

Wormhole Spaces, Connes' "Points, Speaking to Each Other", and the Translocal Structure of Quantum Theory

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Abstract

We amalgamate three seemingly quite different fields of concepts and phenomena and argue that they actually represent closely related aspects of a more primordial space-time structure called by us wormhole spaces. Connes' framework of non-commutative topological spaces and "points, speaking to each other", a translocal web of (cor)relations, being hidden in the depth-structure of our macroscopic space-time and made visible by the application of a new geometric renormalisation process, and the apparent but difficult to understand translocal features of quantum theory. We argue that the conception of our space-time continuum as being basically an aggregate of structureless points is almost surely too poor and has to be extended and that the conceptual structure of quantum theory, in particular its translocal features like e.g. entanglement and complex superposition, are exactly a mesoscopic consequence of this microscopic wormhole structure. We emphasize the close connections with the "small world phenomenon" and rigorously show that the micro state of our space-time, viewed as a dynamical system, has to be critical in a scale free way as recently observed in other fields of network science. We then briefly indicate the mechanisms by which this non-local structure manages to appear in a seemingly local disguise on the surface level, thus invoking a certain Machian spirit.

1 Introduction

We begin our introduction by quoting the following two lucid remarks by von Weizsäcker [1], similar ideas were also entertained by Wheeler, see e.g. [2] and some other people. The quotations are meant to strike the key of our paper.

...space-time is not the background but a surface aspect of reality...It is extremely improbable that this reality (i.e. quantum reality) will be describable as consisting of events which are localized in space and time.

The translocal phase relations are “surplus information” not lack of information. Quantum theory knows more, not less, than local classical physics.

The bulk of the following analysis is concerned with an amalgamation of some important (mathematical) ideas of Connes about a *non-commutative* generalisation of the *fibration* or *quotienting-out* principle in e.g. topology and, on the other hand, of a line of ideas and concepts we developed in recent years in our approach to quantum gravity. Our analysis results in the observation of the emergence of a *translocal* component being hidden in the fine structure of our space-time manifold. We will argue that this translocal substructure has important consequences for a better understanding of the many seemingly non-local features of quantum theory.

We think for example that the famous results of Bell (see [3]) are rather an indication that the substructure of quantum theory is necessarily non-local and not! that the attempts to go beyond the standard interpretation should be abandoned (as has been also remarked by Bohm for several times, see [4],[5]). We will advocate the possibility that quantum theory may emerge as an *effective* theory from a basically translocal microscopic theory of space-time, its main ingredient being a network of *microscopic wormholes*. For that reason we call this class of spaces, we are going to define and investigate, *wormhole spaces*. That is, we favor an approach to quantum gravity which considers quantum theory as a coarse grained consequence of the translocal fine structure of microscopic space-time, being the consequence of the peculiar structure of *pregeometry* underlying our ordinary space-time manifold.

Presently there do exist a variety of routes towards a theory of quantum gravity, ranging from frameworks, which depart from a more or less continuous space-time picture, imposing quantum theory more or less unaltered on the underlying classical structure as an independent, quasi God-given scheme, to working philosophies which try to view both quantum theory and gravity as emerging (low-energy) *effective theories* of a more primordial and basically discrete substructure (with ‘discrete’ not necessarily meaning ‘countable’ but rather the absence of a priori continuum concepts). A short and very incomplete list of papers advocating such more or less discrete approaches is for example [6] to [19].

Our own point of view has been presented in a series of papers in recent years, to which we refer the reader as to more references and a more detailed analysis of the different ideas and concepts being promoted by the various groups sharing this latter working philosophy ([20],[21],[22],[23]). It goes without saying that it takes a much longer route to arrive at testable consequences if one follows this latter more *bottom up oriented* avenue and before the results can be compared with results derived by perhaps more immediate *top down* methods (which sometimes are openly inspired by certain continuum theories such as classical general relativity). In the following we want to represent such a testable consequence, that is, predicting the existence of a microscopic wormhole structure and discussing some of the possible observable effects. We mention in particular a certain Machian flavor of this finding.

Furthermore, while some concepts and/or technical tools are shared to a greater or lesser extent by the various groups working in this field, this does not necessarily mean that also the respective frameworks are more or less identical. To give an example, discrete structures like e.g. (topological) graphs occur of course also elsewhere in (quantum) gravity research but usually as structures embedded in a preexisting smooth manifold, sharing sometimes even some of the metric properties of the ambient space or derive from certain triangulations, that is, being typically very regular. In contrast to such scenarios our *dynamic graphs* are strongly fluctuating irregular dynamical systems (reflecting in a sense the presumed huge vacuum fluctuations of quantum space-time). Smooth continuum structures are only supposed to emerge in a sufficiently coarse grained limit which, on the other hand, is only expected to exist provided the underlying microscopic network is in a very peculiar geometrically critical state (being closely related to the recently found *scale-free small world networks*). Therefore it is perhaps helpful to stress some points which we consider to be characteristic for our own approach.

We represent the primordial substratum as a densely entangled network of elementary relations, interactions or information channels, which is dynamically evolving according to some imposed dynamical law. We share the wide spread working philosophy that complex and collective behavior can or should emerge from some simple looking basic laws and surmise that our space-time or quantum vacuum is no exception. In this sense our approach is *bottom up*, that is, part of our business is it to reconstruct the concepts of modern continuum physics as, so to speak, collective quantities from this primordial substratum (as to similar ideas cf. for example the last pages in [24] or the philosophy expounded in [19] or [26]). What is however special in our framework is that *both* the states living *on* the underlying network and the network architecture itself are dynamically evolving, the two components interacting with each other. Hence the interaction of something like “*geometry*” and “*matter*”, which is playing such a dominant conceptual role in general relativity, is automatically incorporated in nuce in our approach. Insofar it transcends the *cellular automaton* approach advocated by ’t

Hooft.

Technically we implement this property by introducing *dynamic graphs* in which edges can not only reorient themselves under the dynamics but can also be created and deleted (for the sake of brevity we refer the interested reader to [27] [28] or the above cited earlier papers as to the technical definitions and some appropriate evolution laws). We note that in [27] we showed in particular that, while having a richer structure, our networks are also *causal sets*. We avoid however, for the time being, the discussion of the notorious *problem of time* in quantum gravity (see for example [29]). That is, purely for (technical) convenience we let our network evolve in discrete time steps, which, however, does not imply that physical time is assumed to have the nature of an overall discrete clock time. Our philosophy is rather that ultimately also physical time will turn out to be a collective variable emerging on the more coarse grained levels of our hierarchy of scales of resolution.

We recently came upon the following illuminating remark in [30] which beautifully characterizes the continuum version of the kind of spaces we are going to develop.

...But if a wormhole can fluctuate out of existence when its entrances are far apart ...then, by the principle of microscopic reversibility, the fluctuation *into* existence of a wormhole having widely separated entrances ought to occur equally readily. This means that every region of space must, through the quantum principle, be potentially “close” to every other region, something that is certainly not obvious from the operator field equations which, like their classical counterparts, are strictly local... It is difficult to imagine any way in which widely separated regions of space can be “potentially close” to each other unless space-time itself is embedded in a convoluted way in a higher-dimensional manifold. Additionally, a dynamical agency in that higher-dimensional manifold must exist which can transmit a sense of that closeness.

While this is not relevant for the following investigation, we hold the view that, due to our imposed dynamical laws, on a large, cosmological time scale, our network is in a process of unfolding from an essentially maximally connected initial state, in which almost all the degrees of freedom were directly dynamically linked with each other (in mathematical terms, a simplex or complete graph), towards a state, which represents our present universe, viz. having a large classical *diameter* (i.e. lots of possible elementary links being switched off by the imposed microscopic dynamics), and behaving, at least macroscopically, in a (quasi-)classical and apparently local way. The concept of *locality* tells us that sufficiently separated regions are non- or at most weakly interacting on this (quasi-) classical level. In a sense, such a classical background is considered to be the necessary prerequisite for an effective theory like quantum theory to emerge from a more fundamental theory. On the other hand we will argue in the following, that there

exists, in addition to the local structure, an almost hidden additional web of *translocal* (cor)relations between widely separated regions of classical space-time, which represent, so to speak, the remnants of this earlier more primordial and much stronger correlated phase (being prevalent in the big bang era).

These ideas will be expounded and corroborated in sections 5 to 7, the pivotal section being section 6 in which the structure of the *critical (scale free) network states* is rigorously analyzed. We conclude the paper with a brief discussion of the possible consequences of this peculiar two-level structure of space-time for the translocal features of quantum theory.

In the following the two papers [27] and [28] are of particular technical relevance. In [27] we developed in quite some detail both the conceptual and the numerical machinery for extracting this mentioned two-level structure from our underlying network. The framework which makes this possible we called *geometric renormalisation* or *geometric coarse-graining*. Its aim is the construction of a non-trivial geometric fixed point which corresponds to our continuous classical space-time and the distillation of the necessary preconditions (a *critical, scale-free* non-local geometric network state).

These observations immediately lead over to the second paper, [28]. In it we connect our findings with seemingly closely related observations made in a, at first glance, quite different context, the so-called *small world phenomenon* in biological, sociological and other related networks. It turns out that in both fields we frequently seem to have roughly two kinds of ties or links, local ones in closely knitted *friendship neighborhoods* and non-local ones among only loosely connected *acquaintances*, each belonging to a local friendship neighborhood of its own, but which, typically, do not overlap with each other. In addition to that, our network displays an even more peculiar further property which we call a *wormhole structure*.

We start our investigation by elaborating in sections 2 and 3 on some of the ideas of Connes, which are later amalgamated with the other line of our analysis in section 7, the motto being “*points, speaking to each other*”. We note in particular, that in our approach *physical points* (also called *lumps* by us, cf. [22]) have a rich internal structure. It is therefore interesting that the limits of the classical point-concept are also clearly felt in pure mathematics, see for example the beautiful essay by Cartier, [31], who remarks:

... The central problem is that of the points of space...
 ... the only things that matter are their mutual relationships...
 ... To a given order the infinitesimals of the immediately higher order appear to be points without structure, until we open the box that they constitute and that reveals infinitesimals of a higher order playing provisionally the role of points.

While our model systems, when appropriately *coarse-grained*, can presumably lead to something like classical gravity in a low-energy limit, we say however

almost nothing about this important point in the present paper. We only remark that it is obvious that concepts like curvature, dimension and the like are contained in our approach (as to dimensional concepts see e.g. [32], [27] or [28]).

To give an example, a concept closely related to the notion of curvature, i.e. the relation between the number of points lying in a surface, having a certain fixed distance from a given point and the distance itself, can easily be formulated in our network approach. On the one hand remember the famous thought experiment of Einstein of a rotating reference frame (leading to the conclusion that space-time is non-euclidean, cf. [33],[34] or [35] sect.8.3). On the other hand, we studied such a relation in [32] or [28] to introduce the notion of (fractal) graph dimensions. In network or graph theory it is called the *distance degree sequence relative to a vertex*. Concepts like the above can also be discussed in the more abstract and wider setting of *metric spaces* (see the interesting ideas of Gromov in [36] or [37]). Our graphs and networks are natural examples of such (discrete) metric spaces. As our graphs and their coarse grained descendants (lump spaces) are even *geodesic metric spaces*, the same will hold for the continuous limit manifold if it exists. It is then an extremely interesting question under what conditions this limit manifold carries a Riemannian metric (a problem already envisaged by Riemann him self! [38]).

2 Physico-Mathematical Aspects of Point-Set Topology

One of the ideas of Connes is to give the *interior of points*, which, on their side, frequently result from some sort of contraction or identification of subensembles of finer constituents, a non-trivial noncommutative structure.

A large part of modern physics still relies on the ideas of mathematical continuum geometry and point-set topology. Furthermore, in most fields of physics, with the exception of general relativity, space typically occurs as some fixed background structure, not participating in the dynamics of the constituents of matter. But even in general relativity space-time is contrived as a preexisting topological manifold of structureless points which rather play the role of labels of events. In some sense this is a slightly dubious point of view as no one has ever seen these individual points and without coordinate systems and events it would be hard to tell the individual space-time points from each other anyhow.

This manifold is a dynamic agent in general relativity but not so much as a dynamical system (as in our approach) with direct interaction between the constituents, viz, the points. This interaction is rather mediated by matter-fields and/or the metric tensor or connection fields. These are considered to be quantities being attached to the points, but the points themselves appear to be unaffected by the dynamics, nor are these fields usually regarded as actually

encoding the internal structure of points or their infinitesimal neighborhoods. In this sense the points of the manifold are *ideal elements* in a twofold way (as to a discussion of the notion of ideal concepts see e.g. [23] and further references there). They neither do act nor are acted upon, they serve only as carriers of fields. Relating this field approach to our point of view; as our (physical) points are little densely knitted subunits, they are capable of carrying internal states. From a more macroscopic point of view one may regard these states as being located in the *infinitesimal* neighborhood of the points of the space-time continuum. It is then suggestive to view fields *at* points as encoding in a coarse-grained way the fine structure of the unresolved microscopic lumps. The same applies to theories of the Kaluza-Klein type.

While on the physical side, this clean picture is a little bit blurred by the advent of quantum theory, with classical concepts of localized objects and points now becoming slightly obscure, nevertheless, the whole framework is still, *cum grano salis*, moulded in this universal conceptual form of continuous spaces and local fields living on them.

A concept like interaction between points played also no notable role in classical mathematics (apart, perhaps, from graphs, to which we come below). However, there exists the widespread concept of *identification* or *quotienting out*, that is, with π a surjective map

$$\pi : X \rightarrow Y \tag{1}$$

X and Y two spaces, we can decide to identify the set of points, lying in the preimage of $y \in Y$ with y and, by the same token, the partitioning of X by π with Y . We write

$$X/\sim = Y \tag{2}$$

the equivalence relation being induced by π . Correspondingly we can introduce the *quotient space* of X by \sim if we are given an equivalence relation on X . Physically we can equally well view the map as a *sorting* of the points of X by the points of Y or as imposing a value property on X .

Ordinarily, the individual points in the respective equivalence classes are then identified, that is, in general the emerging structure becomes poorer or coarser. As stressed by Connes (see the following two sections) the structure can in fact become so poor and coarse in many relevant cases as to become virtually void and uninteresting (while, on the other hand, the underlying fine structure may be extremely complicated and far from trivial). It was an important observation of Connes that in such extreme situations the fine structure of such *leaf-* and *identification spaces* can be more appropriately encoded in a *noncommutative* structure, living *over* such spaces (or rather, certain extensions being associated with such X/\sim).

This is the mathematical aspect. As to physics, Connes in [39] made the subtle remark as to such identified points, $\{a, b\}$ of some initial space X : "...to

allow them to ‘speak’ to each other”. In physical terms, one may interpret this as *interaction* among the points of a space or manifold.

Orbits, leaves and other subset structure, occurring in the construction of quotient spaces, are examples of equivalence relations. It turns out that, in our approach, this is a too narrow framework. An equivalence relation is a subset, $R \subset X \times X$, X a certain set, with the following properties called, reflexivity, symmetry and transitivity, respectively, i.e.

$$\forall x \in X : (x, x) \in R, (x, y) \in R \rightarrow (y, x) \in R, (x, y), (y, z) \in R \rightarrow (x, z) \in R \quad (3)$$

Other types of relations are, for example, *adjacency*: R is symmetric and irreflexive ($(x, x) \notin R$), *partial order*: R is reflexive, antisymmetric ($(x, y) \in R \rightarrow (y, x) \notin R$) and transitive and so forth.

In our network approach the type of relations which naturally occur are even more special. They are generalisations of adjacencies. Our dynamical systems are assumed to live on (simple) graphs. If these graphs are unoriented they define an adjacency, with edges between the vertices denoting the (symmetric) relation. If we take, in addition the dynamical laws into account, our graphs become oriented graphs, with the orientation being (clock)time dependent. That is, at each instant of time the edges or the relations happen to be oriented anew so that either (x, y) or (y, x) occurs in $R(t)$. Put differently, the type of relations which are also relevant in the following are irreflexive and antisymmetric. At each instant of time and for each vertex x we have a subset $[x]_{in}$ of X influencing x and a subset $[x]_{out}$ being influenced by x , the first set given by edges having x as target, the latter set being given by edges having x as source. It is important that, typically, these relations are no longer transitive.

For convenience we always assume that in the relation R every $x \in X$ occurs as a possible first entry in (x, y) , put differently, each x is related to at least one other element of X . In the directed case this means that each x has at least one *outgoing edge*.

If the base set X is countable, we can represent these relations by (in general) non-symmetric *adjacency matrices*. Labelling the rows and columns by the members of X , the corresponding entries in the row belonging to x are either 1 if the respective column label belongs to $[x]_{out}$ or 0 else. Correspondingly, the column labelled by x has entries with value 1 at the places belonging to $[x]_{in}$. In this way the wiring structure of the oriented graph can be neatly encoded in a matrix. A more detailed analysis of properties of such matrices and (directed) graphs was made in [49].

3 A Road to Noncommutative Spaces

As perhaps not everybody is familiar with the details of the mathematical constructions we briefly review this topic in the following in our own words.

3.1 Mathematical Prerequisites and Motivation

An important conceptual tool in modern mathematics to construct new spaces from given ones, is the quotient operation, that is, dividing a bigger point set by an *equivalence relation*. Starting from a set, X , and a particular subset, $R \subset X \times X$, having the above properties of an equivalence relation we form a new space denoted by X/R , X/\sim or simply \tilde{X} , with points being the equivalence classes, \tilde{x} , defined by R , i.e.

$$y \in \tilde{x} \text{ if } (x, y) \in R \text{ and } \tilde{y} = \tilde{x} \text{ if } y \in \tilde{x} \quad (4)$$

If X carries a *topology* we can endow the new space with the canonical *quotient topology*, being the finest topology on \tilde{X} so that the quotient map

$$\pi : X \rightarrow \tilde{X} \quad , \quad x \rightarrow \tilde{x} \quad (5)$$

is continuous. In other words, a set, $\tilde{O} \subset \tilde{X}$ is open iff $\pi^{-1}(\tilde{O})$ is open in X .

Typical cases in point are identification or quotient spaces derived from the action of a group, G , on X , the equivalence classes being the orbits of the group action, i.e.

$$\tilde{x} := \{g \cdot x, g \in G\} \quad (6)$$

In this case each g is assumed to act as a permutation or bijection on X , that is, the space X is partitioned (or *foliated*; at the moment we do not intend to give the precise definition, see e.g. [40] or [41]) into *orbits* or *leaves*.

In most of classical mathematics, the quotient spaces being studied typically carry a non-singular (e.g. Hausdorff-) quotient topology. On the other hand, as strongly emphasized by Connes, there do exist lots of interesting (quotient) spaces with highly irregular or fragmented orbits, leaves or partitionings. A consequence may be that the ordinary induced topology is trivial, the only open or closed sets being the total space and the empty set, called the *indiscrete* or *coarse* topology. It follows that the associated function spaces are also trivial, consisting only of constant functions (in case the space is connected).

In other words, the ordinary *commutative* philosophy, encoding the topology of quotient spaces in the corresponding function algebra over the space (via the Gelfand-isomorphism), turns out to be completely insufficient as the space may, nevertheless, have an extremely rich internal structure, which is, to express it in physical terms, no longer resolved by the microscope, given by the associated function algebra.

Various paradigmatic examples are discussed in the book of Connes ([40]). A nice review is also [39]. The presumably most thoroughly studied example is the so-called *noncommutative torus* (NCT), ([40],[39] or, as to the purely mathematical aspects, [42] or [43]). A pedagogical review, more adressed to theoretical physicists, is for example [44].

The model itself has already been known in classical mechanics for a long time in connection with ergodic theory (Kronecker foliation, see [45] p.72ff). With coordinates on the two-torus, T^2 , given by

$$(2\pi \cdot x, 2\pi \cdot y) \quad , \quad 0 \leq x, y < 1 \quad (7)$$

or, equivalently

$$T^2 \cong \mathbb{R}^2 / \mathbb{Z}^2 \quad (8)$$

as topological quotient space, one studies the rotation map

$$\dot{x}(t) = \alpha_1 \quad , \quad \dot{y}(t) = \alpha_2 \quad (9)$$

If α_1/α_2 is rational, the induced leaf space, that is T^2/\sim , is a nice topological space in the sense discussed above, as the orbits

$$(x(t), y(t)) \quad , \quad x(0) = x_0, y(0) = y_0 \quad (10)$$

close on themselves after a finite number of cycles.

The situation changes drastically for α_1/α_2 irrational. In that case all the orbits of the flow are dense in T^2 . This is a consequence of the *Poincaré recurrence theorem* (see [45]). This results in a degeneration of the canonically defined quotient topology on T^2/\sim to the indiscrete topology (for a more detailed discussion of topological questions see [46]).

As a consequence the algebra of continuous functions on \tilde{X} degenerates to the constant functions (for \tilde{X} being connected). A parallel result holds for measure theory, based on Borel-measures. The reason is the following. For α_1/α_2 irrational, the corresponding flow is ergodic. Employing the canonical quotient measure, induced by Lebesgue measure on \mathbb{R}^2 , measurable functions on \tilde{X} are functions, being invariant on the leaves. That is, the pull back leads to functions, being invariant under the flow. However we have the important result that invariant measurable functions under an ergodic flow are constant on any set of full measure. Analogously, any invariant measurable set is either of zero measure or measure one (for normalized measure spaces), see [47].

In our particular case this can be visualized as follows. The individual leaves are of course measurable but have Lebesgue measure zero. If we want to have a set with non-vanishing measure, we may for example choose a full interval of initial conditions for the flow. As the flow is ergodic, the corresponding invariant set is however the full 2-torus.

Remark: We note in passing (without discussing this possibly interesting point in more detail at the moment), that one may introduce measures of the *fractal type* on \tilde{X} , leading to a larger class of measurable functions. The relevance for (continuum) physics is however not immediately obvious.

From the above discussion it follows that the functorial identification of (topological) spaces and abelian algebras becomes obsolete in these (not so infrequent)

situations. In the past such spaces have mostly been studied in a more algebraic manner. In the following we want to adopt a slightly more geometric (or topological) point of view and emphasize aspects which will, hopefully, exhibit the relation to our own approach to *quantum space-time physics*.

To do this, a closer inspection of the arguments and ideas, given by Connes in for example the first chapter of [40], called “Noncommutative spaces and Measure Theory” and in particular the subsection I.4: “Geometric Examples of von-Neumann algebras”, is helpful.

3.2 Noncommutative Quotient Spaces, the Construction

It is important for the following to understand in more detail some of the technical subtleties, underlying the construction of operator algebras on e.g. leaf spaces, given in [40]. We will see that, strictly speaking, the “noncommutative” construction is actually performed over a particular *fiber-bundle* with *base space* X and *not* really over the singular quotient space \tilde{X} , which rather plays an intermediary role by supplying the fibers over the points of X .

We simplify the discussion by assuming the leafs, or more generally, equivalence classes of points, to be countable sets. Cases in point are e.g. the action of a discrete group, G , on a manifold, V . We assume the underlying space, X , to be a measure space. In the more general (non-countable) case one may take the *Lebesgue measure class* and deal (in the absence of a canonical volume form) with the Hilbert space of *half-densities* or *half-forms* (cf. [40] or [48]).

With \tilde{X} having no longer an interesting structure as a measure space, we, following Connes, proceed in the following way. We take the initial space, X , a manifold say, and erect a *Hilbert bundle*, \tilde{H} , over X by attaching, in an intermediate step, to each point, $x \in X$, the corresponding equivalence class, \tilde{x} (orbit, leaf) thus forming the subsets (x, \tilde{x}) in $X \times \tilde{X}$. Note that this implies that now all the points, x_i, x_j , belonging to the same leaf, carry the *same* fiber, $\tilde{x}_i = \tilde{x}_j$.

With the fibers being countable, we then erect over each fiber, \tilde{x} , the l_2 -Hilbert space, $H(\tilde{x})$, of sequences

$$\{\tilde{f}(x_i)\}, \quad x_i \in \tilde{x}, \quad \sum_{\tilde{x}} |\tilde{f}(x_i)|^2 < \infty \quad (11)$$

with a basis consisting of the functions

$$\tilde{f}_j, \quad \tilde{f}_j(x_i) = \delta_{ij} \quad (12)$$

That is, instead of the singular space, \tilde{X} , we study the Hilbert bundle, \tilde{H} , with base space the nicer space, X , and fibers being the l_2 -spaces over the leaves, \tilde{x} , indexed by $x \in X$.

Remark: Note that all the Hilbert spaces are isomorphic (but not! canonically isomorphic) to a standard l_2 -space, which may be regarded as *standard fiber*.

It is now easy to construct measurable sections over X in the following way. Pick a \tilde{f}_x in each $H(x) = H(\tilde{x})$, the index, x , running in X , *not* in the fiber over x . By the same token, this defines a function, f , over the space $R \subset X \times X$, R given by the equivalence relation or foliation, $(x, y) \in R$ if $\tilde{x} = \tilde{y}$.

Definition 3.1 *The section, \tilde{f} , of Hilbert vectors, $\tilde{f}(x) = \tilde{f}_x$, is called measurable if the corresponding induced function, f , is measurable over $R \subset X \times X$ with f given by*

$$f(x, y) := \tilde{f}_x(y) \quad (13)$$

i.e. the Hilbert vector \tilde{f}_x evaluated at element y in \tilde{x} . In the same sense we define square integrable sections over X with values in $H(x) = H(\tilde{x})$. We denote this space by $L^2(\tilde{H})$.

It is important to note that this Hilbert bundle is a bundle over X and that even if x, y belong to the same fiber, i.e. $\tilde{x} = \tilde{y}$, the Hilbert vectors in $H(x), H(y)$ can be independently chosen, that is, $\tilde{f}_x \neq \tilde{f}_y$ in general. Consequently, this structure alone does *not* yet reflect the true leaf structure of \tilde{X} . On the other hand, the above amplification construction is technically necessary due to the, in general, degenerated structure of \tilde{X} . The leaf structure will be encoded in the operator algebra constructed below.

It was realized by Connes that we can both get an interesting mathematical structure and a characterisation of the underlying singular leaf space by now taking the natural operator algebras or matrix algebras of bounded operators on $H(x)$ *leafwise*, that is, we define an operator valued function, \tilde{A} , over X with the proviso

$$X \ni x \rightarrow \tilde{A}_x = A_{\tilde{x}} \quad \text{i.e.} \quad \tilde{A}_x = \tilde{A}_y \quad \text{if} \quad \tilde{x} = \tilde{y} \quad (14)$$

with $A_{\tilde{x}}$ a bounded operator in $H(\tilde{x})$.

Remark: Connes calls such operators *random operators*.

Definition 3.2 *We call such a section of operators, \tilde{A} , measurable, if for any pair of measurable Hilbert vector sections, \tilde{f}, \tilde{g} , $(\tilde{f}_x | \tilde{A}_x \cdot \tilde{g}_x)$ is measurable.*

Lemma 3.3 *The random operators with norm given by $\text{ess sup } \|(\tilde{A}_x)\|$ bounded, form a von-Neumann algebra over $L_2(\tilde{H})$ under pointwise multiplication.*

(see [40]). We now see that these random operators or the corresponding von-Neumann algebra characterizes the leaf structure in a particular *noncommutative* way.

4 The Network of Interacting Points

4.1 The Underlying Network QX

In section 2 we argued in favor of a framework which implements the interaction among the points of a manifold and that this is, on the other hand, inherent in some of the ideas of Connes.

The model system we start from is a dynamic discrete graph or network assumed to emulate crucial aspects of (quantum) space-time on the Planck scale. With the help of a coarse-graining or *geometric renormalisation* process ([27]) we undertake to construct a macroscopic *fixed point* representing our continuous space-time on the macroscopic or mesoscopic level. But in contrast to the ordinary continuum employed in classical mathematics or physics, which is assumed to behave in a purely local way, in our approach, the continuum limit develops quasi automatically (brought to light by the renormalisation construction) an intricate and largely hidden extra (non-local) structure among its points.

For notational convenience we only introduce some notation in the following and refer the interested reader to [27] or [28] for more technical details.

Definition 4.1 *A simple, countable, labelled, (un)directed graph, G , consists of a (countable) set of nodes or vertices, V , and a set of edges or bonds, E , each edge connecting two of the nodes. For convenience there exist no multiple edges (i.e. edges, connecting the same pair of nodes) or elementary loops (a bond, starting and ending at the same node). In this situation the bonds can be described by giving the corresponding set of (un)ordered pairs of nodes. The members of V are denoted by x_i , the bonds by e_{ij} , connecting the nodes x_i and x_j .*

Remark: The assumption of a countable vertex set is only made for technical convenience. We could also admit a non-countable vertex set. From a physical point of view one may argue that the *continuum* or uncountable sets are idealisations, anyhow.

We note that graphs carry a natural structure which can be employed to emulate the interaction between the nodes or points. This becomes more apparent if we impose dynamical network laws on these graphs such that they become discrete dynamical systems. Henceforth we denote such a dynamical network, which is supposed to underly our continuous space-time manifold, by QX (“*quantum space*”). We omit a more detailed discussion of the dynamical evolution of states on graphs or networks, which can be found in our above mentioned papers as we want to concentrate primarily on the emergent non-local aspects of our model systems. We only want to emphasize the following point.

It is important that in our approach the bond states are also dynamical degrees of freedom which, a fortiori, can be switched off or on. Therefore the *wiring*, that is, the pure *geometry* (of relations) of the network changes constantly and is hence

also an emergent, dynamical property, *not* given in advance in form of some static background geometry. Furthermore, in the network laws we have studied so far, the individual edges carry states which can take the values $\pm 1, 0$ and which are naturally associated, via the dynamical law, with the two possible orientations of the edge or its silent, inactive state respectively (in the graph framework the edge is simply considered to be temporarily absent if the edge state is zero). These local states are updated after every clock-time step (depending on the state of the nodes in the local environment).

Remark: One sees from this that our *cellular networks are generalisations* of the more common but also geometrically more rigid *cellular automata*.

Consequently, the nodes and bonds are typically *not* arranged in a more or less regular array, a lattice say, with a fixed near-/far-order. It is remarkable that similar ideas have also been entertained in the theory of *cellular automata*, where systems have been studied which selforganize, in a dynamical process, their lattice structure (see the beautiful book of Ilachynski, [50]).

4.2 Dynamical Networks as Random Graphs

As we are dealing with very large graphs, which are, a fortiori, constantly changing their shape, that is, their distribution of active bonds, we make the assumption that the dynamics is sufficiently stochastic so that a point of view may be appropriate, which reminds of the working philosophy of *statistical mechanics*.

It was recently argued ([28] and further literature cited there) that the random graph framework may be too narrow to fully reproduce the observed *near-, far-order* of so-called *scale-free small world networks* which seem to be the crucial prerequisites for the emergence of a non-trivial critical continuum fixed point of our coarse graining process. But nevertheless the random graph picture is still the natural starting point and the basis of a perhaps more advanced theory.

Visualizing the characteristics and patterns being prevalent in large and “typical” graphs was already a notorious problem in *combinatorial graph theory* and led to the invention of the *random graph* framework ([27], [28]). The guiding idea is to deal with graphs of a certain type in a probabilistic sense, that is, forming a probability space with elementary events certain graphs. This turns out to be particularly fruitful as many graph characteristics (or their absence) tend to occur with almost certainty in a probabilistic sense (as has been first observed by Erdős and Rényi). In the following we are dealing with random graphs living over a fixed node set, having the independent *edge probability* $0 < p \leq 1$. The probability that a particular graph, G_m , with m edges occurs is thus

$$pr(G_m) = p^m (1 - p)^{N-m} \quad (15)$$

with $n, N = n(n - 1)/2$ the number of nodes, the maximal possible number of edges respectively. The standard source is [51].

In the above papers (in particular in [27]) we mainly concentrated on properties of so-called *cliques*, their statistical distribution (with respect to their order, r , i.e. number of vertices), degree of mutual overlap etc. We then studied these properties in the consecutive stages and phases of our renormalisation process, being associated to the various levels of magnification or resolution of our space-time manifold.

Definition 4.2 (Subsimplices and Cliques) *With G a given fixed graph and V_i a subset of its vertex set V , the corresponding induced subgraph over V_i is called a subsimplex or complete subgraph, if all its internal pairs of nodes are connected by a bond. In this partially ordered set there exist certain maximal subsimplices, that is, every addition of another node (together with the respective existing bonds to other nodes of the subset) destroys this property. These maximal simplices are called cliques in combinatorics and are the candidates for our physical proto-points. Henceforth we denote them by S_i .*

For the underlying reason why we concentrated on this particular graph characteristic we provided some motivation in the above cited papers. To put it briefly, we will associate these cliques with the nested structure of lumps (or physical points) making up our ordinary space-time. That is, the cliques are assumed to look like ordinary points under low magnification but show their internal (infinitesimal) nested structure under sufficiently high resolution (cf. the remarks by Cartier cited in the introduction). In [28] we related these ideas to astonishingly similar ideas in a, at first glance, quite unrelated field where the local lumps are called *friendship neighborhoods* or *clumps* and the non-local ties *aquaintances*.

We can introduce various *random function* on the above probability space. For each subset $V_i \subset V$ of order r we define the following random variable:

$$X_i(G) := \begin{cases} 1 & \text{if } G_i \text{ is an } r\text{-simplex,} \\ 0 & \text{else} \end{cases} \quad (16)$$

where G_i is the corresponding induced subgraph over V_i in $G \in \mathcal{G}$ (the probability space). Another random variable is then the *number of r -simplices occurring in a given G* , denoted by $Y_r(G)$ and we have:

$$Y_r = \sum_{i=1}^{\binom{n}{r}} X_i \quad (17)$$

with $\binom{n}{r}$ the number of r -subsets $V_i \subset V$. With respect to the probability measure introduced above we have for the *expectation values*:

$$\langle Y_r \rangle = \sum_i \langle X_i \rangle \quad (18)$$

and

$$\langle X_i \rangle = \sum_{G \in \mathcal{G}} X_i(G) \cdot pr(G_i = r\text{-simplex in } G). \quad (19)$$

For $\langle Z_r \rangle$, the *expected number of r -cliques* (i.e. maximal! r -simplices) in the random graph, we have then the following relation

$$\langle Z_r \rangle = \binom{n}{r} \cdot (1 - p^r)^{n-r} \cdot p^{\binom{r}{2}} \quad (20)$$

This quantity, as a function of r (the *order* of the subsimplices) has quite a peculiar numerical behavior. We are interested in the typical *order of cliques* occurring in a generic random graph (where typical is understood in a probabilistic sense).

Definition 4.3 (Clique Number) *The maximal order of occurring cliques contained in G is called its clique number, $cl(G)$. It is another random variable on the probability space $\mathcal{G}(n, p)$.*

It is remarkable that this value is very sharply defined in a typical random graph. Using the above formula for $\langle Z_r \rangle$, we can give an approximative value, r_0 , for its expectation value and get

$$r_0 \approx 2 \log(n) / \log(p^{-1}) + O(\log \log(n)) \quad (21)$$

(cf. chapt. XI.1 of [51]). It holds that practically all the occurring cliques fall in the interval $(r_0/2, r_0)$ (for a quantitative and numerical discussion see [27]). We believe that the random graph picture will reproduce at least the qualitative behavior of such extremely complex dynamical systems, being well aware of the possible limitations and necessary generalisation of this picture ([28]).

5 The Geometric Coarse-Graining or Renormalisation Process

We now are going to set up the connection between the two fields discussed in the preceding sections. That is, on the one hand, the concept of nasty (quotient) spaces having very erratic and in some cases dense orbits, leaves etc., and, on the other hand, our hierarchy of cellular networks or dynamic graphs, which emerges from a certain kind of geometric renormalisation group. As the construction of this nested structure of lumps within lumps was discussed in quite some detail in [27] and the emergence of the small world effect in [28] we can be relatively brief.

The idea to construct a macroscopic (quasi) continuum from an underlying more erratic and discrete primordial substratum via some sort of coarse graining is, as we think, not unnatural (see in particular [22] and our discussion of earlier

work by Menger et al about *random metric spaces*). These considerations led to our concept of a geometric renormalisation group. We start with a graph, G , and, in a first step, pick up its cliques promoting them to the (meta) nodes of a coarser graph, the so-called *clique graph*, G_{cl} . If these cliques are not too small, it should make a physical difference whether two selected cliques have an appreciable overlap of common nodes, if this overlap is only marginal (very few common nodes) or even empty.

In [27] realistic numerical examples were studied in which the typical clique size was roughly of order 10^3 . The random graph framework allowed to calculate the probability distribution of expected clique overlaps and related graph characteristics. The quantitative calculations are however relatively tricky and involved. In the case of a strong overlap the interaction between the respective cliques is more intensive while in the latter cases it is weak and/or indirect, that is, the internal state of the other clique is only feebly felt if the interaction is weak. It is the merit of the renormalisation group that it clearly separates these two different kinds of interaction after several coarse graining or renormalisation steps.

Our above described procedure suggests the physical assumption that *classical* macroscopic behavior is hoped to emerge if we neglect the fine details on small scales (e.g. fluctuations). That is, we only will draw a link between two cliques or lumps, S_i, S_j , if the common overlap is non-marginal compared to the typical order of the cliques on the respective renormalisation level. The graph, thus constructed, we call the *purified clique graph* relative to G . To put it more succinctly:

- Starting from a given fixed graph, G , pick the (generic) *cliques*, S_i , in G , i.e. the subgraphs, forming maximal subsimplices or cliques in G with their order lying in the above mentioned interval, $(r_0/2, r_0)$.
- These cliques form the new nodes of the *clique-graph*, G_{cl} of G . New bonds in the clique graph are drawn between cliques provided they have a sufficient overlap.

Remark 5.1 *The random graph framework shows the highly non-trivial fact that practically all occurring cliques have a number of nodes lying in the above interval, that is, have a typical size.*

We repeat this process of going from a graph to its purified clique graph sufficiently many times until we arrive at a (quasi-)continuous manifold, emerging as a fixed point of our renormalisation process (and being reflected by the emergence of a quasi-static regime in which the graph structure do no longer change appreciably in the consecutive coarse-graining steps). Such a macroscopic fixed point can however only expected to emerge provided the original network has

been in a very peculiar, i.e., (quasi-)critical state as has been described in section 8 of [27].

On each level of coarse-graining, that is, after each renormalisation step, labelled by $l \in \mathbb{Z}$, we get, as in the block spin approach to critical phenomena, a new level set of cliques or lumps, S_i^l , (i labelling the cliques on renormalisation level l), consisting on their sides of $(l - 1)$ -cliques which are the l -nodes of level l , starting from the level $l = 0$ with $G =: G_0$. That is, we have

$$S_j^l = \bigcup_{i \in j} S_i^{(l-1)}, \quad S_i^{(l-1)} = \bigcup_{k \in i} S_k^{(l-2)} \text{ etc.} \quad (22)$$

($i \in j$ denoting the $(l - 1)$ -cliques, belonging, as meta nodes, to the l -clique, S_j). These cliques form the meta nodes in the next step.

Definition 5.2 *The cliques, S_i^0 , of $G =: G_0$ are called zero-cliques. They become the one-nodes, x_i^1 , of level one, i.e. of G_1 . The one-cliques, S_i^1 , are the cliques in G_1 . They become the 2-nodes, x_i^2 , of G_2 etc. Correspondingly, we label the other structural elements, for example, 1-edges, 2-edges or the distance functions, $d_l(x_i^l, x_j^l)$. These higher-level nodes and edges are also called meta-nodes, -edges, respectively.*

If we collapse these new cliques to meta-nodes we do no longer see their internal structure. If, on the other hand, we decide to keep track of their internal organisation (cf. also the remarks of Cartier in the introduction) we have the following (Russian doll like) picture (where for the sake of graphical clarity the mutual overlaps of the occurring cliques of the same level are not represented).

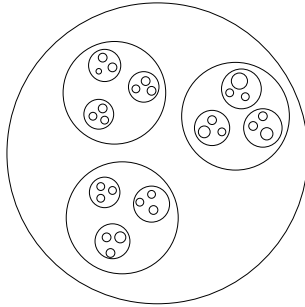


Figure 1: Nested Structure

Each intermediate graph or array of lumps, G_l , carries a certain geometric and *metrical* structure of its own. We can define a *metric*, d_l on G_l (there exist in fact several possibilities) as follows. We can either use the canonical graph metric (distance of nodes measured by the minimal number of edges connecting them) or use a more refined metric which incorporates the varying possible degrees of overlap of cliques (cf. [22]):

$$d_l(S_i^l, S_j^l) := d_{sim}(S_i^l, S_j^l) := \inf_{\gamma} \sum p(S_{k_l}^l, S_{k_{l+1}}^l) \quad (23)$$

where

$$p(A, B) := 1 - \text{sim}(A, B) \quad (24)$$

and

$$\text{sim}(A, B) := [A \wedge B]/[A \vee B] \quad (25)$$

(\wedge, \vee denoting intersection and union of sets).

The above definition is understood as the infimum over the class of paths, γ , connecting the two meta nodes in the respective graph of l -cliques.

Remark: Strictly speaking, the definition in its above form applies only to cliques of finite order. If necessary, corresponding definitions can be made employing measure theoretic concepts (cf. sect. 7 of [22]). One could of course also choose the canonical graph distance which is, however, discrete.

Concerning the importance of a true coarse-graining including an appropriate purification, the following rigorous result is instructive. The picture frequently invoked (*space-time foam*) is the following. On a very primordial scale we have a very erratic space-time structure having not even a stable integer dimension (rather being of a *fractal* type). Smoothing and/or coarse-graining may ultimately lead to a smooth continuous space-time manifold as we know it. In [32] we introduced the concept of (internal) dimension of a graph or network. We later learned that it is also called the *distance degree sequence* in graph theory (cf. [28]). In the above paper we motivated why it should rightly be regarded as a kind of dimension and why it is an important graph characteristic.

To put it briefly, dimension mostly enters physical models via the asymptotic scaling of the number of degrees of freedom which can be reached after a certain number of steps starting from a fixed reference point. This is implemented in the following definition.

Definition 5.3 (Internal Scaling Dimension) *Let x be an arbitrary node of G . Let $\#(U_l(x))$ denote the number of nodes in $U_l(x)$, the neighborhood of nodes with graph distance $\leq l$ from x . We consider the sequence of real numbers $D_l(x) := \frac{\ln(\#(U_l(x)))}{\ln(l)}$. We say $\underline{D}_S(x) := \liminf_{l \rightarrow \infty} D_l(x)$ is the lower and $\overline{D}_S(x) := \limsup_{l \rightarrow \infty} D_l(x)$ the upper internal scaling dimension of G starting from x . If $\underline{D}_S(x) = \overline{D}_S(x) =: D_S(x)$ we say G has internal scaling dimension $D_S(x)$ starting from x . Finally, if $D_S(x) = D_S \forall x$, we simply say G has internal scaling dimension D_S .*

Remark: For a rigorous implementation of this concept of dimension we employ infinite graphs. For practical purposes it is of course sufficient to have very large graphs. Note that a similar attitude is frequently adopted in the theory blockspin renormalisation, where, if one works with large but finite systems, the system would shrink after every step by a certain factor. This is compensated by a rescaling of the system.

If in the preceding construction we decide not to suppress weak (marginal) overlaps, that is, to use the unpurified ordinary clique graph, \hat{G}_{cl} , we have the following remarkable result ([27]).

Theorem 5.4 *Assuming that G has dimension D and globally bounded node degree we have $D_{cl} = D$. That is, the dimension does not change when going from a graph to its non-purified ordinary clique graph.*

This observation reminds one of a similar behavior of the (uncoarse-grained) entropy functional in statistical mechanics which is a constant of motion as the corresponding measure is invariant under time evolution. Only the coarse-grained entropy happens to increase in a non-equilibrium state. Further even more surprising results can be found in the following section.

We now briefly indicate how one may construct a continuous space from extracting information from our scale of networks or lump spaces. We take the lumps of a certain renormalisation level and try to arrange them and their mutual overlaps in some real embedding space in essentially the same way as the corresponding cliques or lumps in the coarse-grained graph, G_{l_0} , say. We discussed such geometric constructions in much more detail in [22], employing, among other things, *fuzzy geometric methods*. As depicted in the preceding picture, we endow these geometric lumps with the same nested structure as the lumps or cliques of our coarse grained graphs, i.e. l_0 -cliques containing $(l_0 - 1)$ -cliques and so forth down to the initial nodes and bonds. In the different context of loop quantum gravity similar constructions are discussed in [9]. Furthermore we want to mention an interesting discussion of a hierarchy of limits of measurability in [52].

In making this association, it becomes obvious that there may arise spatial *obstructions* or *frustrations* in case the different links occurring in the graph G_{l_0} cannot be implemented geometrically by an appropriate *packing* of overlapping balls in a D dimensional continuum. A more rigid implementation is via *simplicial complexes*. This would be a more traditional method in which contact is mediated by having a common face. Put differently, it is not an automatic property that the packing of these lumps fits into some quasi smooth manifold-like structure. Certain crucial properties like a scale free geometric long range order have to be fulfilled, see the next section or section IV of [27] or [28] as to a more detailed analysis of critical network states.

A last remark concerns the relation of the metrics or distance functions in the two scenarios. As described above we have a couple of natural metrics on our (clique) graph at our disposal, either discrete ones or randomized or smooth ones. If one wants to relate such a grainy distance function to a truly continuous version in some final smooth space strict isometry of mappings between metric spaces is certainly not the most natural concept. A weaker notion is frequently (and in particular, in physics) more appropriate. Such a concept is the notion of *rough* or *quasi-isometry* (see, for example, [37] or [53]). This notion is defined as follows.

Definition 5.5 Let F be a map from a metric space, X , to a metric space, Y with metrics d_X, d_Y . It is called quasi-isometric if the following holds: There exist constants, $\lambda \geq 1, \varepsilon \geq 0$, such that

$$\lambda^{-1} \cdot d_X(x, y) - \varepsilon \leq d_Y(F(x), F(y)) \leq \lambda \cdot d_X(x, y) + \varepsilon \quad (26)$$

6 The Translocal Network

We now come to the central part of our geometric renormalisation group analysis. Given a large not too sparsely wired network or graph, G , (that is, the existing generic cliques are not too small), we construct its canonical (unpurified) clique graph, \hat{G}_{cl} , and then delete, according to our coarse graining or purification prescription, certain bonds in \hat{G}_{cl} as described above or in [27].

Each clique or lump, S_0 , lying in G_{cl} , has its own neighborhood structure, its *local group*, given by the cliques, S_i , being directly connected with S_0 in the clique graph G_{cl} , that is, having sufficient overlap with S_0 . We can estimate the cardinality of the typical local group of a given clique and compare it with the total number of cliques in G_{cl} or the number of cliques, not overlapping with S_0 . In [27], extensively using random graph theory, we got the following approximate result.

Conclusion 6.1

$$\langle N_{loc.gr.} \rangle \approx N_{cl} / (n^{l_0} \cdot \bar{r}^{\bar{r}}) \quad (27)$$

with n the number of nodes in the graph, G , N_{cl} the number of generic cliques in the corresponding clique graph, l_0 the assumed sufficient degree of overlap of the generic cliques, \bar{r} some appropriate value lying in the interval $[r_0/2, r_0]$, $n \gg \bar{r} \gg l_0$ being assumed (where the second \gg is not so pronounced as the first one; n is usually gigantic compared to the typical clique size \bar{r} !).

Both N_{cl} and n are typically quite large in our model examples. If r_0 is not too small, l_0 has to be chosen larger than 1. We conclude that in this regime most of the cliques have zero or only marginal overlap with a given clique, S_0 . That is, most of the edges, occurring in the unpurified clique graph, have to be deleted or, in other words, are only weak links (see below). Hence, already after a single coarse-graining step, most of the fine structure happens to be smoothed out. We therefore have the situation that, with $p^{-1} = O(1)$, or p not too small, between two arbitrary non-overlapping cliques, S_i, S_j , there will nevertheless usually exist links (of the preceding level), connecting individual nodes lying in S_i, S_j respectively.

Observation 6.2 Under the above assumptions there exist usually an appreciable number of nodes, x_i, x_j lying in S_i, S_j respectively, such that $d_G(x_i, x_j) = 1$ or, more generally, $d_G(x_i, x_j)$ small, while $d_{G_{cl}}(S_i, S_j)$ may be large in the purified clique graph G_{cl} .

This process of coarse graining is repeated up to the level, l_0 , which is assumed to be sufficiently near to the macroscopic continuum. On every step we observe this phenomenon of the existence of two types of links between lumps or cliques. We thus get in the end a complicated nested hierarchical structure of different types of links between the final infinitesimal neighborhoods on the macroscopic level and coming from all the levels below the final level, l_0 . Remember in particular, that, by construction, nodes on a level l with $1 < l < l_0$ represent full cliques on the preceding level ($l - 1$), and by the same token, edges on level l are given by non-marginal overlaps of cliques on the preceding level.

Observation 6.3 *What we have described above is the two-level (or, rather, multi-level) structure of macroscopic space-time we alluded to in the introduction, that is, a continuous, locally behaving macroscopic space-time manifold, M , and an immersed web, W , of translocal (weak) links, representing, so to speak, short cuts between more distant regions of M . Thinking of the picture of quotient spaces and equivalence relations, discussed in preceding sections, we dubbed this compound structure QX/ST (QX standing for ‘quantum space’ and ST for macroscopic space-time) in our earlier papers.*

In the following figures we try to make this complicated and layered structure among the nodes and cliques a little bit more transparent. We draw, for example, two non-overlapping cliques, S_1, S_2 , together with some members of their local groups. The euclidean distance in the picture is meant to indicate their large relative distance in G_{cl} . We assume that two nodes, x, y , lying in S_1, S_2 respectively, are connected by a link belonging to G .

In any general clique graph we have this difference between weak and strong links. If our graph is a fortiori a sufficiently typical random graph this structure is even a little bit more particular as we learned above that in a typical random graph practically all cliques are almost of the same size (their order lying in the interval $r_0/2, r_0$). That is, each weak link belongs actually to at least one other clique of practically the same order.

Remark 6.4 *This last property may be different in graphs, having a more pronounced near- and far-order as discussed in [28] and further literature given there.*

In our illustration the pointed lines plus the arrow mean that the points x, y , occurring twice, have to be identified and the corresponding lines to be contracted. That is, the third clique has actually a common node both with S_1 and S_2 . A connection via an intermediate clique is one possibility. The following picture describes a direct (weak) contact of S_1 and S_2 via a single common node x .

Up to now we arrived at our conclusions concerning the translocal structure of our coarse-grained macroscopic limit networks mainly with the help of the random graph framework plus certain consequences of the geometric renormalisation process. We will now amend these observations with some precise results which

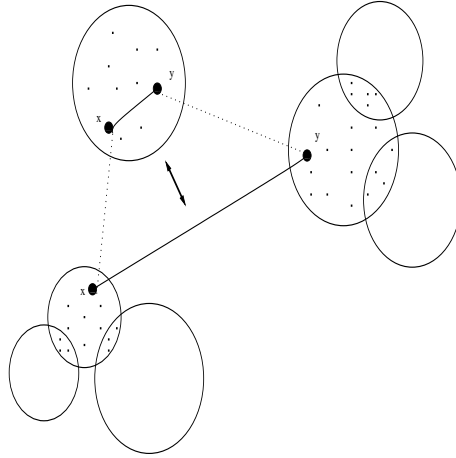


Figure 2: Translocal Links 1

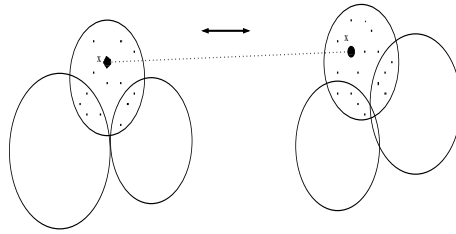


Figure 3: Translocal Links 2

almost rigorously show that these limit networks have to be of a *critical scale-free small world* type.

One of our main conceptual tools will be the behavior of the dimension of a network under coarse-graining. We showed in the preceding section that coarse-graining is absolutely necessary if we want to change or reduce the presumably (fractal) dimension of the initial network on the primordial Planck scale. As a corollary of the above theorem we have:

Corollary 6.5 *For the purified clique graph (in contrast to the unpurified one), we have*

$$D_{cl} \leq D \quad \text{instead of} \quad D_{cl} = D \quad (28)$$

One may be inclined to surmise that generically we will have $D_{cl} < D$, but this is not true! Quite to the contrary, it turns out to be very tricky to really reduce the dimension. *Local* alterations of the wiring diagram will not do. Analysing the situation in which a dimensional reduction can actually take place leads to the concept of *critical network states*. We approach the problem of dimensional reduction by smoothing in two steps. We learned that

$$D_G = D_{\hat{G}_{cl}} \leq D_{G_{cl}} \quad (29)$$

That is, it is sufficient to control the step from \hat{G}_{cl} to G_{cl} . This transition consists in the deletion of a certain fraction of (weak) links.

In [32], sect. 4.1 we proved an interesting theorem which shows that the local insertion of arbitrarily many additional edges does not change the dimension of a graph. More precisely:

Proposition 6.6 *Additional insertions of bonds between arbitrarily many nodes, y, z , having original graph distance, $d(y, z) \leq k$, $k \in \mathbb{N}$ arbitrary but fixed, do not change $\underline{D}(x)$ or $\bar{D}(x)$.*

Whereas edge deletion is not simply the dual operation of edge insertion, we can generalize the above result in a way appropriate for our problem. What we need is a generalisation of the notion *local*.

Both \hat{G}_{cl} and G_{cl} carry their own natural graph metrics, \hat{d} and d , given by the *geodesic* edge path distance.

Definition 6.7 *We assume that we pass over from a graph G to a new graph G' , living on the same node set, by means of a number of edge deletions. These edge deletions are called local of order $\leq k$ if only edges between nodes, x, y , are deleted which have a final distance in G' globally bounded by k .*

We then have the remarkable theorem:

Theorem 6.8 *If the edge deletions in going from \hat{G}_{cl} to G_{cl} are uniformly local of an arbitrary but finite order, the dimension does not change, i.e.*

$$D_{\hat{G}_{cl}} = D_{G_{cl}} \tag{30}$$

Proof: This can be proven by reversing the proof of the above cited proposition, i.e. we envisage the dual process of going from G_{cl} to \hat{G}_{cl} by edge insertions, which are now, by assumption, local of a certain finite order. From our proposition we now can infer that the dimension remains unchanged. \square

Corollary 6.9 *In order to change the dimension by edge deletions (in an infinite graph; see the remarks after definition 5.3) it is therefore necessary (but not sufficient; there are counter examples!) that there exist infinitely many edge deletions with their degree of locality not being boundable by any given number k . Put differently, for any given k there exist infinitely many edge deletions between nodes with final distance greater than k .*

Conclusion 6.10 *We view this kind of scale-free behavior of the occurrence of long-distance weak links as a rigorous formulation of the critical scale-free network states mentioned in the introduction.*

We want to complement these general results with an illustrative example, which shows all the features we mentioned above. We emphasize however that it is *not* meant as an example of a limit state of a true proto space-time of the kind we have discussed in the preceding sections. It is only a toy model! In a first step we embed the one-dimensional line of integers, \mathbb{Z}_1 , in the two-dimensional lattice, \mathbb{Z}_2 , in the way depicted in the following figure. We regard this ambient

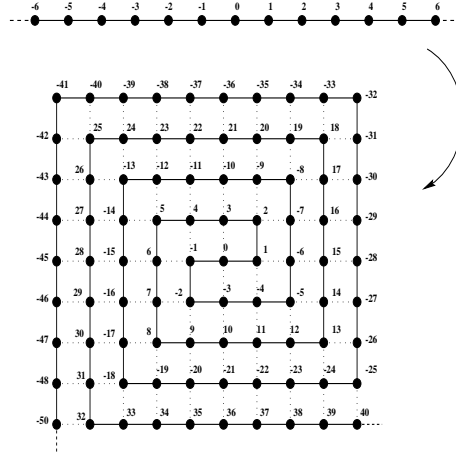


Figure 4:

lattice as the unpurified clique graph, \hat{G}_{cl} , of a certain coarse-graining level, with the nodes being cliques. Strong links, given by a sufficient overlap, are denoted by bold lines, weak bonds, which have to be deleted when going from \hat{G}_{cl} to the purified clique graph, G_{cl} , by dashed lines.

We now see that \hat{G}_{cl} has graph dimension two while G_{cl} has dimension one. We learned from our previous rigorous results that such a change can only occur if \hat{G}_{cl} is in a very peculiar critical state. Inspecting our toy model we see that exactly this is the case (what is actually peculiar is the embedding of \mathbb{Z}_1 in \mathbb{Z}_2). Node distance in G_{cl} is measured by the canonical metric of \mathbb{Z}_1 , while the metric in \hat{G}_{cl} is the one coming from \mathbb{Z}_2 . We see that the bond deletion process really violates the locality assumption for any given k . For any given k there exist an infinity of weak links (to be deleted), connecting nodes having a distance larger than k . Take for example the following sequence of weak links (ordered by increasing node distance)

$$(0, 3), (3, -10), (-10, 21), (21, -36) \dots \quad (31)$$

where the numbers denote the position of the nodes with respect to \mathbb{Z}_1 .

7 Wormhole Spaces or a Continuum Model of Points Speaking to Each Other

In the first sections of the present paper we discussed a general point of view concerning a wider conception of continuous spaces, being of possible relevance for (quantum) space-time physics. We then discussed the subject from a different angle, i.e., as dynamical, densely entangled networks of relations among microscopic constituents, being depicted by nodes, the relations or elementary interactions by edges. If we perform a sequence of specific coarse-graining steps on this network, which, under certain conditions, will finally converge to a smooth macroscopic space or space-time, a detailed analysis shows the following. With the help of the random graph concept we observe the quasi automatic emergence of a new and subliminal web of *translocal* interactions, being immersed in this classical manifold, M . In the following subsections we want to give a brief account of the sort of continuous model spaces we expect to emerge from this construction, that is, ordinary continuous spaces with an embedded web of translocal *short cuts*. We introduce these spaces in a quasi axiomatic way as it may sometimes be easier to start right away from these models, if one wants to work out the consequences for continuum physics (like, for example ordinary quantum theory) instead of going the long way beginning at the Planck scale. Confer also the illuminating remarks by DeWitt cited in the introduction.

7.1 A Class of Continuum Models

We conclude that these spaces which, presumably, emerge in quantum space-time physics, support two modes (or rather a whole hierarchy) of interactions and/or information exchange among their constituents. A *local* one, obeying the “*Nahwirkungsprinzip*” (*no action at a distance*), propagating from (physical) points (or lumps) to their *infinitesimal neighbors* and so on, and, on the other hand, a *translocal* almost quasi-instantaneous (but presumably feeble) interaction with arbitrarily distant regions of the manifold M of a more stochastic type.

This presumed more irregular and *stochastic* behavior is a result of the weak contact (via the weak links) between individual nodes in the various lumps making up the physical points being translocally related in contrast to the more robust interaction (strong local links) given by a more intense overlap of full cliques or lumps as is the case in the infinitesimal neighborhood or (in the clique or lump language) in the local group. One can study this different behavior when imposing a dynamical microscopic law as we discussed it in previous papers. One usually observes large individual fluctuations at individual nodes or along individual links of the primordial level which are then smoothed and averaged out over full cliques or clusters of cliques on the higher levels of the network. This is now the place where our line of argumentation returns to the point we departed

from, that is, the picture of “*points talking to each other*”, which we invoked in the first sections.

To begin with, the implementation of our findings by means of a continuum description is surely not unique! From a mathematical point of view a whole *class* of spaces, all sharing certain basic characteristics, can be invented. So we begin by introducing some models which do not yet share *all* the features we expect from our renormalisation group analysis.

We start with some continuous space, M , like e.g. \mathbb{R}^d , or a manifold, being locally homeomorphic to some \mathbb{R}^d . We assume that in M a countable but dense subset, X , or, alternatively, a partition of non-overlapping, countable and dense subsets, X_ι , is specified:

- X or X_ι are countable and dense in M (note that they do not contain interior points with respect to the topology of M).
- X_ι do not overlap and $\cup X_\iota = M$. In another model situation we may assume that the X_ι are not dense with $\cup X_\iota \neq M$ but dense in M .

The above assumptions describe slightly different models and there certainly do exist more model systems of this kind. We surmise however that, on a more macroscopic scale, the correct choice is perhaps not really crucial. Central is the idea that these delocalized sets should somehow be meager compared to the full continuum, that is, they should have Lebesgue measure zero. On the other hand, sets of the fractal type (cantor dust) may be admissible.

As we invoked in previous sections the renormalisation group picture, the phenomenon called *universality* comes to mind. There may in fact exist different microscopic model systems all converging to the same coarse grained macroscopic fixed point provided that they share certain crucial characteristics, determining the whole class. In our case this is the particular kind of non-local *entanglement*.

In the ‘foliation-model’ (alluding to a situation similar to the non-commutative torus, discussed above), i.e. all X_ι dense in M but non-overlapping, we encounter the following situation. Every point of M belongs to exactly one of the subsets, X_ι . Each of these subsets is spread over the whole manifold M and we have in particular that for each neighborhood, O_y , of some point, $y \in M$

$$X_\iota \cap O_y \text{ dense in } O_y \tag{32}$$

Observation 7.1 *The above partition defines an equivalence relation $R \subset M \times M$ with members (x, y) so that $y \in \tilde{x}$ with \tilde{x} the set X_ι , x is belonging to. As in the introductory sections, we can define a bundle structure, \tilde{M} , with base space M and fibers (x, \tilde{x}) and proceed in the same way as above by e.g. introducing the local Hilbert spaces, $H(\tilde{x})$ which are attached as fibers, H_y to the members, y , of*

the equivalence class. Correspondingly, we can introduce random operators acting in these local Hilbert space fibers.

Conclusion 7.2 *As in Connes approach we see, that the natural structure is not really some kind of quotient space but rather an amplification of the original space, M , to a fiber space over M . The internal spaces describe the subset of points which “can speak to each other” in a translocal way. These subsets are the classes of points of M , which happen to be connected by weak links in the underlying network, as has been described in the preceding section.*

We already remarked in section 2 that the final continuum structure, expected to emerge from our primordial dynamical network, may be more special what concerns the embedded translocal web than a simple equivalence relation (see also [27] and, in particular the section about networks as *causal sets*). As in the dynamical laws, employed by us, the edges typically carry states labelled by $(\pm 1, 0)$ being associated with the two possible orientations of the edge or the non-active state, we have rather a relational structure corresponding to a directed or oriented graph.

Furthermore, the ordinary mathematical model spaces are *static*. As we discussed in earlier papers, the states on the network and hence also the geometric structure follow a dynamical network law which constantly changes the wiring, the orientation of the edges and the shape of the cliques or lumps together with their mutual overlap. We took this into account in [22] by emulating it on the more macroscopic levels in form of *fuzzyness* of shapes and *randomness* of, for example, distances. That is, for each point $x \in M$ we rather have a (dense) set of distant points, $[x]_{in}$, sending information to x and another set, $[x]_{out}$, getting information from the point x , both sets being time dependent.

Note that in contrast to equivalence relations, subsets like $[x]_{in}$ or $[x]_{out}$ do not lead to a partitioning of M . There can exist complicated overlap patterns for the respective sets belonging to points $x \neq y$, nor does there exist a transitivity relation. On the other hand, on a macroscopic level, the differences between the various model spaces of the *wormhole class* to be found on the finer scales may be washed out. As all the above subsets are expected to be dense in M or X , and therefore also in each given region of M , any region O_1 gets translocal (stochastic) information from any other region O_2 and vice versa.

7.2 Microscopic Wormholes and Wheeler’s Space-Time Foam

The chain of thoughts, presented in the preceding sections, leads to a new microscopic picture of space-time and/or the *quantum vacuum*, strongly suggesting a *translocal entanglement* among distant regions of our continuous manifold. This structure is encoded in a web of relations which is largely hidden on the surface

level of (quasi)classical space-time but which, as we think, becomes observable through its expression in various features of quantum non-locality (cf. the remarks of v.Weizsaecker cited in the introduction).

So far our approach was decidedly bottom-up, starting from a presumed underlying microscopic substratum and reconstructing the more macroscopic levels by a renormalisation-like process of coarse-graining. On the other hand, there does exist for already quite some time a more top-down oriented picture, which, coming down from the continuum side of physics, evokes the scenario of a foam-like substructure of space-time on the Planck scale. In this context Wheeler et al developed the idea of microscopic wormholes, connecting distant parts of our ordinary space-time manifold or even different universes (see e.g. the classical book by Misner, Thorne, Wheeler; [24]). A beautiful and more up to date presentation can be found in [55].

We note that both ideas are realized in our framework. The idea of a foamy space which is almost fractal on a truly microscopic scale and has a scale or resolution dependent dimension is realized in our geometric renormalisation procedure provided that the network is in a critical state having a dense web of practically scale-free translocal links. This was rigorously shown in the preceding sections. It is suggestive to associate these translocal links with the presumed microscopic wormholes of Wheeler.

Observation 7.3 *Associating our web of translocal links with the microscopic wormholes of Wheeler we have shown that their existence is crucial in order that the dimension of space-time can become a scale dependent property, decreasing from a presumably large (fractal) microscopic dimension to the small integer dimension of ordinary classical space-time. That is, both ideas belong closely together.*

We remark that the scenario we are envisaging is not so far-fetched as it may seem. There exist, in fact, several recent investigations concerning the possible role of wormholes for the stability of the ordinary vacuum in quantum gravity. The possible effects of a gas of Planckian wormholes on various physical phenomena were studied several times in the past; as an example we mention the paper by Coleman ([56]). In [57] it was argued, that in quantum gravity an array of Planckian wormholes may be the correct ground state. This short list is far from being complete. All these speculations and observations seem to underpin our own line of reasoning.

8 A Brief Outlook on Quantum Entanglement and Other Translocal Quantum Phenomena

One of our motivations, to develop the above framework, is the goal to reach a better (and more realistic) understanding of the many mysteries being inherent

in the various phenomena of *quantum non-locality* and *entanglement*, the evident, but not well understood, necessity of *complex* superposition, *interference* and the peculiarities of the *measuring process*. Some of these points have been already discussed in a preliminary form in [23]. Now, with the concept of wormholespaces at our disposal, we are able to analyze these phenomena with greater rigor. In order not to blow up the size of the present paper beyond reasonable length, we will however make only some general remarks.

There exist several papers in the more recent past, which strike a similar key as far as the general working philosophy or parts of the present analysis are concerned (while the technical framework may be quite different). The following brief remarks are not meant as a full discussion of the field. We mention only a few points of view which seem to be particularly close to our own approach. An interesting approach has been developed by Smolin ([58],[18],[59]). It is perhaps intriguing to relate the *matrix-model* approach in the latter paper to our *bundle* or *foliation structure*. In both cases we have an array of countable subspaces which interact with each other. At the end of [18], on the other hand, one can find a brief discussion of a relational description of space-time in form of graphs.

A technically slightly different line of ideas is pursued in the following papers of 't Hooft ([25],[26],[19]). In this approach a deterministic *cellular automaton*-like primordial substratum is introduced which is similar to but more regular and static than our dynamical cellular network, *QX*. It is argued that quantum theory might emerge on a larger scale from such a deterministic and regular array. This approach has also been briefly discussed by us in [23].

Before we come to the more obvious consequences of our presumed wormhole structure we want to briefly mention that we think that the (somewhat mysterious) need of employing a *complex* superposition principle and a *complex structure* in general ([64]) is also a (subtle and not so obvious) consequence of this additional translocal web, being embedded in the ordinary continuous and locally behaving space structure.

It is quite funny in this respect to see that Schroedinger himself, in his row of five or six epochal papers about wave mechanics, for a very long time hold the view that, as in ordinary classical undulatory physics, one can always go over to the real part of the complex wave function if one wants to. Only in the last paper of this series ([60]) it began to dawn on him (possibly inspired by a letter from Lorentz) that the complex structure of quantum theory is inevitable and is buried deeply in its foundational structure (see also the beautiful essay by Yang [61]). The possible consequences and deficits of quantum mechanics over a *real* Hilbert space were analyzed in some detail in [63], see also the remarks in section 8 of [62]. Very illuminating in this respect is the paper of Dirac ([65]) in which he rightly argues that the emergence of a *phase quantity* is presumably even more important than the emergence of non-commuting observables.

More obvious is the effect which the translocal web of weak bonds will have on the understanding of the quantum mechanical measurement process, on *entangle-*

ment and other related phenomena. It was exactly the phenomenon of seemingly *instantaneous collapse* which stood in the way of a more *realistic* interpretation of the extended complex wave pattern.

Assuming that Einstein causality also holds sway in the quantum regime, quasi-instantaneous destruction of those parts of the wave, being located outside the region of direct measurement interference, could only be explained by granting the wave function, or more generally, the quantum state only the ontological status of a mere *bookkeeping device* of the *(non)-knowledge* of the observer. Looked upon from a slightly different angle, this explains the dominance of the *ensemble picture*.

On the other hand, if, in addition to the ordinary *local* and *causal* propagation between neighboring lumps and taking place with a *finite* velocity, we have a further, more subliminal *translocal* information transport through the web of weak bonds or, in more popular terms, through *hyperspace*, the almost instantaneous destruction of a real and existing excitation pattern of the vacuum becomes possible. Similar considerations hold in the context of entanglement. The details of these processes need of course a subtle analysis.

As a last point to mention we want to briefly comment on the seeming dichotomy between the presumed underlying and almost hidden translocal substructure and, on the other hand, the apparent local representation in form of partial differential equation. It is one of our findings that such non-local contributions and effects may come in a *local* disguise, such that, without an underlying more fundamental theory, it turns out to be difficult or nearly impossible to detect the translocal pieces of a model theory and separate them from the local ones. To give an example what we have in mind. An observer, prepared to take only local interactions into account, may formulate an effective theory of locally interacting fields, $A_i(x, t)$. However, these seemingly local fields may, for example, represent (x, t) -dependent integrals over distant contribution of the field configuration or even over quantities, which do occur on a deeper, more primordial level (as discussed in our coarse graining process). It could in particular happen that coupling constants turn out to be such integrated non-local quantities, an idea which obviously carry a strongly Machian spirit. To show that quantum theory would exactly be such an effective theory will represent the next logical step.

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