Banach Algebras – Bare Bones Basics

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1. Introduction

A Banach algebra $\mathcal{B}$ is a Banach space (over $\mathbb{C}$), equipped with a product (making $\mathcal{B}$ an algebra over $\mathbb{C}$), satisfying

$$\|xy\| \le \|x\|\,\|y\|. \tag{1.1}$$

We say $\mathcal{B}$ has a unit I if $I \in \mathcal{B}$ satisfies

$$Ix = xI = x, \quad \forall\, x \in \mathcal{B}, \quad \|I\| = 1. \tag{1.2}$$

Banach algebras arise in a variety of settings. Examples include $\mathcal{L}(V)$, the space of bounded linear operators on a Banach space V (with the operator norm), $C(X)$, the space of continuous functions on a compact space X (with the sup norm), the space of functions on the circle S^1 with absolutely summable Fourier series,

$$\mathcal{A}(S^1) = \{f \in C(S^1) : \sum |\hat{f}(k)| < \infty\}, \quad \|f\| = \sum |\hat{f}(k)|, \tag{1.2A}$$

where $\hat{f}(k)$ are the Fourier coefficients of f, and many others, such as algebras of bounded holomorphic functions on a complex domain, or closed subalgebras of such algebras as mentioned above. It is useful and informative to have a general theory of Banach algebras that encompasses such examples.

The purpose of these notes is to discuss some basic aspects of such a theory. One major theme we take up centers on the question of when an element x of a Banach algebra $\mathcal{B}$ is invertible, i.e., when there exists $x^{-1} \in \mathcal{B}$ such that $xx^{-1} = x^{-1}x = I$. This study gets started with the following simple observation. Let $y \in \mathcal{B}$; then

$$\|y\| < 1 \Longrightarrow I - y \ \text{ is invertible.} \tag{1.3}$$

In fact,

$$(I - y)^{-1} = \sum_{k=0}^{\infty} y^k. \tag{1.4}$$

To proceed, it is useful to introduce the *resolvent set* $\rho(x)$ and *spectrum* $\sigma(x)$ of $x \in \mathcal{B}$, defined as follows. For $\zeta \in \mathbb{C}$,

$$\zeta \in \rho(x) \Leftrightarrow \zeta - x \ \text{ is invertible}, \quad \sigma(x) = \mathbb{C} \setminus \rho(x). \tag{1.5}$$

If $\zeta \in \rho(x)$, let us set $R_\zeta = (\zeta - x)^{-1}$. Note that if $\zeta_0 \in \rho(x)$,

$$\zeta - x = \zeta_0 - x + (\zeta - \zeta_0) = (\zeta_0 - x)(I + (\zeta - \zeta_0)R_{\zeta_0}), \tag{1.6}$$

and, by (1.3), this is invertible as long as $|\zeta - \zeta_0| \le 1/\|R_{\zeta_0}\|$, and one has a convergent power series in $\zeta - \zeta_0$. Hence

Proposition 1.1. *If $\mathcal{B}$ is a Banach algebra with unit and $x \in \mathcal{B}$, then $\rho(x) \subset \mathbb{C}$ is open, and $(\zeta - x)^{-1}$ is holomorphic on $\rho(x)$.*

Note also that, for $x \in \mathcal{B}$,

$$\zeta - x = \zeta(I - \zeta^{-1}x), \tag{1.7}$$

and, by (1.3), this is invertible as long as $|\zeta| > \|x\|$. Hence

Proposition 1.2. *In the setting of Proposition 1.1, $\sigma(x)$ is a compact set, contained in $\{\zeta \in \mathbb{C} : |\zeta| \le \|x\|\}$.*

It follows readily from (1.7) that

$$\|(\zeta - x)^{-1}\| \le \frac{C}{|\zeta|}, \quad \text{for } \ |\zeta| > 2\|x\|. \tag{1.8}$$

If $\sigma(x) = \emptyset$, $\zeta - x)^{-1}$ would be an entire holomorphic function. Then (1.8) would contradict Liouville's theorem. This yields

Proposition 1.3. *In the setting of Proposition 1.1, $\sigma(x) \neq \emptyset$.*

It is useful to have a more precise knowledge of $\sigma(x)$. One ingredient in the study of $\sigma(x)$ is the *spectral radius*:

$$r(x) = \sup \{|\zeta| : \zeta \in \sigma(x)\}. \tag{1.9}$$

From (1.7), we have

$$(\zeta - x)^{-1} = \frac{1}{\zeta} \sum_{k=0}^{\infty} \frac{1}{\zeta^k} x^k, \tag{1.10}$$

for $|\zeta| > \|x\|$. Hence (as already stated in Proposition 1.2)

$$r(x) \le \|x\|. \tag{1.11}$$

Some sharper results will be mentioned below.

These notes are structured as follows. In §2 we discuss some results of I. Gelfand on commutative Banach algebras $\mathcal{B}$ (satisfying $xy = yx$ for all $x, y \in \mathcal{B}$), with unit. This theory centers in the study of *characters* on $\mathcal{B}$, i.e., linear maps satisfying

$$\varphi : \mathcal{B} \longrightarrow \mathbb{C}, \quad \varphi(xy) = \varphi(x)\varphi(y), \quad \varphi(I) = 1. \tag{1.12}$$

The set $\mathcal{M}(\mathcal{B})$ of such characters is shown to be a compact subset of the dual space $\mathcal{B}'$ (with the weak* topology), and the central result is that, for all $x \in \mathcal{B}$,

$$\sigma(x) = \{\varphi(x) : \varphi \in \mathcal{M}(\mathcal{B})\}. \tag{1.13}$$

These characters fit together to yield the Gelfand transform

$$\gamma : \mathcal{B} \longrightarrow C(\mathcal{M}(\mathcal{B})), \quad \gamma(x)(\varphi) = \varphi(x). \tag{1.14}$$

This map need not be injective (though it often is), and it need not have dense range (though it often does). There is an important class of algebras, described below, for which (1.14) is an *isomorphism.*

Given the results of §2, one is motivated to classify the characters of a given commutative Banach algebra with unit. In §3 we discuss some classes of Banach algebras of continuous functions on the circle S^1 for which we can produce a natural one-to-one correspondence $\mathcal{M}(\mathcal{B}) \equiv S^1$. This class includes the algebra $\mathcal{A}(S^1)$, given by (1.2A). In such a case, the result (1.13) establishes a classical theorem of N. Wiener on $\mathcal{A}(S^1)$.

In §4 we discuss C^* algebras, which are Banach algebras $\mathcal{C}$ with a conjugate linear involution $x \mapsto x^*$, satisfying

$$(xy)^* = y^*x^*, \quad \|x^*x\| = \|x\|^2, \quad \forall\, x, y \in \mathcal{C} \tag{1.15}$$

(and $I^* = I$ if $\mathcal{C}$ has the unit I). Examples include $C(X)$ ($f^* = \overline{f}$), $\mathcal{L}(H)$, the space of bounded linear operators on a Hilbert space H (with the operator adjoint), and closed subalgebras of such examples, provided such a subalgebra is invariant under $x \mapsto x^*$. In this setting, self-adjoint elements $a \in \mathcal{C}$ (satisfying $a = a^*$) play a major role. Key results of §4 include

$$a = a^* \Longrightarrow \sigma(a) \subset \mathbb{R} \ \text{ and } \ r(a) = \|a\|. \tag{1.16}$$

In §5, we concentrate on commutative C^* algebras, and refine results of §2 in this setting. The central result is that if $\mathcal{A}$ is a commutative C^* algebra with unit, then the Gelfand map

$$\gamma : \mathcal{A} \longrightarrow C(\mathcal{M}(\mathcal{A})) \tag{1.17}$$

is an isometric $*$-isomorphism of algebras. The results in (1.16) play an important role in proving this.

In §6, we apply the results of §5 to prove the spectral theorem for a bounded self-adjoint operator on a Hilbert space H, or more generally for a family $\{A_j\}$ of mutually commuting self-adjoint operators in $\mathcal{L}(H)$. This family generates a commutative C^* algebra $\mathcal{A}$. The version of the spectral theorem we prove can be stated as follows. There exists a measure space $(\mathfrak{X}, \mu)$, a unitary map $\Phi : H \to L^2(\mathfrak{X}, \mu)$, and an isometric $*$-homomorphism $\Gamma : \mathcal{A} \to L^\infty(\mathfrak{X}, \mu)$, such that

$$\Phi A \Phi^{-1} f = \Gamma(A) f, \quad \forall\, A \in \mathcal{A}, \ f \in L^2(\mathfrak{X}, \mu). \tag{1.18}$$

The $*$-isomorphism (1.17) plays a key role in the proof of this result; $\mathfrak{X}$ arises as a disjoint union of copies of $\mathcal{M}(\mathcal{A})$, each carrying a certain positive Radon measure.

These notes end with several appendices. Appendix A discusses the holomorphic functional calculus, which defines $f(x) \in \mathcal{B}$ whenever $x \in \mathcal{B}$ and f is holomorphic on a neighborhood of $\sigma(X)$, as follows:

$$f(x) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} f(\zeta)(\zeta - x)^{-1}\, d\zeta, \tag{1.19}$$

where Ω is a smoothly bounded neighborhood of $\sigma(x)$ and f is holomorphic on a neighborhood of $\overline{\Omega}$. Results of this appendix are applied in remarks at the end of §2, and to properties of e^{tx} for $x \in \mathcal{B}$ in §4.

Appendix B establishes the identity

$$r(x) = \lim_{k\to\infty} \|x^k\|^{1/k}, \tag{1.20}$$

which is more precise than (1.11). This is useful in §4, perhaps ironically, to prove the latter result in (1.16).

Appendices A and B cover material basic to the development in §§2–4. The next appendices provide material supplementary to that of our main text. (One who needs only "bare bones" might skip this material.) Appendix C extends the conclusion of Proposition 3.2 to a class of Banach algebras we call "rich Banach algebras of continuous functions." Appendices D and E derive some further extensions of Proposition 3.2.

As the title of this set of notes indicates, the material here touches on just a small subset of the lore on Banach algebras. For more on this, one can consult Chapters 7–9 of [C] and Chapter 9 of [Y], which are useful texts on functional analysis. In addition, one can consult [HR] and [L], which apply Banach algebra theory to the study of harmonic analysis on topological groups, and [M], which gives a general treatment of C^* algebra theory.

2. The Gelfand transform

Throughout this section, $\mathcal{B}$ is a Banach algebra with unit I, satisfying $\|I\| = 1$, and we assume $\mathcal{B}$ is commutative, i.e., $xy = yx$, for all $x, y \in \mathcal{B}$.

A character on $\mathcal{B}$ is an algebraic homomorphism,

$$\varphi : \mathcal{B} \longrightarrow \mathbb{C}, \quad \varphi(xy) = \varphi(x)\varphi(y), \quad \varphi(I) = 1. \tag{2.1}$$

Such a linear map is automatically bounded. This is seen as follows (using (1.3)):

$$\|y\| < 1 \Rightarrow (I - y)^{-1} \in \mathcal{B} \Rightarrow \varphi(I - y) \neq 0 \Rightarrow \varphi(y) \neq 1. \tag{2.2}$$

Applying this to $y = x/\alpha$,

$$\varphi(x) = \alpha \neq 0 \Rightarrow \varphi(\alpha^{-1}x) = 1 \Rightarrow \|\alpha^{-1}x\| \geq 1 \Rightarrow |\varphi(x)| \leq \|x\|, \tag{2.3}$$

so $\|\varphi\| = 1$. We denote the set of characters of $\mathcal{B}$ by $\mathcal{M}(\mathcal{B})$. This is a subset of the unit ball in $\mathcal{B}'$, closed with respect to the weak* topology, so it is a compact space.

Given such φ,

$$\mathcal{I} = \operatorname{Ker} \varphi \tag{2.4}$$

is a closed linear subspace of $\mathcal{B}$, and it is an ideal:

$$x \in \mathcal{I},\ y \in \mathcal{B} \Longrightarrow xy \in \mathcal{I}. \tag{2.5}$$

Since (2.4) has codimension 1 in $\mathcal{B}$ it must be a maximal ideal.

It is useful to linger on the concept of an ideal, so let $\mathcal{I} \subset \mathcal{B}$ be a proper ideal. Then $x \in \mathcal{I} \Rightarrow x$ is not invertible. On the other hand, $\|I - x\| < 1 \Rightarrow x$ is invertible, so

$$\operatorname{dist}(I, \mathcal{I}) = 1. \tag{2.6}$$

Hence the closure of such $\mathcal{I}$ is also a proper ideal. When $\mathcal{I} \subset \mathcal{B}$ is a closed ideal, the quotient $\mathcal{B}/\mathcal{I}$ is a Banach space, with norm $\|[x]\| = \inf\{\|x - z\| : z \in \mathcal{I}\}$. It has a product:

$$\begin{aligned} [x],\ [y] \in \mathcal{I} \Rightarrow [x][y] &= (x + \mathcal{I})(y + \mathcal{I}) \\ &= xy + x\mathcal{I} + \mathcal{I}y + \mathcal{I}\mathcal{I} \\ &= [xy]. \end{aligned} \tag{2.7}$$

Furthermore, one readily verifies

$$\|[x]\,[y]\| \leq \|[x]\|\,\|[y]\|, \tag{2.8}$$

so $\mathcal{B}/\mathcal{I}$ is a Banach algebra.

As we have noted, each element of a proper ideal $\mathcal{I} \subset \mathcal{B}$ is not invertible. Conversely, if $x \in \mathcal{B}$ is not invertible, $(x) = \{xy : y \in \mathcal{B}\}$ is a proper ideal in $\mathcal{B}$. Zorn's lemma gives

Proposition 2.1. *Each proper ideal $\mathcal{B}$ is contained in a maximal (proper) ideal in $\mathcal{B}$.*

Hence we have

Corollary 2.2. *If $x \in \mathcal{B}$ is not invertible, there exists a maximal ideal $\mathcal{I} \subset \mathcal{B}$ such that $x \in \mathcal{I}$.*

We now relate this to characters.

Proposition 2.3. *If $\mathcal{I} \subset \mathcal{B}$ is a maximal ideal, there exists a character $\varphi : \mathcal{B} \to \mathbb{C}$ such that $\mathcal{I} = \operatorname{Ker} \varphi$.*

The proof makes use of the following result, called the Gelfand-Mazur theorem.

Proposition 2.4. *If $\mathcal{I} \subset \mathcal{B}$ is a maximal ideal, then there is a canonical isomorphism*

$$\mathcal{B}/\mathcal{I} \equiv \mathbb{C}. \tag{2.9}$$

Proof. $\mathcal{B}/\mathcal{I}$ is a Banach algebra with unit. Given $[x] \in \mathcal{B}/\mathcal{I}$, we claim that

$$[x] \neq 0 \Longrightarrow [x]^{-1} \in \mathcal{B}/\mathcal{I}. \tag{2.10}$$

In fact, $x \notin \mathcal{I} \Rightarrow (x) + \mathcal{I} = \{xy + \mathcal{I} : y \in \mathcal{B}\}$ is a larger ideal, hence $= \mathcal{B}$, so there exists $y \in \mathcal{B}$ such that $xy = I \bmod \mathcal{I}$. This gives (2.10). In other words, $\mathcal{B}/\mathcal{I}$ is a *field*. Now we know that the spectrum $\sigma([x]) \neq \emptyset$, and

$$\lambda \in \sigma([x]) \Longrightarrow \lambda - [x] \ \text{ not invertible in } \ \mathcal{B}/\mathcal{I}, \tag{2.11}$$

so (2.10) gives $[x] = \lambda$. This proves (2.9).

Proof of Proposition 2.3. The character φ is given by the composition

$$\mathcal{B} \longrightarrow \mathcal{B}/\mathcal{I} \xrightarrow{\approx} \mathbb{C}. \tag{2.12}$$

NOTE. There is just one such character; if ψ is also a character, $\operatorname{Ker} \varphi = \operatorname{Ker} \psi \Rightarrow \varphi \equiv \psi$.

Having Proposition 2.3, we can restate Corollary 2.2:

Corollary 2.5. *If $x \in \mathcal{B}$ is not invertible, there exists a character $\varphi : \mathcal{B} \to \mathbb{C}$ such that $\varphi(x) = 0$.*

Note that the converse is clear; if x is invertible and φ is a character, then $1 = \varphi(xx^{-1}) = \varphi(x)\varphi(x^{-1})$, so $\varphi(x) \neq 0$. We hence have the following important result.

Proposition 2.6. *Given* $x \in \mathcal{B}$,

$$\sigma(x) = \{\varphi(x) : \varphi \in \mathcal{M}(\mathcal{B})\}. \tag{2.13}$$

Proof. The identity (2.13) is equivalent to the assertion that, for $\lambda \in \mathbb{C}$, $\lambda - x$ is not invertible if and only if there exists $\varphi \in \mathcal{M}(\mathcal{B})$ such that $\varphi(\lambda - x) = 0$, an assertion which clearly follows from Corollary 2.5.

The Gelfand transform is the map

$$\begin{aligned} &\gamma : \mathcal{B} \longrightarrow C(\mathcal{M}(\mathcal{B})), \\ &\gamma(x)(\varphi) = \varphi(x). \end{aligned} \tag{2.14}$$

From (2.13) (plus (1.20)) we have

$$\sup_{\varphi} |\gamma(x)(\varphi)| = r(x) = \lim_{k\to\infty} \|x^k\|^{1/k}. \tag{2.15}$$

In particular,

$$\operatorname{Ker}\gamma = \{x \in \mathcal{B} : r(x) = 0\}. \tag{2.16}$$

Frequently, this is 0, but not always. The range of γ is sometimes dense in $C(\mathcal{M}(\mathcal{B}))$ and sometimes not.

Recall that if $x \in \mathcal{B}$ and we have a smoothly bounded open $\Omega \supset \sigma(x)$ and f is holomorphic on a neighborhood of $\overline{\Omega}$, then $f(x) \in \mathcal{B}$ is given by

$$f(x) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta)(\zeta - x)^{-1}\, d\zeta. \tag{2.17}$$

This is related to the action of characters as follows.

Proposition 2.7. *Given* $\varphi \in \mathcal{M}(\mathcal{B})$,

$$\varphi(f(x)) = f(\varphi(x)). \tag{2.18}$$

Proof. For $\zeta \in \rho(x)$, $1 = \varphi((\zeta - x)(\zeta - x)^{-1}) = (\zeta - \varphi(x))\varphi((\zeta - x)^{-1})$, so

$$\varphi((\zeta - x)^{-1}) = \frac{1}{\zeta - \varphi(x)}. \tag{2.19}$$

Hence, applying φ to (2.17) gives

$$\varphi(f(x)) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta) \frac{1}{\zeta - \varphi(x)}\, d\zeta = f(\varphi(x)), \tag{2.20}$$

as asserted.

To restate Proposition 2.7 in the language of (2.14), we have

$$\gamma(f(x)) = f(\gamma(x)), \tag{2.21}$$

whenever $x \in \mathcal{B}$ and f is holomorphic on a neighborhood of $\sigma(x)$.

The following consequence of Propositions 2.6–2.7 is a special case of the spectral mapping theorem; see Appendix A for a more general version, in the setting of noncommutative Banach algebras.

Proposition 2.8. *Given* $x \in \mathcal{B}$ *(a commutative Banach algebra with unit) and* f *holomorphic on a neighborhood of* $\sigma(x)$,

$$\sigma(f(x)) = f(\sigma(x)). \tag{2.22}$$

3. Banach algebras of functions on S^1

Let $\mathcal{B}$ be an algebra of continuous functions on the circle S^1, with pointwise sum and product. We assume $\mathcal{B}$ is equipped with a norm making it a Banach algebra. We also assume $1 \in \mathcal{B}$ and $\|1\| = 1$. Note that for each $\zeta \in S^1$, $f \mapsto f(\zeta)$ is a character, so (2.3) implies

$$\sup_{\zeta \in S^1} |f(\zeta)| \le \|f\|, \quad \forall\, f \in \mathcal{B}. \tag{3.1}$$

We place the following somewhat restrictive hypotheses on $\mathcal{B}$. Taking $e_k(\zeta) = \zeta^k,\ k \in \mathbb{Z}$, we assume

$$e_{\pm 1} \in \mathcal{B}, \quad \|e_{\pm 1}\| = 1. \tag{3.2}$$

We also assume

$$\mathcal{P} = \operatorname{Span}\{e_k : k \in \mathbb{Z}\} \ \text{ is dense in } \ \mathcal{B}. \tag{3.3}$$

Examples of Banach algebras satisfying these hypotheses include $C(S^1)$, and also

$$\mathcal{A}(S^1) = \{f \in C(S^1) : \sum |\hat{f}(k)| < \infty\}, \quad \|f\| = \sum |\hat{f}(k)|. \tag{3.4}$$

We note parenthetically that (3.2) $\Rightarrow \|e_{\pm k}\| \le 1$. Since also $1 = e_k e_{-k}$, this yields

$$\|e_{\pm k}\| = 1, \quad \forall\, k \in \mathbb{Z}. \tag{3.5}$$

We seek to classify all the characters $\varphi : \mathcal{B} \to \mathbb{C}$ for such a Banach algebra $\mathcal{B}$. This works as follows. Given such a character, set

$$\zeta = \varphi(e_1), \quad \zeta \in \mathbb{C}. \tag{3.6}$$

It follows that $\varphi(e_k) = \zeta^k,\ \forall\, k \in \mathbb{Z}$. By (3.6) plus (2.3),

$$|\zeta| = |\varphi(e_1)| \le \|e_1\| = 1, \quad |\zeta^{-1}| = |\varphi(e_{-1})| \le \|e_{-1}\| = 1, \tag{3.7}$$

so

$$\varphi(e_1) = \zeta \in S^1, \quad \forall\, \varphi \in \mathcal{M}(\mathcal{B}). \tag{3.8}$$

Hence $\varphi(e_1) = e_1(\zeta)$ and, more generally, $\varphi(e_k) = e_k(\zeta),\ \forall\, k \in \mathbb{Z}$, so

$$\varphi(f) = f(\zeta), \tag{3.9}$$

for all $f \in \mathcal{P}$, given in (3.3), so (3.9) holds for all $f \in \mathcal{B}$. Together with Proposition 2.6, this gives the following.

Proposition 3.1. *If $\mathcal{B}$ satisfies (3.2)–(3.3), then*

$$f \in \mathcal{B} \Longrightarrow \sigma(f) = \{f(\zeta) : \zeta \in S^1\}. \tag{3.10}$$

Such a result is obvious for $\mathcal{B} = C(S^1)$. For $\mathcal{B} = \mathcal{A}(S^1)$, given by (3.4), it is a classical result of N. Wiener, often stated in the following form.

Proposition 3.2. *If $f \in \mathcal{A}(S^1)$ and $f(\zeta) \neq 0,\ \forall\, \zeta \in S^1$, then $1/f \in \mathcal{A}(S^1)$.*

We want to extend the scope of Proposition 3.1 to a larger class of Banach algebras. We retain the hypothesis (3.3) but relax (3.2) to the following:

$$\|e_{\pm k}\| \le C_\varepsilon e^{\varepsilon |k|}, \quad \forall\, \varepsilon > 0,\ k \in \mathbb{Z}. \tag{3.11}$$

Now let $\varphi : \mathcal{B} \to \mathbb{C}$ be a character, and define $\zeta \in \mathbb{C}$ by (3.6). We still have $\varphi(e_k) = \zeta^k,\ \forall\, k \in \mathbb{Z}$, and this time, in place of (3.7),

$$|\zeta^k| = |\varphi(e_k)| \le \|e_k\| \le C_\varepsilon e^{\varepsilon|k|},\ \forall\, k \in \mathbb{Z} \Longrightarrow |\zeta| = 1. \tag{3.12}$$

Thus the conclusions (3.8) and (3.9) continue to hold, and we have

Proposition 3.3. *If $\mathcal{B}$ satisfies (3.3) and (3.11), the conclusion (3.10) holds.*

See Appendix C for a substantial generalization of Proposition 3.2.

4. C^* algebras

A C^* algebra $\mathcal{C}$ is a Banach algebra, equipped with a conjugate linear involution $x \mapsto x^*$, satisfying

$$(xy)^* = y^* x^*, \quad \|x^* x\| = \|x\|^2, \quad \forall\, x, y \in \mathcal{C}. \tag{4.1}$$

The paradigm example of a C^* algebra is $\mathcal{L}(H)$, the space of bounded linear operators on a Hilbert space H. In such a case, $x : H \to H$ has an adjoint x^* defined by $(xv, w) = (v, x^* w)$, for $v, w \in H$. Note that $\|x^* x\| \le \|x^*\|\,\|x\|$ plus (4.1) implies $\|x\| \le \|x^*\|$, for all $x \in \mathcal{C}$, hence

$$\|x^*\| = \|x\|, \quad \forall\, x \in \mathcal{C}. \tag{4.2}$$

We typically consider C^* algebras with unit, so $\mathcal{C}$ has a unit I, satisfying $\|I\| = 1$, and $I^* = I$. In such a case, recall the definitions of the resolvent set $\rho(x)$ and spectrum $\sigma(x)$ of $x \in \mathcal{C}$ from §1. As shown there, $\rho(x)$ is open and $\sigma(x)$ is compact and nonempty, for each $x \in \mathcal{C}$.

We say $a \in \mathcal{C}$ is self adjoint provided $a = a^*$. One of the first basic results we aim to establish is

$$a = a^* \Longrightarrow \sigma(a) \subset \mathbb{R}. \tag{4.3}$$

This is more straightforward for $\mathcal{C} = \mathcal{L}(H)$ than for general C^* algebras. It turns out to be convenient first to address the analogous issue of $\sigma(u)$ when $u \in \mathcal{C}$ is unitary, i.e.,

$$u^* u = u u^* = I. \tag{4.4}$$

(We say $u \in \mathfrak{U}$.) Note that (4.4) and (4.2) imply

$$\|u\| = \|u^*\| = \|u^{-1}\| = 1, \tag{4.5}$$

so certainly $\sigma(u) \subset \{\zeta \in \mathbb{C} : |\zeta| \le 1\}$, by Proposition 1.2. Also, writing

$$\zeta - u = -u(I - \zeta u^{-1}), \tag{4.6}$$

we see that

$$|\zeta| < 1 \Longrightarrow (\zeta - u)^{-1} = -(I - \zeta u^{-1})^{-1} u^{-1}, \tag{4.6A}$$

and $(I - \zeta u^{-1})^{-1}$ is given by a convergent power series, since $\|\zeta u^{-1}\| = |\zeta| < 1$. Hence

$$u \in \mathfrak{U} \Longrightarrow \sigma(u) \subset S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}. \tag{4.7}$$

We now prove

Proposition 4.1. *The implication (4.3) holds.*

Proof. Using the power series

$$u(t) = e^{ita} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} a^k, \tag{4.8}$$

we readily verify that

$$a = a^* \Longrightarrow u(t)^* = u(-t), \quad \forall\, t \in \mathbb{R}. \tag{4.9}$$

Also, basic results on the exponential (cf. (A.25) and (A.30)) give

$$u(s+t) = u(s)u(t), \quad \forall\, s, t \in \mathbb{R}, \tag{4.10}$$

and of course $u(0) = I$, hence $u(-t) = u(t)^{-1}$. Thus

$$a = a^* \Longrightarrow u = e^{ia} \in \mathfrak{U}. \tag{4.11}$$

Now, for $\lambda \in \mathbb{C}$,

$$\begin{aligned} e^{ia} - e^{i\lambda} &= (e^{i(a-\lambda)} - I)e^{i\lambda} \\ &= i(a-\lambda)be^{i\lambda}, \end{aligned} \tag{4.12}$$

with

$$b = \sum_{k=1}^{\infty} \frac{1}{k!} (i(a-\lambda))^{k-1} \in \mathcal{C}, \tag{4.13}$$

so

$$\lambda \in \sigma(a) \Longrightarrow e^{i\lambda} \in \sigma(e^{ia}). \tag{4.14}$$

By (4.7) and (4.11), $\sigma(e^{ia}) \subset S^1$ if $a = a^*$, so (4.14) implies (4.3).

Now that we have (4.3), we can associate to a self-adjoint $a \in \mathcal{C}$ another unitary element, known as the Cauchy transform:

$$u = (a+i)(a-i)^{-1}, \quad a = a^*. \tag{4.15}$$

To see that this is unitary, note that $(a-i)(a-i)^{-1} = (a-i)^{-1}(a-i)$ implies $(a-i)^{-1}$ commutes with a, hence with $a+i$, so also

$$u = (a-i)^{-1}(a+i). \tag{4.16}$$

If $x \in \mathcal{C}$ is invertible with inverse $y \in \mathcal{C}$, then $(xy)^* = y^*x^*$ yields $(x^*)^{-1} = (x^{-1})^*$, so (4.15) yields

$$u^* = ((a-i)^{-1})^*(a+i)^* = (a+i)^{-1}(a-i), \tag{4.17}$$

and comparison with (4.16) gives

$$u^*u = uu^* = I, \tag{4.18}$$

proving unitarity.

In general, if $\mathcal{B}$ is a Banach algebra with unit I and $\mathcal{A} \subset \mathcal{B}$ a closed subalgebra, containing I, then $x \in \mathcal{A} \subset \mathcal{B}$ might be invertible in $\mathcal{B}$ but not in $\mathcal{A}$, so its spectrum might depend essentially on which Banach algebra one is concerned with. It is useful to know that this does not happen in the C^* algebra context. We have the following.

Proposition 4.2. *Let $\mathcal{C}$ be a C^* algebra with unit I and $\mathcal{A} \subset \mathcal{C}$ a closed subalgebra, invariant under $x \mapsto x^*$ and containing I. Then $x \in \mathcal{A} \subset \mathcal{C}$ is invertible in $\mathcal{C}$ if and only if it is invertible in $\mathcal{A}$.*

Proof. Clearly x is invertible in $\mathcal{C}$ (resp., $\mathcal{A}$) if and only if x^*x is, so it suffices to prove the result when $x = x^*$. Then (with obvious notation)

$$\sigma_{\mathcal{C}}(x) \subset \sigma_{\mathcal{A}}(x) \subset \mathbb{R}, \tag{4.19}$$

and we desire to show that, in this setting, $\sigma_{\mathcal{C}}(x) = \sigma_{\mathcal{A}}(x)$. The key observation is that

$$R_\zeta = (\zeta - x)^{-1} \ \ \text{(inverse in } \mathcal{C}\text{) is holomorphic on } \ \mathbb{C} \setminus \sigma_{\mathcal{C}}(x), \tag{4.20}$$

and it clearly agrees with

$$\widetilde{R}_\zeta = (\zeta - x)^{-1} \ \ \text{(inverse in } \mathcal{A}\text{)}, \tag{4.21}$$

for $|\zeta| > \|x\|$, since in both cases

$$R_\zeta = \widetilde{R}_\zeta = \frac{1}{\zeta} \sum_{k=0}^{\infty} \frac{1}{\zeta^k} x^k. \tag{4.22}$$

We conclude that, if $\omega \in \mathcal{C}'$ and $\omega \perp \mathcal{A}$, then

$$\langle R_\zeta, \omega \rangle = 0, \tag{4.23}$$

whenever $|\zeta| > \|x\|$. By analytic continuation (since $\mathbb{C} \setminus \sigma_{\mathcal{C}}(x)$ is *connected*), (4.23) holds for all $\zeta \in \mathbb{C} \setminus \sigma_{\mathcal{C}}(x)$, and Proposition 4.2 is proven.

The following is a useful result on the spectral radius of a self-adjoint element $a \in \mathcal{C}$.

Proposition 4.3. *If $\mathcal{C}$ is a C^* algebra with unit and $a \in \mathcal{C}$, then*

$$a = a^* \Longrightarrow r(a) = \|a\|. \tag{4.24}$$

Proof. As shown in Appendix B, Proposition B.1,

$$\begin{aligned} r(a) &= \lim_{k \to \infty} \|a^k\|^{1/k} \\ &= \lim_{n \to \infty} \|a^{2^n}\|^{1/2^n}. \end{aligned} \tag{4.25}$$

Now (4.1) implies, for $a = a^*$,

$$\|a^{2^n}\| = \|a^{2^{n-1}}\|^2 = \cdots = \|a\|^{2^n}, \tag{4.26}$$

so

$$\lim_{n \to \infty} \|a^{2^n}\|^{1/2^n} = \|a\|, \tag{4.27}$$

and we have (4.24).

5. Commutative C^* algebras

Throughout this section, $\mathcal{A}$ will be a commutative C^* algebra, with unit I. We derive further results on the Gelfand transform $\gamma : \mathcal{A} \to C(\mathcal{M}(\mathcal{A}))$ in this setting. Recall that, for $x \in \mathcal{A}$,

$$\gamma(x)(\varphi) = \varphi(x), \tag{5.1}$$

where

$$\varphi : \mathcal{A} \longrightarrow \mathbb{C} \ \text{ is a character,} \tag{5.2}$$

so φ is linear, $\varphi(xy) = \varphi(x)\varphi(y),\ \varphi(I) = 1$. $\mathcal{M}(\mathcal{A})$ denotes the set of characters, which is a compact subset of the dual space $\mathcal{A}'$, with the weak* topology. By Proposition 2.6,

$$x \in \mathcal{A} \Longrightarrow \sigma(x) = \{\varphi(x) : x \in \mathcal{M}(\mathcal{A})\}. \tag{5.3}$$

In particular, by Proposition 4.1,

$$a = a^* \in \mathcal{A},\ \varphi \in \mathcal{M}(\mathcal{A}) \Longrightarrow \varphi(a) \in \mathbb{R}. \tag{5.4}$$

This gives the following.

Proposition 5.1. *If $x \in \mathcal{A}$ and $\varphi \in \mathcal{M}(\mathcal{A})$, then*

$$\varphi(x^*) = \overline{\varphi(x)}. \tag{5.5}$$

Proof. We can write

$$x = a + ib, \quad a = a^*,\ b = b^*,$$

taking $a = (x + x^*)/2,\ b = (x - x^*)/2i$. Then $x^* = a - ib$, so

$$\varphi(x) = \varphi(a) + i\varphi(b), \quad \varphi(x^*) = \varphi(a) - i\varphi(b). \tag{5.6}$$

Since $\varphi(a),\ \varphi(b) \in \mathbb{R}$, (5.5) follows.

The following is a crucial estimate.

Proposition 5.2. *If $x \in \mathcal{A}$, then*

$$\|x\| = \sup_{\varphi \in \mathcal{M}(\mathcal{A})} |\varphi(x)|. \tag{5.7}$$

Proof. Since $\varphi(x^*x) = |\varphi(x)|^2$ and $\|x^*x\| = \|x\|^2$, (5.7) is equivalent to

$$\sup_{\varphi \in \mathcal{M}(\mathcal{A})} |\varphi(x^*x)| = \|x^*x\|, \quad \forall\, x \in \mathcal{A}. \tag{5.8}$$

This in turn would follow from

$$\sup_{\varphi \in \mathcal{M}(\mathcal{A})} |\varphi(a)| = \|a\|, \quad \text{when} \ \ a = a^* \in \mathcal{A}. \tag{5.9}$$

Now (2.13) implies

$$\sup_{\varphi \in \mathcal{M}(\mathcal{A})} |\varphi(x)| = r(x), \quad \forall\, x \in \mathcal{A}. \tag{5.10}$$

and specializing to $x = a = a^*$, this shows that (5.9) holds provided

$$\|a\| = r(a), \quad \text{when} \ \ a = a^* \in \mathcal{A}. \tag{5.11}$$

Now (4.24) gives the result (5.11), so Proposition 5.2 is proven.

Recall the Gelfand transform

$$\gamma : \mathcal{A} \longrightarrow C(\mathcal{M}(\mathcal{A})), \quad \gamma(x)(\varphi) = \varphi(x). \tag{5.12}$$

From (5.5) we have

$$\gamma(x^*) = \overline{\gamma(x)}, \quad \forall\, x \in \mathcal{A}, \tag{5.13}$$

and from (5.7) we have

$$\|x\| = \sup_{\varphi \in \mathcal{M}(\mathcal{A})} |\gamma(x)(\varphi)|, \quad \forall\, x \in \mathcal{A}. \tag{5.14}$$

This leads to the following.

Proposition 5.3. *If $\mathcal{A}$ is a commutative C^* algebra with unit,*

$$\gamma : \mathcal{A} \longrightarrow C(\mathcal{M}(\mathcal{A})) \tag{5.15}$$

is an isometric $*$-isomorphism of C^* algebras.

Proof. From (5.13)–(5.14), we see that γ is a $*$-homomorphism of C^* algebras, and it is an isometry. Hence it is an isomorphism of $\mathcal{A}$ onto its image under γ in $C(\mathcal{M}(\mathcal{A}))$. The image (call it $\widetilde{\mathcal{A}}$) is an algebra of functions on the compact space $\mathcal{M}(\mathcal{A})$, containing 1 (since $\varphi(I) = 1$), invariant under conjugation (by (5.13)), and closed in $C(\mathcal{M}(\mathcal{A}))$. Furthermore, the image separates points in $\mathcal{M}(\mathcal{A})$. That is,

$$\varphi \neq \psi \in \mathcal{M}(\mathcal{A}) \Longrightarrow \varphi(x) \neq \psi(x), \quad \text{for some} \ \ x \in \mathcal{A}. \tag{5.16}$$

(This is a tautology.) It follows by the Stone-Weierstrass theorem that $\widetilde{\mathcal{A}} = C(\mathcal{M}(\mathcal{A}))$, and this concludes the proof of Proposition 5.3.

6. Applications to the spectral theorem

Given a Hilbert space H, let $\{A_j : j \in J\} \subset \mathcal{L}(H)$ be a family of commuting self-adjoint operators:

$$A_j^* = A_j \quad A_j A_k = A_k A_j, \quad \forall\, j, k \in J. \tag{6.1}$$

Let $\mathcal{A} \subset \mathcal{L}(H)$ be the Banach algebra with unit generated by $\{A_j\}$; $\mathcal{A}$ is the norm closure of the space of polynomials in the operators A_j (with complex coefficients). Clearly $\mathcal{A}$ is commutative. The self-adjointness implies that if $T \in \mathcal{A}$, then $T^* \in \mathcal{A}$, so $\mathcal{A}$ is a commutative C^* algebra. From §5 we have the isometric isomorphism of C^* algebras

$$\gamma : \mathcal{A} \longrightarrow C(X), \quad X = \mathcal{M}(\mathcal{A}). \tag{6.2}$$

We will use this to establish a "spectral representation" of $\mathcal{A}$, by an algebra of multiplication operators on some L^2 space.

To proceed, we pick $v \in H$ and define

$$W : C(X) \longrightarrow H \tag{6.3}$$

($W = W_{\mathcal{A},v}$) as follows:

$$W(f) = \tau(f)v, \quad \tau = \gamma^{-1} : C(X) \longrightarrow \mathcal{A}. \tag{6.4}$$

We also define a linear functional

$$\mu : C(X) \to \mathbb{C}, \quad \mu(f) = (W(f), v) = (\tau(f)v, v). \tag{6.5}$$

The following positivity result gets us on our way.

Proposition 6.1. *If $f \in C(X)$ and $f \geq 0$, then $\mu(f) \geq 0$.*

Proof. The claim is that if $f \geq 0$, then $\tau(f)$ is a positive semi-definite operator on H. To see this, take

$$g = f^{1/2} \in C(X), \quad g \geq 0. \tag{6.6}$$

Since γ and τ are $*$-isomorphisms, $A = \tau(g)$ is self-adjoint on H, and $\tau(f) = \tau(g^2) = A^2$, so $(\tau(f)v, v) = (A^2 v, v) = \|Av\|^2 \geq 0$. This completes the proof.

Hence $\mu = \mu_v$ defines a positive Radon measure on X:

$$(W(f), v) = (\tau(f)v, v) = \int_X f \, d\mu. \tag{6.7}$$

We are now set up to perform an inner product computation. Take $f, g \in C(X)$. Then

$$
\begin{aligned}
(W(f), W(g))_H &= (\tau(f)v, \tau(g)v) \\
&= (\tau(f\overline{g})v, v) \\
&= \int_X f\overline{g}\, d\mu \\
&= (f, g)_{L^2(X,\mu)}.
\end{aligned}
\tag{6.8}
$$

It follows that W in (6.3)–(6.4) has a unique extension to a linear isometry

$$
W : L^2(X, \mu) \longrightarrow H. \tag{6.9}
$$

The range of W is

$$
H_v = \text{ closure in } H \text{ of } \{Av : A \in \mathcal{A}\}. \tag{6.10}
$$

What is interesting about W is that it intertwines the action of an operator $A \in \mathcal{A}$ on H with the action of multiplication by $\gamma(A)$ on $L^2(X, \mu)$. In fact, given $A \in \mathcal{A}$ and $f \in C(X)$,

$$
\begin{aligned}
W(\gamma(A)f) &= \tau(\gamma(A)f)v \\
&= A\tau(f)v \\
&= AW(f).
\end{aligned}
\tag{6.11}
$$

This extends by continuity from $f \in C(X)$ to all $f \in L^2(X, \mu)$.

We call H_v the cyclic subspace of H generated by $\mathcal{A}$ and v. If $H_v = H$, we say v is a cyclic vector for $\mathcal{A}$. The following is a variant of the spectral theorem for the case where there is a cyclic vector.

Proposition 6.2. *If $\mathcal{A} \subset \mathcal{L}(H)$ is a commutative C^* algebra with unit and $v \in H$ is a cyclic vector for $\mathcal{A}$, then*

$$
W : L^2(X, \mu) \longrightarrow H \tag{6.12}
$$

is unitary, and

$$
W^{-1}AWf = \gamma(A)f, \quad \forall\, A \in \mathcal{A},\ f \in L^2(X, \mu). \tag{6.13}
$$

In general, we cannot say that $\mathcal{A}$ has a cyclic vector, but we have the following. For simplicity, we assume H is separable.

Proposition 6.3. *If H is separable and $\mathcal{A} \subset \mathcal{L}(H)$ is a commutative C^* algebra with unit, then there exist $v_j \in H$ such that H_{v_j} are mutually orthogonal subspaces of H, with span dense in H.*

Proof. Let $\{w_j : j \in \mathbb{N}\}$ be a dense subset of H, all $w_j \neq 0$. Take $v_1 = w_1$, and construct $H_1 = H_{v_1}$ as above. Note that

$$A : H_1 \longrightarrow H_1, \quad \forall\, A \in \mathcal{A}. \tag{6.14}$$

If $H_1 = H$, we are done. If not, we proceed as follows. We claim that, when $H_1 \subset H$ is a linear subspace,

$$A \in \mathcal{A},\ A : H_1 \to H_1 \Longrightarrow A^* : H_1^\perp \to H_1^\perp. \tag{6.15}$$

In fact, if $v \in H_1,\ w \in H_1^\perp$, then

$$(v, A^* w) = (Av, w) = 0, \quad (\text{given } Av \in H_1), \tag{6.16}$$

so (6.15) follows.

To continue, consider the first $j \geq 2$ such that $w_j \notin H_1$, and let v_2 denote the orthogonal projection of w_j onto $H_1^\perp$. Then set

$$H_2 = \ \text{closure in } H \text{ of } \ \{Av_2 : A \in \mathcal{A}\}, \tag{6.17}$$

which is a linear subspace of $H_1^\perp$, by (6.15). Clearly $H_1 \oplus H_2$ contains Span $\{w_k : 1 \leq k \leq j\}$. If $H_1 \oplus H_2 = H$, we are done. If not,

$$A : (H_1 \oplus H_2)^\perp \longrightarrow (H_1 \oplus H_2)^\perp, \quad \forall\, A \in \mathcal{A}. \tag{6.18}$$

Take the first $j_3 > j$ such that $w_{j_3} \notin H_1 \oplus H_2$, and let v_3 denote the orthogonal projection of w_{j_3} onto $(H_1 \oplus H_2)^\perp$. Then set

$$H_3 = \ \text{closure in } H \text{ of } \ \{Av_3 : A \in \mathcal{A}\}. \tag{6.19}$$

Continue. If for some K, $H_1 \oplus \cdots \oplus H_K = H$, we are done. If not, we get a countable sequence of mutually orthogonal spaces

$$H_k = \ \text{closure in } H \text{ of } \ \{Av_k : A \in \mathcal{A}\}, \tag{6.20}$$

whose span contains w_j for all $j \in \mathbb{N}$, so is dense in H. This proves Proposition 6.3.

We now extend Proposition 6.2 to the following version of the spectral theorem. The reader can compare this with Theorems 1.1–1.2 of [T2].

Proposition 6.4. *Let H be a separable Hilbert space. If $\mathcal{A} \subset \mathcal{L}(H)$ is a commutative C^* algebra with unit, there exists a sigma-compact space $\mathfrak{X}$, equipped with a locally finite measure μ, a unitary map $\Phi : H \to L^2(\mathfrak{X}, \mu)$, and an isometric $*$-homomorphism of algebras*

$$\Gamma : \mathcal{A} \longrightarrow L^\infty(\mathfrak{X}, \mu), \tag{6.21}$$

such that

$$\Phi A \Phi^{-1} f = \Gamma(A) f, \quad \forall\, A \in \mathcal{A},\ f \in L^2(\mathfrak{X}, \mu). \tag{6.22}$$

Proof. With v_j as in Proposition 6.3, write

$$H = \bigoplus_{j \ge 1} H_j, \quad H_j = H_{v_j}, \tag{6.23}$$

and

$$W_j = W_{\mathcal{A}, v_j} : C(X) \longrightarrow H_j, \quad W_j(f) = \tau(f) v_j, \tag{6.24}$$

extending to unitary maps

$$W_j : L^2(X, \mu_j) \longrightarrow H_j, \quad \mu_j(f) = (\tau(f) v_j, v_j), \tag{6.25}$$

satisfying

$$W_j^{-1} A W_j f = \gamma(A) f, \quad \forall\, A \in \mathcal{A},\ f \in L^2(X, \mu_j). \tag{6.26}$$

Thus we can define the measure space $(\mathfrak{X}, \mu)$ as the disjoint union

$$(\mathfrak{X}, \mu) = \bigcup_{j \ge 1} (X_j, \mu_j), \quad X_j = X, \tag{6.27}$$

so

$$L^2(\mathfrak{X}, \mu) = \bigoplus_{j \ge 1} L^2(X_j, \mu_j), \tag{6.28}$$

and the W_j in (3.15) fit together to give a unitary map

$$W : L^2(\mathfrak{X}, \mu) \longrightarrow H, \tag{6.29}$$

satisfying

$$W^{-1} A W f = \Gamma(A) f, \quad \forall\, A \in \mathcal{A},\ f \in L^2(\mathfrak{X}, \mu), \tag{6.30}$$

where $\Gamma : \mathcal{A} \to L^\infty(\mathfrak{X}, \mu)$ is given by

$$\Gamma(A)(x) = \gamma(A)(x), \quad x \in X_j. \tag{6.31}$$

Then $\Phi : H \to L^2(\mathfrak{X}, \mu)$ is $\Phi = W^{-1}$.

If H is not separable, one can produce a suitable replacement for Proposition 6.3 using Zorn's lemma. An uncountable number of copies of X might be involved. We omit details.

A. Holomorphic functional calculus

Let $\mathcal{B}$ be a Banach algebra, with unit I. Given $x \in \mathcal{B}$, we have defined the resolvent set $\rho(x)$ and spectrum $\sigma(x)$ in §1. We have seen that $\sigma(x) \subset \mathbb{C}$ is compact and that $R_\zeta = (\zeta - x)^{-1}$ is holomorphic in $\zeta \in \rho(x)$.

Let f be holomorphic on a neighborhood of $\sigma(x)$. In fact, let Ω be a smoothly bounded neighborhood of $\sigma(x)$ and assume f is holomorphic on a neighborhood of $\overline{\Omega}$. We set

$$f(x) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta)(\zeta - x)^{-1}\, d\zeta. \tag{A.1}$$

If $\mathcal{B} = \mathbb{C}$, this is just Cauchy's formula. Note that the element of $\mathcal{B}$ defined by (A.1) is independent of the choice of Ω satisfying the conditions stated above, by Cauchy's theorem, which is valid in the setting of Banach space valued holomorphic functions.

In case f is holomorphic on a neighborhood of $\{\zeta : |\zeta| \le \|x\|\}$, or even $\{\zeta : |\zeta| \le r(x)\}$, we can let Ω be a disk centered at the origin and replace R_ζ by its power series expansion:

$$R_\zeta = (\zeta - x)^{-1} = \frac{1}{\zeta} \sum_{k=0}^{\infty} \frac{1}{\zeta^k} x^k. \tag{A.2}$$

For example,

$$\begin{aligned} p_n(\zeta) &= \zeta^n \quad (n \in \mathbb{Z}^+) \\ \Rightarrow p_n(x) &= \frac{1}{2\pi i} \int_{|\zeta|=R} \zeta^n \zeta^{-1} \sum_{k=0}^{\infty} \zeta^{-k} x^k\, d\zeta \quad (\text{take } R > \|x\|) \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{|\zeta|=R} \zeta^{n-1-k}\, d\zeta\, x^k \\ &= x^n. \end{aligned} \tag{A.3}$$

To establish further properties of this functional calculus, it will be useful to have the following result, known as the *resolvent identity*:

Proposition A.1. *If $x \in \mathcal{B}$ and $z, \zeta \in \rho(x)$, then*

$$R_z - R_\zeta = (\zeta - z) R_z R_\zeta. \tag{A.4}$$

Proof. Note that R_ζ commutes with $\zeta - x$, hence with x, hence with $z - x$. Hence $R_z R_\zeta (z - x) = R_z (z - x) R_\zeta = R_\zeta$, and multiplying on the right by R_z gives

$$R_z R_\zeta = R_\zeta R_z. \tag{A.5}$$

Thus

$$\begin{aligned} R_z - R_\zeta &= (\zeta - x) R_\zeta R_z - (z - x) R_z R_\zeta \\ &= (\zeta - z) R_\zeta R_z, \end{aligned} \tag{A.6}$$

proving (A.4).

Now for our multiplicative property.

Proposition A.2. *If $x \in \mathcal{B}$ and f and g are holomorphic on a neighborhood of $\sigma(x)$, then*

$$f(x)g(x) = (fg)(x). \tag{A.7}$$

Proof. Suppose $\sigma(x) \subset \Omega \subset \overline{\Omega} \subset \Omega_1$ and f and g are holomorphic on a neighborhood of $\overline{\Omega}_1$. Write

$$g(x) = \frac{1}{2\pi i} \int_{\partial\Omega_1} g(z)(z - x)^{-1}\, dz, \tag{A.8}$$

and, using (A.1), write $f(x)g(x)$ as a double integral. The product $R_\zeta R_z$ of resolvents appears in the integral. Using the resolvent identity (A.4), we obtain

$$f(x)g(x) = \frac{1}{(2\pi i)^2} \int_{\partial\Omega_1} \int_{\partial\Omega} (\zeta - z)^{-1} f(\zeta) g(z) (R_z - R_\zeta)\, d\zeta\, dz. \tag{A.9}$$

The term involving R_z as a factor has $d\zeta$-integral equal to zero, by Cauchy's theorem. Doing the dz-integral for the other term, using Cauchy's identity

$$g(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega_1} (z - \zeta)^{-1} g(z)\, dz, \tag{A.10}$$

we obtain from (A.9)

$$f(x)g(x) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta) g(\zeta) R_\zeta\, d\zeta, \tag{A.11}$$

which gives (A.7).

Corollary A.3. *If $x \in \mathcal{B}$, f is holomorphic in a neighborhood of $\sigma(x)$, and $z \notin f(\sigma(x))$, then $z - f(x)$ is invertible.*

Proof. Set

$$g_z(\zeta) = \frac{1}{z - f(\zeta)}, \quad \text{holomorphic in } \zeta \text{ on a neighborhood of } \sigma(x). \tag{A.12}$$

Then (A.7) gives

$$g_z(x)(z - f(x)) = (z - f(x))g_z(x) = I. \tag{A.13}$$

Another way to phrase Corollary A.3 is that

$$\sigma(f(x)) \subset f(\sigma(x)). \tag{A.14}$$

This is completed by the following result, known as the spectral mapping theorem. (Compare Proposition 2.8, valid when $\mathcal{B}$ is commutative.)

Proposition A.4. *In the setting of Corollary A.3,*

$$\sigma(f(x)) = f(\sigma(x)). \tag{A.15}$$

Proof. Say f is holomorphic on a neighborhood Ω of $\sigma(x)$. Taking $\lambda \in \sigma(x)$, we have

$$f(x) - f(\lambda) = (x - \lambda)G_\lambda(x), \tag{A.16}$$

where

$$G_\lambda(\zeta) = \frac{f(\zeta) - f(\lambda)}{\zeta - \lambda}, \tag{A.17}$$

which is holomorphic in $\zeta \in \Omega$ (with a removable singularity at $\zeta = \lambda$). Clearly, if $\lambda \in \sigma(x)$, the right side of (A.16) is not invertible, so the left side is not invertible. This yields

$$\lambda \in \sigma(x) \Longrightarrow f(\lambda) \in \sigma(f(x)), \tag{A.18}$$

which together with (A.14) gives (A.15).

Remark. A special case of the argument (A.16)–(A.18) appears in (4.12)–(4.14); see also (B.11)–(B.12).

We next have a composition identity.

Proposition A.5. *Given $x \in \mathcal{B}$, f holomorphic on a neighborhood of $\sigma(x)$, and h holomorphic on a neighborhood of $f(\sigma(x))$ (so $h \circ f$ is holomorphic on a neighborhood of $\sigma(x)$), we have*

$$(h \circ f)(x) = h(f(x)). \tag{A.19}$$

Proof. There is no loss in assuming $\sigma(x) \subset \Omega$, f holomorphic on a neighborhood of $\overline{\Omega}$, and h holomorphic on a neighborhood of $f(\overline{\Omega})$.

First, for $\zeta \in \overline{\Omega}$, γ the boundary of some neighborhood of $f(\overline{\Omega})$, we have

$$\begin{aligned} (h \circ f)(\zeta) = h(f(\zeta)) &= \frac{1}{2\pi i} \int\limits_{\gamma} h(z)(z - f(\zeta))^{-1} \, dz \\ &= \frac{1}{2\pi i} \int\limits_{\gamma} h(z) g_z(\zeta) \, dz, \end{aligned} \tag{A.20}$$

where, as in (A.12),

$$g_z(\zeta) = \frac{1}{z - f(\zeta)}. \tag{A.21}$$

Hence

$$\begin{aligned} (h \circ f)(x) &= \frac{1}{2\pi i} \int\limits_{\partial\Omega} (h \circ f)(\zeta)(\zeta - x)^{-1} \, d\zeta \\ &= \frac{1}{(2\pi i)^2} \int\limits_{\partial\Omega} \int\limits_{\gamma} h(z) g_z(\zeta)(\zeta - x)^{-1} \, dz \, d\zeta. \end{aligned} \tag{A.22}$$

Reversing the order of integration (doing the $d\zeta$-integral first) gives

$$\begin{aligned} (h \circ f)(x) &= \frac{1}{2\pi i} \int\limits_{\gamma} h(z) g_z(x) \, dz \\ &= \frac{1}{2\pi i} \int\limits_{\gamma} h(z)(z - f(x))^{-1} \, dz \quad \text{(by (A.13))} \\ &= h(f(x)), \end{aligned} \tag{A.23}$$

as desired.

The following is an important family of holomorphic functions to apply to elements $x \in \mathcal{B}$, namely $e_t(\zeta) = e^{t\zeta}$. We have the power series

$$e^{tx} = \sum_{k=0}^{\infty} \frac{t^k}{k!} x^k, \quad t \in \mathbb{R}, \ x \in \mathcal{B}. \tag{A.24}$$

Applying (A.7) to $f(\zeta) = e^{s\zeta},\ g(\zeta) = e^{t\zeta}$ gives

$$e^{(s+t)x} = e^{sx}e^{tx}, \quad \forall\, s,t \in \mathbb{R},\ x \in \mathcal{B}, \tag{A.25}$$

from the standard identity $e^{(s+t)\zeta} = e^{s\zeta}e^{t\zeta}$, valid for $s,t \in \mathbb{R},\ \zeta \in \mathbb{C}$. This is used in (4.10). A direct proof of (A.25) can be given as follows. Applying d/dt to (A.24) gives

$$\frac{d}{dt}e^{tx} = xe^{tx} = e^{tx}x, \quad \forall t \in \mathbb{R},\ x \in \mathcal{B}. \tag{A.26}$$

Hence, via the product rule,

$$\frac{d}{dt}\Big(e^{(s+t)x}e^{-tx}\Big) = e^{(s+t)x}xe^{-tx} - e^{(s+t)x}xe^{-tx} = 0, \tag{A.27}$$

so $e^{(s+t)x}e^{-tx}$ is independent of $t \in \mathbb{R}$. Taking $t = 0$ gives

$$e^{(s+t)x}e^{-tx} = e^{sx}, \quad \forall\, s,t \in \mathbb{R},\ x \in \mathcal{B}. \tag{A.28}$$

Taking $s = 0$ in (A.28) gives

$$e^{tx}e^{-tx} = I, \quad \forall\, t \in \mathbb{R},\ x \in \mathcal{B}, \tag{A.29}$$

so we can multiply each side of (A.28) on the right by e^{tx} and get (A.25).

A variant of this last argument yields the following useful result (justifying the first identity in (4.12)).

Proposition A.6. *If x and $y \in \mathcal{B}$ commute (i.e., $xy = yx$), then*

$$e^{t(x+y)} = e^{tx}e^{ty}, \quad \forall\, t \in \mathbb{R}. \tag{A.30}$$

Proof. Using (A.28) and the product rule, we compute

$$\begin{aligned} &\frac{d}{dt}\Big(e^{t(x+y)}e^{-ty}e^{-tx}\Big) \\ &= e^{t(x+y)}(x+y)e^{-ty}e^{-tx} - e^{t(x+y)}ye^{-ty}e^{-tx} - e^{-t(x+y)}e^{-ty}xe^{-tx} \\ &= e^{t(x+y)}xe^{-ty}e^{-tx} - e^{t(x+y)}e^{-ty}xe^{-tx}. \end{aligned} \tag{A.31}$$

If we show that

$$xe^{-ty} = e^{-ty}x, \quad \forall\, t \in \mathbb{R}, \quad \text{provided} \quad xy = yx, \tag{A.32}$$

it will follow that (A.31) is zero, so $e^{t(x+y)}e^{-ty}e^{-tx}$ is independent of t, hence

$$e^{t(x+y)}e^{-ty}e^{-tx} = I, \quad \forall\, t \in \mathbb{R}, \tag{A.33}$$

from which (A.30) follows upon right multiplication, first by e^{tx} (using (A.29)), then by e^{ty}. As for (A.32), we have

$$e^{-ty}x = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}y^k x. \tag{A.34}$$

Provided $xy = yx$, we also have $y^k x = xy^k$, and (A.32) readily follows. Proposition A.6 is proven.

B. The spectral radius

Let $\mathcal{B}$ be a Banach algebra with unit I. Recall from §1 that if $x \in \mathcal{B}$ we define its spectral radius as

$$r(x) = \sup \{|\zeta| : \zeta \in \sigma(x)\}. \tag{B.1}$$

From (1.7), we have

$$(\zeta - x)^{-1} = \frac{1}{\zeta} \sum_{k=0}^{\infty} \frac{1}{\zeta^k} x^k, \tag{B.2}$$

for $|\zeta| > \|x\|$. Hence, as noted in §1

$$r(x) \le \|x\|. \tag{B.3}$$

Here we establish a precise identity for $r(x)$, which will be of particular use in §§4–5.

To begin, note that the series (B.2) converges as long as there exist $C \in (0, \infty)$ and $\xi \in (0, 1)$ such that

$$\frac{1}{|\zeta|^k} \|x^k\| \le C \xi^k, \quad \forall\, k \ge 1, \tag{B.4}$$

i.e., as long as

$$\frac{1}{|\zeta|} \|x^k\|^{1/k} \le C^{1/k} \xi, \quad \forall\, k \ge 1. \tag{B.5}$$

This in turn holds provided

$$\limsup_{k \to \infty} \|x^k\|^{1/k} < |\zeta|, \tag{B.6}$$

so we have

$$\forall\, x \in \mathcal{B}, \quad r(x) \le \limsup_{k \to \infty} \|x^k\|^{1/k}. \tag{B.7}$$

The main result of this appendix is the following more precise result.

Proposition B.1. *If $\mathcal{B}$ is a Banach algebra with unit and $x \in \mathcal{B}$,*

$$r(x) = \lim_{k\to\infty} \|x^k\|^{1/k}. \tag{B.8}$$

Proof. For one part, $(\zeta - x)^{-1}$ is given by a convergent Laurent series for $|\zeta| > r(x)$, and this series must be the one in (B.2). Hence the terms in this series must be uniformly bounded in norm, so, for each $R > r(x)$, there exists $C = C(R) < \infty$ such that

$$\frac{1}{R^k}\|x^k\| \le C, \quad \forall\, k \in \mathbb{N}, \tag{B.9}$$

hence $\|x^k\|^{1/k} \le C^{1/k}R$, so

$$\limsup_{k\to\infty} \|x^k\|^{1/k} \le r(k). \tag{B.10}$$

This together with (B.7) gives $r(x) = \limsup_{k\to\infty} \|x^k\|^{1/k}$. To get (B.8), we note the following. From

$$\zeta^k - x^k = (\zeta - x)(\zeta^{k-1} + \zeta^{k-2}x + \cdots + z^{k-1}), \tag{B.11}$$

we have the first implication in

$$\begin{aligned} \zeta \in \sigma(x) &\Longrightarrow \zeta^k \in \sigma(x^k) \\ &\Longrightarrow |\zeta^k| \le \|x^k\| \\ &\Longrightarrow |\zeta| \le \|x^k\|^{1/k}, \end{aligned} \tag{B.12}$$

the second implication by (B.3) applied to x^k. Hence

$$r(x) \le \inf_{k\ge 1} \|x^k\|^{1/k}, \tag{B.13}$$

which together with (B.10) gives (B.8).

C. Rich Banach algebras of continuous functions

Let M be a compact Hausdorff space, and let $\mathcal{B}$ be a Banach algebra of continuous functions on M, with pointwise sum and product, and norm $f \mapsto \|f\|$. For each $p \in M$, $f \mapsto f(p)$ is a character on $\mathcal{B}$, so (2.3) implies

$$\sup_{p \in M} |f(p)| \le \|f\|, \quad \forall\, f \in \mathcal{B}. \tag{C.1}$$

We say such $\mathcal{B}$ is a *rich Banach algebra* of continuous functions on M provided there exists an algebra $\mathcal{C}$ of continuous functions on M, closed under $f \mapsto \overline{f}$, such that

$$\mathcal{C} \subset \mathcal{B} \ \text{ is dense,} \tag{C.2}$$

and

$$f \in \mathcal{C} \ \text{ nowhere vanishing on } \ M \Longrightarrow f^{-1} \in \mathcal{C}. \tag{C.3}$$

We have the following considerable generalization of Proposition 3.2.

Proposition C.1. *Let $\mathcal{B}$ be a rich Banach algebra of continuous functions on a compact Hausdorff space M. Then, for each character $\varphi : \mathcal{B} \to \mathbb{C}$, there exists $p \in M$ such that*

$$\varphi(p) = f(p), \quad \forall\, f \in \mathcal{B}. \tag{C.4}$$

Proof. If we compose φ with (C.2), we get a linear map $\varphi : \mathcal{C} \to \mathbb{C}$ satisfying, for all $f, g \in \mathcal{C}$,

$$\varphi(fg) = \varphi(f)\varphi(g), \quad \varphi(1) = 1. \tag{C.5}$$

Hence Proposition C.1 is a consequence of the following.

Proposition C.2. *With $\mathcal{C}$ as above, if $\varphi : \mathcal{C} \to \mathbb{C}$ is a linear map satisfying (C.5), then there exists $p \in M$ such that*

$$\varphi(p) = f(p), \quad \forall\, f \in \mathcal{C}. \tag{C.6}$$

Proof. To begin, we claim there exists $p \in M$ such that

$$f \in \mathcal{C}, \ \varphi(f) = 0 \Longrightarrow f(p) = 0. \tag{C.7}$$

In fact, if no such p exists, we can cover M with a finite number of open sets $U_j,\ 1 \le j \le K$, and take $f_j \in \mathcal{C}$, nowhere vanishing on U_j, such that $\varphi(f_j) = 0$. Then

$$f = \sum_{j=1}^{K} |f_j|^2 \Longrightarrow f \in \mathcal{C},\ \varphi(f) = 0. \tag{C.8}$$

However, $f > 0$ on M, so $f^{-1} \in \mathcal{C}$, so the assertion $\varphi(f) = 0$ contradicts $\varphi(f^{-1}f) = \varphi(1) = 1$.

Now, take $p \in M$ such that (C.7) holds, and set

$$\psi(f) = f(p), \quad \psi : \mathcal{C} \to \mathbb{C}. \tag{C.9}$$

We see that both ψ and φ obey (C.5), and

$$f \in \mathcal{C},\ \varphi(f) = 0 \Longrightarrow \psi(f) = 0. \tag{C.10}$$

Given this, if $f \in \mathcal{C}$,

$$\varphi(f) = \alpha \Rightarrow \varphi(f - \alpha) = 0 \Rightarrow \psi(f - \alpha) = 0 \Rightarrow f(p) = \alpha, \tag{C.11}$$

and we have (C.6).

To apply Proposition C.1 to Proposition 3.2, i.e., to

$$M = S^1, \quad \mathcal{B} = \mathcal{A}(S^1), \tag{C.12}$$

we can take

$$\mathcal{C} = C^\infty(S^1). \tag{C.13}$$

Note that we are not assuming in Proposition C.1 that $\mathcal{C}$ is a Banach algebra.

More generally, Proposition C.1 applies to

$$M = \mathbb{T}^k, \quad \mathcal{B} = \mathcal{A}(\mathbb{T}^k), \quad \mathcal{C} = C^\infty(\mathbb{T}^k), \tag{C.14}$$

when $\mathbb{T}^k$ is the k-dimensional torus, and $\mathcal{A}(\mathbb{T}^k)$ consists of functions on $\mathbb{T}^k$ whose Fourier coefficients $\hat{f}(\ell)$ satisfy $\sum_{\ell \in \mathbb{Z}^k} |\hat{f}(\ell)| < \infty$. See Appendix E for a further generalization, replacing $\mathbb{T}^k$ by a more general compact group.

D. Variants of a theorem of Bochner and Phillips

Let $\mathcal{B}$ be a Banach algebra with unit. Set

$$\mathcal{A}(S^1,\mathcal{B}) = \Big\{ f \in C(S^1,\mathcal{B}) : \sum_{k\in\mathbb{Z}} \|\hat{f}(k)\|_{\mathcal{B}} < \infty \Big\}. \tag{D.1}$$

In [BP], the following was proven.

Proposition D.1. *Let $f \in \mathcal{A}(S^1,\mathcal{B})$ and assume $f(\zeta)$ is invertible in $\mathcal{B}$ for each $\zeta \in S^1$, so $g(\zeta) = f(\zeta)^{-1}$ gives $g \in C(S^1,\mathcal{B})$. Then in fact $g \in \mathcal{A}(S^1,\mathcal{B})$.*

The case $\mathcal{B} = \mathbb{C}$ is a classical result of Wiener (cf. Proposition 3.2). We aim to provide a proof of Proposition D.1 and explore some variants.

To proceed, let M be a compact Riemannian manifold. With $\mathcal{B}$ as above, let $\mathcal{A}$ be a Banach algebra of continuous functions on M, with values in $\mathcal{B}$. We assume that

$$C^\infty(M,\mathcal{B}) \hookrightarrow \mathcal{A} \hookrightarrow C(M,\mathcal{B}), \tag{D.2}$$

with continuous injections. Given

$$f \in \mathcal{A}, \quad f(p) \ \text{ invertible for all } \ p \in M, \tag{D.3}$$

so

$$g(p) = f(p)^{-1} \Longrightarrow g \in C(M,\mathcal{B}), \tag{D.4}$$

we seek criteria that imply

$$g \in \mathcal{A}. \tag{D.5}$$

We make the following hypothesis. There exists $N \in \mathbb{N}$ such that for $n \geq N$, $p \in M$, there are

$$\varphi_{np} \in C_0^\infty(B_{1/n}(p)), \quad 0 \leq \varphi_{np} \leq 1, \quad \varphi_{np} = 1 \ \text{ on } \ B_{1/2n}(p), \tag{D.6}$$

such that

$$\forall\, f \in \mathcal{A}, \ \ \|(f - f(p))\varphi_{np}\|_{\mathcal{A}} \to 0, \ \text{as } n \to \infty. \tag{D.7}$$

Note that, for g as in (D.4), $x \in M$,

$$\begin{aligned} g(x)\varphi_{2n,p}(x) &= \Big[f(p) + f(x)\varphi_{np}(x) - f(p)\varphi_{np}(x) \Big]^{-1} \varphi_{2n,p}(x) \\ &= \Big(I + g_{np}(x) \Big)^{-1} f(p)^{-1} \varphi_{2n,p}(x), \end{aligned} \tag{D.8}$$

where

$$g_{np}(x) = f(p)^{-1}(f(x) - f(p))\varphi_{np}(x). \tag{D.9}$$

Given (D.7), we see that there exists $n = n(p)$ (depending on f) such that

$$\|g_{np}\|_{\mathcal{A}} \le \frac{1}{2}. \tag{D.10}$$

Hence, for each $p \in M$, there exists $n = n(p)$ such that

$$g\varphi_{2n,p} \in \mathcal{A}. \tag{D.11}$$

Now $\{B_{1/4n(p)}(p) : p \in M\}$ covers M, so there is a finite subcover, i.e., points $p_j,\ 1 \le j \le K$ (depending on $f \in \mathcal{A}$) such that

$$\varphi_j(x) = \varphi_{2n(p_j),p_j} \Longrightarrow \Phi = \sum_{j=1}^{K} \varphi_j \ge 1 \ \text{ on } \ M. \tag{D.12}$$

We have

$$g\Phi \in \mathcal{A}, \tag{D.13}$$

and of course $1/\Phi \in C^\infty(M) \subset \mathcal{A}$. We record our result.

Proposition D.2. *Let $\mathcal{A}$ be a Banach algebra of $\mathcal{B}$-valued functions on M, satisfying (D.2). Assume that for each $p \in M$, $n \ge N$, there exist φ_{np} satisfying (D.6)–(D.7). Then*

$$f \in \mathcal{A},\ f^{-1} \in C(M, \mathcal{B}) \Longrightarrow f^{-1} \in \mathcal{A}. \tag{D.14}$$

Regarding cases where the hypothesis (D.7) applies, clearly it holds for $\mathcal{A} = C(M, \mathcal{B})$. On the other hand, it fails for $\mathcal{A} = \mathrm{Lip}^\alpha(M, \mathcal{B}),\ \alpha \in (0, 1]$. Since the conclusion (D.14) holds for such $\mathcal{A}$, Proposition D.2 is by no means definitive. It applies to a Banach algebra $\mathcal{A}$ of $\mathcal{B}$-valued functions with a topology barely stronger than that of $C(M, \mathcal{B})$. We next show that the hypothesis (D.7) holds for $\mathcal{A} = \mathcal{A}(S^1, \mathcal{B})$, given by (D.1).

Regard S^1 as $[-\pi, \pi]$, with endpoints identified. It suffices to take $p = 0$. We take $\varphi \in C_0^\infty(-1, 1),\ \varphi = 1$ on $[-1/2, 1/2],\ 0 \le \varphi \le 1$, and set $\varphi_n(x) = \varphi(nx)$. We claim that, for all $f \in \mathcal{A} = \mathcal{A}(S^1, \mathcal{B})$,

$$\|(f - f(0))\varphi_n\|_{\mathcal{A}} \longrightarrow 0, \quad \text{as} \ \ n \to \infty. \tag{D.15}$$

It suffices to establish this when $f \in \mathcal{A}$ satisfies

$$\operatorname{supp} f \subset [-1, 1]. \tag{D.16}$$

We can then consider f as a function on $\mathbb{R}$. As such, its Fourier transform $\hat{f}(\xi)$ exists and is holomorpic in $\xi \in \mathbb{C}$.

Lemma D.3. *There exists $C \in (1, \infty)$ such that for all $f \in \mathcal{A} = \mathcal{A}(S^1, \mathcal{B})$ satisfying (D.16),*

$$\frac{1}{C} \int_{-\infty}^{\infty} \|\hat{f}(\xi)\|_{\mathcal{B}}\, d\xi \le \sum_{k=-\infty}^{\infty} \|\hat{f}(k)\|_{\mathcal{B}} \le C \int_{-\infty}^{\infty} \|\hat{f}(\xi)\|_{\mathcal{B}}\, d\xi. \tag{D.17}$$

Proof. With φ as above, set $\psi(x) = \varphi(x/2)$ and, for $\eta \in \mathbb{C}$, set

$$\psi_\eta(x) = e^{-i\eta x}\psi(x), \quad \text{so} \quad \widehat{\psi_\eta f}(\xi) = \hat{f}(\xi + \eta), \tag{D.18}$$

for $\xi \in \mathbb{C}$. We have

$$\widehat{\psi_\eta f}(k) = \sum_{\ell} \hat{\psi}_\eta(\ell) \hat{f}(k - \ell), \tag{D.19}$$

and $\psi_\eta \in \mathcal{A}(S^1, \mathcal{B})$, locally bounded in η, hence $\psi_\eta f$ belongs to $\mathcal{A}(S^1, \mathcal{B})$ and is locally bounded in η. Now, for $k \in \mathbb{Z}$,

$$\begin{aligned} \int_k^{k+1} \|\hat{f}(\xi)\|_{\mathcal{B}}\, d\xi &= \int_0^1 \|\hat{f}(k + \eta)\|_{\mathcal{B}}\, d\eta \\ &= \int_0^1 \|\widehat{\psi_\eta f}(k)\|_{\mathcal{B}}\, d\eta, \end{aligned} \tag{D.20}$$

and (by (D.19))

$$\sum_k \|\widehat{\psi_\eta f}(k)\|_{\mathcal{B}} \le C \sum_k \|\hat{f}(k)\|_{\mathcal{B}}, \quad |\eta| \le 1, \tag{D.21}$$

so

$$\int_{-\infty}^{\infty} \|\hat{f}(\xi)\|_{\mathcal{B}}\, d\xi \le C \sum_k \|\hat{f}(k)\|_{\mathcal{B}}. \tag{D.22}$$

This gives the first inequality in (D.17).

For the converse, we have

$$\hat{f}(k) = \widehat{\psi_{-\eta} f}(k + \eta), \tag{D.23}$$

hence there exists $C < \infty$ such that

$$\sum_k \|\hat{f}(k)\|_{\mathcal{B}} \le C \sum_k \|\hat{f}(k + \eta)\|_{\mathcal{B}}, \quad \forall\, \eta \in [0, 1]. \tag{D.24}$$

Integrating over $\eta \in [0, 1]$ gives the second inequality in (D.17).

Having Lemma D.3, we proceed as follows. To prove (D.15), it suffices to show that

$$\int_{-\infty}^{\infty} \|\hat{f} * \hat{\varphi}_n(\xi) - f(0)\hat{\varphi}_n(\xi)\|_{\mathcal{B}}\, d\xi \longrightarrow 0, \quad \text{as} \quad n \to \infty, \tag{D.28}$$

when $f \in C(\mathbb{R}, \mathcal{B})$ satisfies (D.16) and

$$\int_{-\infty}^{\infty} \|\hat{f}(\xi)\|_{\mathcal{B}}\, d\xi < \infty. \tag{D.26}$$

Note that $\hat{\varphi}_n(\xi) = n^{-1}\hat{\varphi}(\xi/n)$. Since

$$\begin{aligned} &\hat{f} * \varphi_n(\xi) - f(0)\hat{\varphi}_n(\xi) \\ &= \int_{-\infty}^{\infty} \Big(\hat{\varphi}_n(\xi - \eta)\hat{f}(\eta) - \hat{\varphi}_n(\xi)\hat{f}(\eta)\Big)\, d\eta, \end{aligned} \tag{D.27}$$

we see that the integral in (D.25) is

$$\begin{aligned} &\leq \iint |\hat{\varphi}_n(\xi - \eta) - \hat{\varphi}_n(\xi)| \cdot \|\hat{f}(\eta)\|_{\mathcal{B}}\, d\eta\, d\xi \\ &= \iint \Big|\hat{\varphi}\Big(\zeta - \frac{1}{n}\eta\Big) - \hat{\varphi}(\zeta)\Big| \cdot \|\hat{f}(\eta)\|_{\mathcal{B}}\, d\zeta\, d\eta, \\ &\longrightarrow 0, \quad \text{as} \quad n \to \infty, \end{aligned} \tag{D.28}$$

the limit holding by the Lebesgue dominated convergence theorem, as long as (D.26) holds. This completes the proof of (D.15), and hence proves Proposition D.1.

E. The spaces $\mathcal{A}(G)$

Let G be a compact group, $\widehat{G}$ the set of (equivalence classes of) irreducible unitary representations of G. Given $\pi \in \widehat{G}$, we say π represents G on V_π, of dimension d_π. Given $f \in L^1(G)$, we set

$$\pi(f) = \int_G f(x)\pi(x)\,dx, \tag{E.1}$$

dx denoting Haar measure on G. The Plancherel formula (following from the Weyl orthogonality relations and the Peter-Weyl theorem) is

$$\|f\|_{L^2}^2 = \sum_\pi d_\pi \|\pi(f)\|_{\mathrm{HS}}^2. \tag{E.2}$$

Here and in sums below, π runs over $\widehat{G}$. By polarization,

$$(f,g)_{L^2} = \sum_\pi d_\pi \operatorname{Tr}(\pi(f)\pi(g)^*). \tag{E.3}$$

For sufficiently "regular" functions u on G, there is the "Fourier inversion formula,"

$$f(x) = \sum_\pi d_\pi \operatorname{Tr}(\pi(u)\pi(x)^*). \tag{E.4}$$

For a condition guaranteeing absolute and unifirm convergence in (E.4), note that

$$\sum_\pi d_\pi |\operatorname{Tr}(\pi(u)\pi(x)^*)| \le \sum_\pi d_\pi \|\pi(u)\|_{\mathrm{Tr}}. \tag{E.5}$$

We say

$$u \in \mathcal{A}(G) \Longleftrightarrow \sum_\pi d_\pi \|\pi(u)\|_{\mathrm{Tr}} < \infty, \tag{E.6}$$

and set

$$\|u\|_{\mathcal{A}(G)} = \sum_\pi d_\pi \|\pi(u)\|_{\mathrm{Tr}}. \tag{E.7}$$

Clearly $\mathcal{A}(G)$ is a Banach space, and $\mathcal{A}(G) \subset C(G)$, densely. In case $G = S^1$ or more generally $G = \mathbb{T}^k$, we get the spaces treated in §3 and Appendix C.

The following is proven in [E].

Proposition E.1. *If $u, v \in \mathcal{A}(G)$, then $uv \in \mathcal{A}(G)$.*

The proof of Proposition E.1 seems to be harder for general compact G than for $G = \mathbb{T}^k$. (Furthermore, [E] treats locally compact G.) It follows from Proposition E.1 that (if G is a compact group) $\mathcal{A}(G)$ is a Banach algebra. To achieve $\beta = 1$ in $\|uv\|_{\mathcal{A}} \le \beta \|u\|_{\mathcal{A}} \|v\|_{\mathcal{A}}$, it might be necessary to replace (E.7) by an equivalent norm.

If $\mathcal{F}(G)$ denotes the set of finite sums of the form (E.4), we easily see that

$$\mathcal{F}(G) \subset \mathcal{A}(G) \ \text{ is dense.} \tag{E.8}$$

Also (decompose tensor products) $\mathcal{F}(G)$ is an algebra. However, generally $u \in \mathcal{F}(G)$, $u^{-1} \in C(G)$ does not imply $u^{-1} \in \mathcal{F}(G)$. If G is a compact Lie group, we have

$$\mathcal{F}(G) \subset C^\infty(G) \subset \mathcal{A}(G). \tag{E.8}$$

Then we can apply Proposition C.1, with $\mathcal{B} = \mathcal{A}(G)$, $\mathcal{C} = C^\infty(G)$, obtaining the following.

Proposition E.2. *Let G be a compact Lie group. For each character $\varphi : \mathcal{A}(G) \to \mathbb{C}$, there exists $p \in G$ such that*

$$\varphi(u) = u(p), \quad \forall\, u \in \mathcal{A}(G). \tag{E.10}$$

Consequently,

$$u \in \mathcal{A}(G),\ u^{-1} \in C(G) \Longrightarrow u^{-1} \in \mathcal{A}(G). \tag{E.11}$$

In [E], such a result is established for an arbitrary compact group, not necessarily a Lie group.

We record some further properties of $\mathcal{A}(G)$. First, note that

$$A \in \mathrm{End}(V_\pi) \Longrightarrow \|A\|_{\mathrm{Tr}} = \inf\{\|B\|_{\mathrm{HS}} \|C\|_{\mathrm{HS}} : A = BC\}. \tag{E.12}$$

From this it is readily deduced, via (E.2), plus

$$\pi(f * g) = \pi(f)\pi(g), \tag{E.13}$$

that

$$\mathcal{A}(G) = L^2(G) * L^2(G), \tag{E.14}$$

and

$$\|u\|_{\mathcal{A}(G)} = \inf\{\|f\|_{L^2(G)} \|g\|_{L^2(G)} : u = f * g\}. \tag{E.15}$$

An important class of functions on G is the class of positive definite functions $\mathcal{P}(G)$. Given $u \in L^1(G)$, we say

$$u \in \mathcal{P}(G) \Longleftrightarrow \pi(u) \ge 0, \quad \forall\, \pi \in \widehat{G}. \tag{E.16}$$

We have

$$\mathcal{P}(G) \cap C(G) \subset \mathcal{A}(G), \tag{E.17}$$

and, for $u \in \mathcal{P}(G) \cap C(G)$,

$$u(e) = \sum_\pi d_\pi \,\mathrm{Tr}\, \pi(u) = \|u\|_{\mathcal{A}(G)}. \tag{E.18}$$

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