*. Then $\iota(s)=\tau(s)=v(s)$ on *I*.

Observed which in the show of Theorem 3.2, $\iota(s)$ *and* $\tau(s)$ are in quasi-solutions for the *FIHDE*(6), since we have that

$$D^{\alpha}\left[\iota(s)-\psi\left(s,\iota(s),\iota\left(\iota(s)\right)\right)\right]=\aleph\left(s,\tau(s),\tau\left(\tau(s)\right)\right),\quad s\in I,\ \iota(s_0)=v_0, \tag{27}$$

and

$$D^{\alpha}\left[\tau(s)-\psi\left(s,\tau(s),\tau\left(\tau(s)\right)\right)\right]=\aleph\left(s,\iota(s),\iota\left(\iota(s)\right)\right),\quad s\in I,\ \tau(s_0)=v_0. \tag{28}$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors jointly worked on deriving the results and approved the final manuscript.

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