SUPERSTABILITY IN ABSTRACT ELEMENTARY CLASSES

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Abstract. We prove that several definitions of superstability in abstract elementary classes (AECs) are equivalent under the assumption that the class is stable, tame, has amalgamation, joint embedding, and arbitrarily large models. This partially answers questions of Shelah.

Theorem 0.1. Let K be a tame AEC with amalgamation, joint embedding, and arbitrarily large models. Assume K is stable. Then the following are equivalent:

(1) For all high-enough λ, there exists $\kappa \leq \lambda$ such that there is a good λ-frame on the class of κ-saturated models in K_λ.
(2) For all high-enough λ, K has a unique limit model of cardinality λ.
(3) For all high-enough λ, K has a superlimit model of cardinality λ.
(4) For all high-enough λ, the union of a chain of λ-saturated models is λ-saturated.
(5) There exists θ such that for all high-enough λ, K is (λ, θ)-solvable.

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1. Introduction

In the context of classification theory for AECs, a notion analog to the first-order notion of *stability* exists: it is defined as one might expect[1] (by counting Galois types). However it has been unclear what a parallel notion to superstability might be. Recall that for first-order theories we have:

Fact 1.1. Let T be a first-order complete theory. The following are equivalent:

(1) T is stable in every cardinal $\lambda \geq 2^{|T|}$.
(2) For all λ, the union of an increasing chain of λ-saturated models is λ-saturated.
(3) $\kappa(T) = \aleph_0$ and T is stable.
(4) T has a saturated model of cardinality λ for every $\lambda \geq 2^{|T|}$.
(5) T is stable and $\mathrm{D}^n[\bar{x} = \bar{x}, L(T), \infty] < \infty$.
(6) There does not exists a set of formulas $\Phi = \{\varphi_n(\bar{x}; \bar{y}_n) \mid n < \omega\}$ such that Φ can be used to code the structure $(\omega^{\leq \omega}, <, <_{lex})$

All the implications appear in Shelah's book [She90] with the exception of (2) $\implies$ (6) which was established by Albert and Grossberg [AG90, Theorem 13.2].

In the last 30 years, in the context of classification theory for non elementary classes, several notions that generalize that of first-order superstablity have been considered. See papers by Grossberg, Shelah, VanDieren, Vasey and Villaveces: [GS86, Gro88], [She99], [SV99], [Van06, Van13], [GVV], [Vasa, Vasb].

In [Shea, Discussion 2.9] Shelah mentions that part of the program of classification theory for AECs is to show that all the various notions of first-order saturation (limit, superlimit, or model-homogeneity, see Section 3) are equivalent under the assumption of superstablity. A possible definition of superstability is *solvability*, which appears in the

[1] A justification for the definition appears in [Vasc], showing that it is equivalent (under tameness) to no order property.

introduction to [She09a] and is hailed as a true counterpart to first-order superstability. Full justification is delayed to [Sheb] but [She09a, Chapter IV] already uses it. Other definitions of superstability analog to the ones in Fact 1.1 can also be formulated. The main result of this paper is to accomplish the above program of Shelah (showing that all the notions of saturated are equivalent) for tame AECs, and that in addition several definitions of superstability that previously appeared in the literature are equivalent in this context.

Theorem 1.2 (Main Theorem)**.** Let K be a tame AEC with amalgamation, joint embedding, and arbitrarily large models. Assume K is stable. Then the following are equivalent:

(1) There exists $\mu_1 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_1$, for all $\delta < \lambda^+$, for all increasing continuous $\langle M_i : i \leq \delta \rangle$ in K_λ and all $p \in \text{gS}(M_\delta)$, if M_{i+1} is universal over M_i for all $i < \delta$, then there exists $i < \delta$ such that p does not λ-split over M_i.
(2) There exists $\mu_2 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_2$, for some $\kappa \leq \lambda$, there is a good λ-frame on $K_\lambda^{\kappa\text{-sat}}$.
(3) There exists $\mu_3 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_3$, K has uniqueness of limit models in cardinality λ.
(4) There exists $\mu_4 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_4$, K has a superlimit model of cardinality λ.
(5) There exists $\mu_5 \geq \text{LS}(K)$ such that for every $\lambda \geq \mu_5$, the union of a chain of λ-saturated models is λ-saturated.
(6) There exists $\mu_6 \geq \text{LS}(K)$ such that for all $\lambda \geq \mu_6$, K is (λ, μ_6)-solvable.

Proof. Combine Theorem 4.8 and Theorem 5.44. □

Remark 1.3. The uniqueness of limit models in (3) is the "strong" version requiring isomorphism fixing the base (see Definition 3.4). However the proof shows that it could be replaced by the weak version where the isomorphism is not required to fix the base.

Remark 1.4. Theorem 1.2 has a global assumption of stability (see Definition 2.1). While stability is implied by some of the equivalent conditions (e.g. by (2) or (6)) other conditions may be vacuously true if stability fails (e.g. (3) or (1)). Thus for simplicity we just require stability outright.

At present, we do not know how to prove an analogs to the last two properties of Fact 1.1. More on this in Section 7.

Interestingly, the proof of Theorem 1.2 does not tell us that the threshold cardinals μ_ℓ above are equal. In fact, it uses tameness heavily to move from one cardinal to the next and uses e.g. that one equivalent definition holds below λ to prove that another definition holds at λ. Showing equivalence of these definitions cardinal by cardinals, or even just showing that we can take $\mu_1 = \mu_2 = \ldots = \mu_6$ seems much harder. A first step is in Section 6, where we show that the statements are still equivalent if we require $\mu_\ell < \beth_{(2^{\text{LS}(K)})^+}$, provided that the class is $(< \text{LS}(K))$-tame (see Definition 2.2). In a forthcoming paper [Vanb] VanDieren gives some relationships between versions of (3) and (5) in a single cardinal (with (1) as a background assumption). This is done without assuming tameness, using very different technologies than in this paper.

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2. Preliminaries

We now review some of the basics of abstract elementary classes and fix some notation. The reader is advised to skim through this section quickly and go back to it as needed. We assume familiarity with a basic text on AECs such as [Bal09] or [Gro] and refer the reader to the preliminaries of [Vasc] for more details and motivations on the concepts used in this paper. Throughout this section, K is an AEC. For λ an infinite cardinal, define $h(\lambda) := \beth_{(2^\lambda)^+}$. For the convenience of the reader, we briefly recall the definition of Galois types and stability:

Definition 2.1.

(1) For $N \in K$, $\bar{b} \in {}^{<\infty}|N|$, and $A \subseteq |N|$, write $\text{gtp}(\bar{b}/A; N)$ for the Galois type of $\bar{b}$ over A in N (so we also make use of Galois types over sets, which are defined just like Galois types over models, see [Vasc, Definition 2.16]).
(2) We let $\text{gS}(A; N) := \{\text{gtp}(b/A; N) \mid b \in |N|\}$ be the set of Galois types of length one over A in N.
(3) We say that K is *stable in* λ if for any $N \in K$ and any $A \subseteq |N|$ with $|A| \leq \lambda$, $|\text{gS}(A; N)| \leq \lambda$.
(4) We say K is *stable* if it is stable in a proper class of cardinals.

Shelah's program of classification theory for abstract elementary classes started in 1977 with a circulation of a draft of [She87] (a revised version is [She09a, Chapter I]). As a full classification theory seems impossible due to various counterexamples (e.g. [HS90]) and immense technical difficulties of addressing some of the main conjectures, all known non-trivial results are obtained under some additional model-theoretic or even set-theoretic assumptions on the family of classes we try to develop structure/non-structure results for. In July 2001, Grossberg and VanDieren circulated a draft of a paper titled "Morley Sequences in Abstract Elementary Classes" (a revised version was published as [GV06b]). In that paper, they introduced tameness as a useful assumption to prove upward stability results as well as existence of Morley sequence with respect to non-splitting in stable AECs.

Definition 2.2 (Definitions 3.2 in [GV06b])**.** Let χ be an infinite cardinal. K is $(<\chi)$-*tame* if for any $M \in K$ and any $p \neq q$ in $\text{gS}(M)$, there exists $A \subseteq |M|$ such that $|A| < \chi$ and $p \upharpoonright A \neq q \upharpoonright A$. K is χ-*tame* if it is $(<\chi^+)$-tame.

We say that K *is tame* provided there exists a cardinal χ such that the class K is $(<\chi)$-tame.

Remark 2.3. The definition makes sense also when $\chi \leq \text{LS}(K)$. This will be used e.g. in the statement of Theorem 6.1. For example, we use the fact that if K is stable, has amalgamation and is $(<\text{LS}(K))$-tame, then there exists $\lambda_0 < h(\text{LS}(K))$ such that K is stable in λ_0 ([Vasc, Theorem 0.2]). The reader confused by this point can simply replace χ by $\chi + \text{LS}(K)^+$ (this will result only in slightly higher bounds).

Remark 2.4. In [GV06c] and [GV06a] Grossberg and VanDieren established several cases of Shelah's categoricity conjecture (which is after 40 years still the best known open problem in the field of AECs). At the time, the main justification for the tameness assumption was that it appears in all known cases of structural results and it seems to be difficult to construct non-tame classes. In 2013, Will Boney [Bon14b] derived from the existence of a class of strongly compact cardinals that all AECs are tame. In a preprint from 2014, Lieberman and Rosický [LR] pointed out that this theorem of Boney follows from a 25 year old theorem of Makkai and Paré ([MP89, Theorem 5.5.1]). In a forthcoming paper Boney and Unger [BU] establish that if every AEC is tame then a proper class of large cardinals exists. Thus tameness (for all AECs) is a large cardinal axioms. We believe that this is evidence for the assertion that tameness is a new interesting model-theoretic property, a new dichotomy[2], that should follow (see [GV06a, Conjecture 1.5]) from categoricity in a "high-enough" cardinal.

A definition of superstability analog to $\kappa(T) = \aleph_0$ in first-order model theory has been studied in AECs (see [SV99, GVV, Van06, Van13, Vasa, Vasb]). Since it is not immediately obvious what forking should be in that framework, the simpler independence relation of μ-splitting is used for the purpose of the definition. Moreover in AECs, types over models are much better behaved than types over sets, so it does not make sense in general to ask for every type to not split over a finite set[3]. Thus we require that every type over the union of a chain does not split over a model in the chain. For technical reasons (essentially because it makes it possible to prove that the condition follows from categoricity), we require the chain to be increasing with respect to universal extension. This rephrases (1) in Theorem 1.2:

Definition 2.5.

(1) For $M, N \in K$, say $M <_{\text{univ}} N$ (N is *universal over* M) if and only if $M < N$ and whenever we have $M' \geq M$ such that $\|M'\| \leq \|N\|$, then there exists $f : M' \xrightarrow[M]{} N$.
(2) Let $\lambda \geq \text{LS}(K)$. We say K has $(*)_\lambda$ if for any regular $\delta < \lambda^+$, any $\langle M_i : i < \delta \rangle$ in K_λ with $M_i <_{\text{univ}} M_{i+1}$ for all $i < \delta$, any $p \in \text{gS}(\bigcup_{i<\delta} M_i)$, there exists $i < \delta$ such that p does not λ-split over M_i.

Remark 2.6. In the notation[4] of [Vasb, Definition 3.14], $(*)_\lambda$ holds if and only if $\kappa_1(\mathfrak{i}_{\lambda\text{-ns}}(K_\lambda), <_{\text{univ}}) = \aleph_0$.

Definition 2.7 (Superstability)**.** K is μ-superstable if:

(1) $\text{LS}(K) \leq \mu$.
(2) There exists $M \in K_\mu$ such that for any $M' \in K_\mu$ there is $f : M' \to M$ with $f[M'] <_{\text{univ}} M$.
(3) $(*)_\mu$ holds.

[2]Consider, for example, the statement that every sufficiently long sequence in a monster model for a first-order theory contains an indiscernible subsequence. In general, this is a large cardinal axiom, but it is known to be true when the theory is on the good side of a dividing line (in this case stability). We believe that the situation for tameness is analog.

[3]But see [Vasd, Theorem C.15] where a notion of forking over set is constructed from categoricity in a universal class.

[4]Of course, the κ notation has a long history, appearing first in [She70].

Remark 2.8. It is easy to check that Condition (2) is equivalent to "K_μ is nonempty, has amalgamation, joint embedding, no maximal models, and is stable in μ". Thus if K is nonempty, has amalgamation, joint embedding, and no maximal models, then for $\mu \geq \text{LS}(K)$, K is μ-superstable if and only if K is stable in μ and $(*)_\mu$ holds.

Remark 2.9. While Definition 2.7 makes sense in any AEC, here we focus on tame AECs with amalgamation, and will not study what happens to Definition 2.7 without these assumptions (although, as said above, the notion was first introduced in [SV99] without even amalgamation, and it has been further studied in [GVV] or even more generally [Van06, Van13], see also the forthcoming [Vana]).

For the convenience of the reader, we recall some facts about superstability for tame AECs with amalgamation.

Fact 2.10. Let K be an AEC with amalgamation.

(1) [Vasb, Proposition 10.10] If K is μ-superstable, μ-tame, and $\mu' \geq \mu$, then K is μ'-superstable. In particular, $K_{\geq \mu}$ has joint embedding, no maximal models, and is stable in all cardinals.
(2) [Vasb, Theorem 10.16][5] If K is $(< \kappa)$-tame with $\kappa = \beth_\kappa > \text{LS}(K)$ and categorical in a $\lambda > \kappa$, then K is κ-superstable.

3. Definitions of saturated

In this section, we introduce various definitions of saturation and prove implications between them. The search for a good definition of "saturated" in AECs is central. Perhaps the most natural one is:

Definition 3.1. Let $M \in K$ and let μ be a cardinal. M is *μ-saturated* if for any $N \geq M$, any $A \subseteq |M|$ of size less than μ, any $p \in \text{gS}(A; N)$ is realized in M. When $\mu = \|M\|$, we omit it.

We write $K^{\mu\text{-sat}}$ for the class of μ-saturated models in $K_{\geq \mu}$.

Remark 3.2. We could have called this *Galois* μ-saturated to differentiate it from the first-order notion. Since we never work with syntactic types in this paper, we will not use this terminology.

Remark 3.3. When $\mu = 0$, $K^{\mu\text{-sat}} = K$. This is used in (2) of Theorem 1.2: we allow $\kappa = 0$, which would mean that there is a good λ-frame on K_λ. This is useful since $K_\lambda^{\kappa\text{-sat}}$ might not be closed under unions of chains if $\kappa > 0$.

[5]The proof uses [SV99, Theorem 2.2.1] and indeed it turns out that this theorem suffices to get an even stronger result, see Theorem 6.3.

In [She01, Lemma 0.26] (see also [Gro02, Theorem 6.7] for a proof), it is observed that (under the amalgamation property) M saturated is *equivalent* to M model-homogeneous. This provides some justification for Definition 3.1 under amalgamation.

However when there is no amalgamation the following notion has played a central role (their study without amalgamation really started with [SV99] and was continued in [Van06, Van13]).

Definition 3.4 (Limit model)**.**

(1) $M \in K$ is *limit* over M_0 if $M_0 \leq M$, $\|M_0\| = \|M\|$, and there exists a limit ordinal δ and an increasing continuous $\langle N_i : i \leq \delta \rangle$ such that $M_0 = N_0$, $M = N_\delta$, and $N_i <_{\text{univ}} N_{i+1}$ for all $i < \delta$. M is *limit* if it is limit over some M_0.
(2) We say K has *uniqueness of limit models in* λ if whenever $M_0 \in K_\lambda$ and M_1, M_2 are limit over M_0 in K_λ then $M_1 \cong_{M_0} M_2$. We say that K has *weak uniqueness of limit models in* λ if we only require $M_1 \cong M_2$.

Even with the amalgamation property, uniqueness of limit models is a key concept which is equivalent to superstability in first-order model theory (see [GVV, Theorem 6.1]). In fact, limit models are saturated precisely when this holds:

Fact 3.5. Let $\lambda > \text{LS}(K)$. Assume K_λ has amalgamation, joint embedding, no maximal models, and is stable in λ.

(1) For any $M \in K_\lambda$, there exists $N \in K_\lambda$ such that $M <_{\text{univ}} N$. Thus there exists a limit model over M in K_λ.
(2) K has weak uniqueness of limit models in λ if and only if any limit $M \in K_\lambda$ is saturated.

Proof.

(1) See[6] [She09a, Claim II.1.16] or [GV06b, Theorem 2.12].
(2) This is folklore, so we include a proof. The right to left direction is by uniqueness of saturated models. For the left to right, assume weak uniqueness of limit models in K_λ and let $M \in K$ be limit. Let $A \subseteq |M|$ have size less than λ. Let $\delta := |A|^+$. Note that $\delta \leq \lambda$. By weak uniqueness of limit models, there exists $\langle M_i : i \leq \delta \rangle$ increasing continuous such that $M_\delta = M$

[6] The result first appeared without proofs in early versions of [She09a, Chapter II].

and $M_i <_{\text{univ}} M_{i+1}$ for all $i < \delta$. Pick $i < \delta$ such that $A \subseteq |M_i|$. Then any type over A is realized in M_{i+1}, as needed.

□

Another notion of saturation appears in [She87, Definition 3.1.1][7]. The idea is to encode a generalization of the fact that a union of saturated models should be saturated.

Definition 3.6. Let $M \in K$ and let $\lambda \geq \text{LS}(K)$. We say M is a *superlimit in* λ *if:*

(1) $M \in K_\lambda$.
(2) M is "properly universal": For any $N \in K_\lambda$, there exists $f : N \to M$ such that $f[N] < M$.
(3) Whenever $\langle M_i : i < \delta \rangle$ is an increasing chain in K_λ, $\delta < \lambda^+$ and $M_i \cong M$ for all $i < \delta$, then $\bigcup_{i<\delta} M_i \cong M$.

Note that the superlimit model is unique (this also seems to be folklore):

Proposition 3.7. If M and N are superlimit in λ, then $M \cong N$.

Proof. We build $\langle M_i : i \leq \omega \rangle$ increasing continuous in K_λ such that for every $i < \omega$, $M_{2i} \cong M$ and $M_{2i+1} \cong N$. This is possible using universality. This is enough since then by the last condition in the definition of a superlimit M_ω must be isomorphic to both M and N, hence $M \cong N$. □

Again we can ask when superlimits are saturated. The next lemma is a generalization of [Dru13, Corollary 2.3.12] (there $\chi = \lambda$, so λ is required to be regular).

Lemma 3.8. Assume K has amalgamation, joint embedding, and no maximal models. Let $\lambda > \text{LS}(K)$ be such that:

- There is a saturated model in K_λ.
- There exists a regular $\chi \leq \lambda$ such that for any increasing $\langle M_i : i < \chi \rangle$ in K_λ, if M_i is saturated for all $i < \chi$, then $\bigcup_{i<\chi} M_i$ is saturated.

The following are equivalent:

(1) There is a superlimit model in K_λ.
(2) In K_λ, the union of a chain of saturated models is saturated.

[7]We use the definition in [She09a, Definition 2.4.4] which requires in addition that the model be universal.

Proof. If in K_λ the union of a chain of saturated models is saturated, then the saturated model of size λ is a superlimit. Conversely, if K has a superlimit M in λ, it is enough to show that M is saturated. We build $\langle M_i : i < \chi \rangle$, $\langle N_i : i < \chi \rangle$ increasing in K_λ such that for all $i < \chi$, $M_i \le N_i \le M_{i+1}$, $M_i \cong M$ is superlimit, and N_i is saturated. In the end, $\bigcup_{i<\chi} M_i = \bigcup_{i<\chi} N_i$ is superlimit since it is a union of superlimit and saturated by definition of χ. Moreover, it is isomorphic to M, hence M is saturated. □

Thus (by the proof) under the assumptions of the lemma, superlimit and saturated coincide if chains of saturated models are saturated (another equivalent definition of superstability in the first-order case). In the remainder of this sections, we establish more implications between uniqueness of limit models, union of saturated being saturated, and existence of a superlimit. We assume:

Hypothesis 3.9. *K is a stable $(< \mathrm{LS}(K))$-tame AEC with amalgamation, joint embedding, and arbitrarily large models.*

Results on uniqueness of limit models can be related to chains of saturated models as follows[8]:

Lemma 3.10. *Let $\lambda > \mathrm{LS}(K)$ be a limit cardinal. Assume that for unboundedly many $\mu < \lambda$, K is stable in μ and has weak uniqueness of limit models in μ. Then the union of any increasing chain of λ-saturated models is λ-saturated.*

Proof. Let $\langle M_i : i < \delta \rangle$ be an increasing chain of λ-saturated models with (without loss of generality) $\delta = \mathrm{cf}(\delta) < \lambda$ and let $M_\delta := \bigcup_{i<\delta} M_i$. We want to see that M_δ is λ-saturated. Let $A \subseteq |M_\delta|$ have size less than λ. Let $\mu_0 := (\delta + \mathrm{LS}(K) + |A|)^+$. Since λ is limit, $\mu_0 < \lambda$. Let $\mu \ge \mu_0$ be such that $\mu < \lambda$, K is stable in μ, and K has weak uniqueness of limit models in μ. Let $\langle M_i' : i \le \delta \rangle$ be increasing continuous in K_μ such that for all $i < \delta$, $M_i' \le M_i$, $(A \cap |M_i|) \subseteq |M_i'|$, and $M_i' <_{\mathrm{univ}} M_{i+1}'$. Then M_δ' is (μ, δ)-limit, so by uniqueness of limit models is also (μ, μ_0)-limit. Also, M_δ' contains A, so by cofinality consideration, it must realize all types over A. As $M_\delta' \le M_\delta$, M_δ realizes all types over A. □

We now want to relate chains of saturated models and superlimit using Lemma 3.8. For this, we recall that the assumptions of this lemma hold in our context:

[8]The argument appears already as [Bal09, Theorem 10.22]

Fact 3.11 (Theorem 6.10 in [BV])**.** There exists $\chi < h(\mathrm{LS}(K))$ such that if $\langle M_i : i < \delta \rangle$ is an increasing chain of λ-saturated models and:

(1) $\mathrm{cf}(\delta) \geq \chi$.
(2) K is stable in unboundedly many $\mu < \lambda$.

Then $\bigcup_{i<\delta} M_i$ is λ-saturated.

Fact 3.12 (Theorem 4.13 in [Vasc])**.** There exists $\chi < h(\mathrm{LS}(K))$ such that K is stable in any $\lambda = \lambda^{<\chi}$.

Lemma 3.13. There exists a regular $\chi < h(\mathrm{LS}(K))$ and unboundedly many cardinals λ such that:

(1) K is stable in λ
(2) Any $M \in K_\lambda$ extends to a saturated $N \in K_\lambda$.
(3) If $\langle M_i : i < \chi \rangle$ is an increasing chain of saturated models in K_λ, then $\bigcup_{i<\chi} M_i$ is saturated.

Proof. Fix $\chi < h(\mathrm{LS}(K))$ regular satisfying the conclusions of both Fact 3.11 and Fact 3.12. Let $\lambda = \lambda^{<\chi}$ be such that $\mu^{<\chi} < \lambda$ for all $\mu < \lambda$. There are unboundedly many such limit λ by an easy "catching your tail" argument. Then K is stable in λ and in unboundedly many $\mu < \lambda$. Thus it is easy to check that any $M \in K_\lambda$ extends to a saturated model of size λ. □

Theorem 3.14. Assume either of the following conditions hold:

(1) K has weak uniqueness of limit models in all high-enough cardinals.
(2) K has a superlimit model in all high-enough cardinals.

Then for unboundedly many λ, K is stable in λ and has a saturated superlimit model in λ.

Proof. If the first condition holds, then by Lemma 3.10, in any limit cardinal λ a chain of saturated models is saturated. In particular we can take λ as given by Lemma 3.13. Then K has a saturated model in λ and it is clearly superlimit. If the second condition holds, let again λ be high-enough satisfying the conclusion of Lemma 3.13. By Lemma 3.8, the union of a chain of saturated models in K_λ is saturated, and thus the saturated model in K_λ is superlimit. □

In the next section we show how existence of a saturated superlimit implies superstability.

4. Chain local character over saturated models

Hypothesis 4.1. K is an AEC with amalgamation.

For background, we cite the following result, proven in [MS90, Proposition 4.12] for models of an $L_{\kappa,\omega}$ theory, κ strongly compact, and in [BG, Theorem 8.2.2] for AECs.

Fact 4.2. Let $\kappa > \text{LS}(K)$ be strongly compact. Let $\langle M_i : i \leq \delta \rangle$ be an increasing continuous chain of κ-saturated models (so M_i is κ-saturated *also for limit* i*, including* $i = \delta$). Let $p \in \text{gS}(M_\delta)$. If M_δ is κ-saturated, then there exists $i < \delta$ such that p is $(< \kappa)$-satisfiable over M_i.

The assumption above that M_δ is κ-saturated is crucial (otherwise we would have proven superstability from just stability, which is impossible even in the first-order case). The proof uses the strongly compact to build an appropriate ultrafilter and taking an ultraproduct of the chain $\langle M_i : i \leq \delta \rangle$ in which p is realized. Here, we give a simpler proof that does not need that κ is strongly compact but only that κ is regular uncountable.

Lemma 4.3. Let $\kappa > \aleph_0$ be a regular cardinal. Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous. Let $p \in \text{gS}^\alpha(M_\delta)$ with $|\alpha|^+ < \kappa$. If for all $i \leq \delta$ (*also limit* i*,including* $i = \delta$), $p \upharpoonright M_i$ is $(< \kappa)$-satisfiable over M_i and $\alpha < \text{cf}(\delta)$, then there exists $i < \delta$ such that p is $(< \kappa)$-satisfiable over M_i.

Proof. Without loss of generality, $\delta = \text{cf}(\delta)$. Suppose for a contradiction that the conclusion fails, i.e. for every $i < \delta$, p is *not* $(< \kappa)$-satisfiable over M_i. We consider two cases:

- Case 1: $\delta < \kappa$ Build $\langle A_i : i < \delta \rangle$ increasing such that for all $i < \delta$:
  (1) $A_i \subseteq |M_\delta|$.
  (2) $|A_i| < \kappa$.
  (3) $p \upharpoonright A_i$ is not realized in M_i.

  This is possible by the assumption on p and δ (we are also using that κ is regular to ensure that $|A_i| < \kappa$ is preserved at limit steps). This is enough: let $A := \bigcup_{i<\delta} A_i$. Note that $|A| < \kappa$ since $\delta < \kappa = \text{cf}(\kappa)$. As p is $(< \kappa)$-satisfiable over M_δ, $p \upharpoonright A$ is realized in M_δ, say by $\bar{b}$. As $\ell(\bar{b}) = \alpha < \delta$, there exists $i < \delta$ such that $\bar{b} \in {}^\alpha|M_i|$. But then $p \upharpoonright A$, and therefore $p \upharpoonright A_i$, is realized in M_i by $\bar{b}$, contradicting (3).

- Case 2: $\delta \geq \kappa$ Let $\gamma := |\alpha|^+ + \aleph_0$. Note that γ is regular and (since κ is uncountable and $|\alpha|^+ < \kappa$), $\gamma < \kappa$. Build $\langle i_j : j \leq \gamma \rangle$ increasing continuous in δ such that for all $j < \gamma$, $p \upharpoonright M_{i_{j+1}}$ is *not* $(< \kappa)$-satisfiable over M_{i_j}. This is possible by the assumption on p and δ (and the fact that whenever a type is not $(< \kappa)$-satisfiable, there is a witness of size less than κ). This is enough: by construction, $p \upharpoonright M_{i_\gamma}$ is not $(< \kappa)$-satisfiable over M_{i_j} for all $j < \gamma$. Since $\gamma = \mathrm{cf}(\gamma) < \kappa$, this contradicts the first part.

□

Recall that in [Vasb, Definition 3.14] (or Definition 2.5), the locality cardinal for chain was defined without assuming that the union of the chain was in the class. The above results shows that this is a necessary choice: Otherwise we could get strictly stable elementary classes in which $\kappa_1 = \aleph_0$. This also outlines the subtle difference between the chain and set local character cardinals, even in the elementary context. For example:

Corollary 4.4. Let T be a stable first-order theory. If $\langle M_i : i \leq \delta \rangle$ is an increasing continuous chain of $\aleph_1$-saturated models (so M_i is $\aleph_1$-saturated *also for limit* i*, including* $i = \delta$), $p \in \mathrm{S}(M_\delta)$, then there exists $i < \delta$ such that p does not fork over M_i.

Proof. Set $\kappa = \aleph_1$ and $(K, \leq) = (\mathrm{Mod}(T), \preceq)$ in Lemma 4.3. □

Remark 4.5. This gives a quicker, more general, proof of [AG90, Theorem 13.2].

Question 4.6. Does Corollary 4.4 say anything nontrivial? For example, let T be a countable first-order theory and assume it is stable but not superstable. Let $\lambda \geq \aleph_1$. When can we build an increasing continuous chain $\langle M_i : i \leq \delta \rangle$ of $\aleph_1$-saturated models of T of size λ?

We can now prove most conditions in the main theorem. For a start, we point out that among the definitions in the statement of Theorem 1.2, the first one has been used as a starting hypothesis many times previously. In [Vasa], λ-superstability was shown to imply the existence of a good λ^+-frame on the class of saturated models of size λ^+, except that this class may not be an AEC. The result was later generalized in [Vasb]. Building on these papers, [BV] gave conditions under which a chain of saturated models is saturated, culminating in:

Fact 4.7 (Theorem 7.1 in [BV])**.** If K is μ-superstable (for some μ), then in all high-enough cardinals λ, $K_\lambda^{\lambda\text{-sat}}$ has a type-full good λ-frame, K has uniqueness of limit models, has a saturated superlimit model, and any chain of λ-saturated models is λ-saturated.

Thus the first condition in Theorem 1.2 implies all the other ones, except perhaps solvability (examined in Section 5).

We now restate and prove Theorem 1.2 with more conditions added (but without solvability)[9]:

Theorem 4.8. Let K be a tame AEC with amalgamation, joint embedding, and arbitrarily large models. Assume K is stable. Then the following are equivalent:

(1) For all high-enough λ, $(*)_\lambda$ holds.
(2) For all high-enough cardinal λ, for some $\kappa \leq \lambda$, there is a good λ-frame on $K_\lambda^{\kappa\text{-sat}}$.
(3) K has uniqueness of limit models in all high-enough cardinals.
(4) K has a superlimit model in all high-enough cardinals.
(5) For all high-enough cardinal λ, the union of a chain of λ-saturated models is λ-saturated.
(6) For unboundedly many λ, K is stable in λ and has a saturated superlimit model in λ.

Proof. Without loss of generality, K is $(< \text{LS}(K))$-tame. Note that with our background assumptions, $(*)_\lambda$ together with stability in λ is equivalent to K being λ-superstable (Remark 2.8).

First assume (1). Since K is stable, we can pick $\lambda \geq \text{LS}(K)$ such that K is stable in λ and satisfies $(*)_\lambda$, and hence is λ-superstable. By Fact 4.7, (2)-(6) all follow. Assume (2)[10]. By [She09a, Lemma II.4.8] (or see [Bon14a, Theorem 9.2]), (3) holds. Also, (5) directly implies (4). Moreover by Theorem 3.14, both (3) and (4) imply (6). To sum up, we have shown:

[9]On good frames, see [She09a, Definition II.2.1] or [JS13, Definition 2.1.1]. The later definition is the one we use: as opposed to the definition in Shelah's book, it does not require the existence of a superlimit. When we say that there is a good λ-frame on $K_\lambda^{\kappa\text{-sat}}$ we mean that the frame's independence relation is defined on the κ-saturated models of size λ. The definition of a good frame implies that if $\delta < \lambda^+$ and $\langle M_i : i < \delta \rangle$ is an increasing chain of κ-saturated models, then $\bigcup_{i<\delta} M_i$ is κ-saturated.

[10]Note that by the definition of a good frame (see [She09a, Definition II.2.1]), this implies in particular that chains of κ-saturated models in K_λ have κ-saturated unions.

- (1) implies (5).
- (1) implies (2) implies (3) implies (6).
- (5) implies (4) implies (6).

Thus all the conditions imply (6) and (1) implies all the conditions, so it only remains to show (6) implies (1). Assume (6), fix a $\kappa > \text{LS}(K)$ such that $\kappa = \beth_\kappa$ and let $\lambda > \kappa$ witness (6). Let $\langle M_i : i < \delta \rangle$ be an increasing chain of saturated models in K_λ, $\delta < \lambda^+$. Then $M_\delta := \bigcup_{i<\delta} M_i$ is saturated. Therefore by Lemma 4.3 (with κ there standing for κ^+ here), any $p \in \text{gS}(M_\delta)$ is $(< \kappa^+)$-satisfiable (and hence $(< \kappa)$-satisfiable) over some M_i, $i < \delta$. By [Vasb, Fact 3.17.(2d)] (originally proven as [Vasc, Theorem 5.13]), the $(< \kappa)$-satisfiability independence relation has the uniqueness property[11]. By [BGKV, Proposition 3.12, Lemma 4.2], $(< \kappa)$-satisfiability must be extended by non-λ-splitting. Hence any $p \in \text{gS}(M_\delta)$ does not λ-split over M_i for some $i < \delta$. Thus $(*)_\lambda$ holds (really we have only proven it for saturated models, however using a back and forth argument as in the proof of uniqueness of limit models of the same cofinality, we get it for all models, see [Vasb, Proposition 10.6]) so K is λ-superstable. By Fact 2.10, K is also λ'-superstable for all $\lambda' \geq \lambda$, so (1) holds. $\square$

Remark 4.9. In (2), we do *not* assume that the good frame is type-full (i.e. it may be that there exists some nonalgebraic types which are not basic, so fork over their domain). However if (1) holds, then the proof of (1) implies (2) shows we can take the frame to be type-full. Therefore, in the presence of tameness, the existence of a good frame implies the existence of a *type-full* good frame (in a potentially much higher cardinal, and over more saturated models).

5. Solvability

Solvability appears as a possible definition of superstability for AECs in [She09a, Chapter IV]. In the introduction to the book, Shelah claims (without proof) that it is equivalent to first-order superstability. We give a proof here and actually show (assuming amalgamation, stability, and tameness) that solvability is equivalent to any of the definitions in the main theorem. This partially answers some questions of Shelah on [She09a, p. 56].

We assume some familiarity with Ehrenfeucht-Mostowski models, see for example [Bal09, Section 6.2] or [She09a, Definition IV.0.8].

[11] Note that the no order property hypothesis there follows from [She99, Claim 4.5.3], or see [Vasb, Fact 2.20].

Definition 5.1. Let K be an AEC and let $\theta \leq \lambda$ be such that $\text{LS}(K) \leq \theta$.

(1) [She09a, Definition IV.0.8.2] Let $\Upsilon_\theta[K]$ be the set of Φ proper for linear orders with:
  (a) $|L(\Phi)| \leq \theta$.
  (b) For I a linear order, $\text{EM}_{L(K)}(I, \Phi) \in K$.
  (c) For $I \subseteq J$ linear orders, $\text{EM}_{L(K)}(I, \Phi) \leq \text{EM}_{L(K)}(J, \Phi)$.

(2) [She09a, Definition IV.1.4.1] We say that Φ *witnesses* (λ, θ)-*solvability* if:
  (a) $\Phi \in \Upsilon_\theta[K]$.
  (b) If I is a linear order of size λ, then $\text{EM}_{L(K)}(I, \Phi)$ is superlimit of size λ.

  K is (λ, θ)-*solvable* if there exists Φ witnessing (λ, θ)-solvability.

(3) K is *uniformly* (λ, θ)-*solvable* if there exists Φ such that for all $\mu \geq \lambda$, Φ witnesses (μ, θ)-solvability.

Remark 5.2. If K is uniformly (λ, θ)-solvable, then K is (μ, θ)-solvable for all $\mu \geq \lambda$.

Fact 5.3. Let K be an AEC and let $\theta \geq \text{LS}(K)$. Then K has arbitrarily large models if and only if $\Upsilon_\theta[K] \neq \emptyset$.

We start by giving some more manageable definitions of solvability ((3 is the one we will use). Shelah already mentions one of them on [She09a, p. 53] (but does not prove it is equivalent).

Proposition 5.4. Let K be an AEC and let $\text{LS}(K) \leq \theta \leq \lambda$. The following are equivalent.

(1) K is [uniformly] (λ, θ)-solvable.

(2) There exists $L' \supseteq L(K)$ with $|L'| \leq \theta$ and $\psi \in L'_{\theta^+, \omega}$ such that:
  (a) ψ has arbitrarily large models.
  (b) [For all $\mu \geq \lambda$], if $M \models \psi$ and $\|M\| = \lambda$ [$\|M\| = \mu$], then $M \upharpoonright L(K)$ is in K and is superlimit.

(3) There exists $L' \supseteq L(K)$ and an AEC K' with $L(K') = L'$, $\text{LS}(K') \leq \theta$ such that:
  (a) K' has arbitrarily large models.
  (b) [For all $\mu \geq \lambda$], if $M \in K'$ and $\|M\| = \lambda$ [$\|M\| = \mu$], then $M \upharpoonright L(K)$ is in K and is superlimit.

Proof.

- <u>(1) implies (2)</u>: Let Φ witness (λ, θ)-solvability and write $\Phi = \{p_n \mid n < \omega\}$. Let $L' := L(\Phi) \cup \{P, <\}$, where P, $<$ are symbols for a unary predicate and a binary relation respectively. Let $\psi \in L'_{\theta^+,\omega}$ say:

(1) $(P, <)$ is a linear order.
(2) For all $n < \omega$ and all $x_0 < \cdots < x_{n-1}$ in P, $x_0 \dots x_{n-1}$ realizes p_n.
(3) For all y, there exists $n < \omega$, $x_0 < \cdots < x_{n-1}$ in P, and τ an n-ary term of $L(\Phi)$ such that $y = \tau(x_0, \dots, x_{n-1})$.

Then if $M \models \psi$, $M \upharpoonright L = \text{EM}(P^M, \Phi)$. Conversely, if $M = \text{EM}(I, \Phi)$, we can expand M to an L'-structure by letting $(P^M, <^M) := (I, <)$. Thus ψ is as desired.

- (2) implies (3): Given L' and ψ as given by (2), Let Ψ be a fragment of L' containing ψ of size θ and let K' be $\text{Mod}(\psi)$ ordered by $\preceq_\Psi$. Then K' is as desired for (3).
- (3) implies (1): Directly from Fact 5.3.

□

Let K be an AEC and assume there exists θ such that K is (λ, θ)-solvable for all high-enough λ, then in particular K has a superlimit in all high-enough λ, so we obtain one of the conditions in the main theorem. We now work toward a converse. The proof is similar to that in [BGS99]: we code saturated models using their characterization with average of sequences (the crucial result for this is Lemma 5.33). We recall some of the theory of averages in AEC (as developed by Shelah in [She09b, Chapter V.A] and by Boney and the second author in [BV]), and give a new characterization of forking using averages (Lemma 5.26). All throughout, we assume:

Hypothesis 5.5.

(1) K is an AEC with amalgamation, joint embedding, and arbitrarily large models.
(2) K is stable.
(3) K is $(< \kappa)$-tame, where $\kappa := \text{LS}(K)$.
(4) We work inside a big monster model $\mathfrak{C}$.

Eventually, we will also assume Hypothesis 5.35.

By "working inside $\mathfrak{C}$" we mean as usual that whenever it makes sense, all the definitions are relativized to $\mathfrak{C}$. For example "A is a set" means $A \subseteq |\mathfrak{C}|$ and $\text{gtp}(\bar{b}/A)$ means $\text{gtp}(\bar{b}/A; \mathfrak{C})$. We will use without much comments results about Galois-Morleyization and averages as defined in [Vasc, BV]. Still we have tried to give a syntax-free presentation. The letters $\mathbf{I}$, $\mathbf{J}$ will denote sequences of tuples of length less than κ.

We will use the same conventions as in [BV, Section 5]. Note that the sequences *may be indexed by arbitrary linear orders.* Recall:

Definition 5.6 (Definition V.A.2.1 in [She09b])**.** $\mathbf{I}$ is *χ-convergent* if $|\mathbf{I}| \geq \chi$ and for any $p \in \mathrm{gS}^{<\kappa}(A)$, $|A| < \kappa$, the set of elements of $\mathbf{I}$ realizing p either has fewer than χ elements or its complement has fewer than χ elements.

Definition 5.7 (Definition V.A.2.6 in [She09b])**.** For $\mathbf{I}$ a sequence, χ an infinite cardinal such that $|\mathbf{I}| \geq \chi$, and A a set, define $\mathrm{Av}_\chi(\mathbf{I}/A)$ to be the set of $p_0 \in \mathrm{gS}^{<\kappa}(A_0)$ such that $A_0 \subseteq A$ has size less than κ and the set $\{\bar{b} \in \mathbf{I} \mid \bar{b} \not\models p_0\}$ has size less than χ. When there is a unique $p \in \mathrm{gS}^{<\kappa}(A)$ such that $p \upharpoonright A_0$ is in $\mathrm{Av}_\chi(\mathbf{I}/A)$ for all $A_0 \subseteq A$ of size less than κ, we identify the average with p.

Remark 5.8 (Monotonicity)**.** If $\mathbf{I}$ is χ-convergent, $\chi' \geq \chi$, and $\mathbf{J} \subseteq \mathbf{I}$ is such that $|\mathbf{J}| \geq \chi'$, then $\mathbf{J}$ is χ'-convergent and for any A, $\mathrm{Av}_\chi(\mathbf{I}/A) = \mathrm{Av}_{\chi'}(\mathbf{J}/A)$.

Definition 5.9. $p \in \mathrm{gS}^{<\kappa}(B)$ *does not syntactically split* over $A \subseteq B$ if it does not split (in the syntactic sense, see [BV, Definition 5.7]) in the Galois Morleyization. In semantic term, this means that for all $\bar{b}, \bar{b}' \in {}^{<\kappa}B$, if $\mathrm{gtp}(\bar{b}/A) E_{<\kappa} \mathrm{gtp}(\bar{b}'/A)$, then $(p \upharpoonright \bar{b}) E_{<\kappa} (p \upharpoonright \bar{b}')$. Here, $q_1 E_{<\kappa} q_2$ if and only if $q_1 \upharpoonright A_0 = q_2 \upharpoonright A_0$ for all A_0 of size less than κ.

Remark 5.10. By tameness, $E_{<\kappa}$ is equality *for types of length one*, but may not be equality for longer types.

It turns out that Morley sequences (defined below) are always convergent. The parameters represent respectively a bound on the size of the domain, the degree of saturation of the models, and the length of the sequence.

Definition 5.11 (Definition 5.14 in [BV])**.** We say $\langle \bar{a}_i : i \in I \rangle \frown \langle N_i : i \in I \rangle$ is a (χ_0, χ_1, χ_2)-*Morley sequence for p over A* if:

(1) $\chi_0 \leq \chi_1 \leq \chi_2$ are infinite cardinals, I is a linear order, A is a set, p is a Galois type with $\ell(\bar{x}) < \kappa$, and there is $\alpha < \kappa$ such that for all $i < \delta$, $\bar{a}_i \in {}^\alpha\mathfrak{C}$.
(2) For all $i \in I$, $A \subseteq |N_i|$ and $|A| < \chi_0$.
(3) $\langle N_i : i \in I \rangle$ is increasing, and each N_i is χ_1-saturated.
(4) For all $i \in I$, $\bar{a}_i$ realizes[12] $p \upharpoonright N_i$ and for all $j > i$ in I, $\bar{a}_i \in {}^\alpha N_j$.
(5) $i < j$ in I implies $\bar{a}_i \neq \bar{a}_j$.
(6) $|I| \geq \chi_2$.

[12]Note that $\mathrm{dom}(p)$ might be smaller than N_i.

(7) For all $i < j$ in I, $\text{gtp}(\bar{a}_i/N_i) = \text{gtp}(\bar{a}_j/N_i)$.
(8) For all $i \in I$, $\text{gtp}(\bar{a}_i/N_i)$ does not syntactically split over A.

When p or A is omitted, we mean "for some p or A". We call $\langle N_i : i \in I \rangle$ the *witnesses* to $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle$ being Morley, and when we omit them we simply mean that $\mathbf{I} \frown \langle N_i : i \in I \rangle$ is Morley for some witnesses $\langle N_i : i \in I \rangle$.

We also define a χ-*Morley sequence* to be a (χ, χ, χ)-Morley sequence.

Remark 5.12 (Monotonicity)**.** Let $\langle \bar{a}_i : i \in I \rangle \frown \langle N_i : i \in I \rangle$ be (χ_0, χ_1, χ_2)-Morley for p over A. Let $\chi_0' \geq \chi_0$, $\chi_1' \leq \chi_1$, and $\chi_2' \leq \chi_2$. Let $I' \subseteq I$ be such that $|I'| \geq \chi_2'$, then $\langle \bar{a}_i : i \in I' \rangle \frown \langle N_i : i \in I' \rangle$ is $(\chi_0', \chi_1', \chi_2')$-Morley for p over A.

Remark 5.13. By the proof of [She90, Lemma I.2.5], a Morley sequence is indiscernible (this will not be used).

To state a relationship between averages and stability, we will use:

Definition 5.14. We say $\mathfrak{C}$ has the *syntactic order property of length* χ if there exists $\alpha < \kappa$, $p \in \text{gS}^{\alpha+\alpha}(\emptyset)$ and elements $\langle \bar{a}_i : i < \chi \rangle$ of arity α such that for $i, j < \chi$, $\text{gtp}(\bar{a}_i \bar{a}_j/\emptyset) = p$ if and only if $i < j$.

As explained in [Vasc, Proposition 4.4], the syntactic order property and the Galois order property of [She99, Definition 4.3] are "almost" the same, in particular by [She99, Claim 4.5.3, 4.7.2] (see [BGKV, Fact 5.13] for proofs) we have:

Fact 5.15. There exists $\chi_0 < h(\text{LS}(K))$ such that $\mathfrak{C}$ does not have the syntactic order property of length χ_0.

The next result is key in the treatment of average of [BV]:

Fact 5.16 (Theorem 5.21 in [BV])**.** Let $\chi_0 \geq 2^{\text{LS}(K)}$ be such that $\mathfrak{C}$ does not have the syntactic order property of length χ_0^+. Let $\chi := \left(2^{2^{\chi_0}}\right)^+$. If $\mathbf{I}$ is a $(\chi_0^+, \chi_0^+, \chi)$-Morley sequence, then $\mathbf{I}$ is χ-convergent.

Notation 5.17. Let χ_0 be the least regular cardinal μ such that $\mu \geq 2^{\text{LS}(K)}$ and $\mathfrak{C}$ does not have the syntactic order property of length μ^+ (so χ_0 depends on K). For a cardinal λ, let $\gamma(\lambda) := (2^{2^\lambda})^+$. We write $\chi_0' := \gamma(\chi_0)$.

Remark 5.18. By Fact 5.15, $\chi_0 < h(\text{LS}(K))$. Hence also $\chi_0' < h(\text{LS}(K))$.

Another property of χ_0 is:

Fact 5.19 (Theorem V.A.1.19 in [She09b])**.** If $\lambda = \lambda^{\chi_0}$, then K is stable in λ. In particular, K is stable in χ_0'.

Next, we want to relate average and forking.

Definition 5.20. Let $M_0, M \in K^{(\chi_0')^+\text{-sat}}$ be such that $M_0 \leq M$. Let $p \in \text{gS}(M)$. We say that p *does not fork over* M_0 if there exists $M_0' \in K_{\chi_0'}$ such that $M_0' \leq M_0$ and p does not χ_0'-split over M_0'.

We will use without comments:

Fact 5.21. Forking has the following properties:

(1) Invariance under isomorphisms and monotonicity: if $M_0 \leq M_1 \leq M_2$ are all $(\chi_0')^+$-saturated and $p \in \text{gS}(M_2)$ does not fork over M_0, then $p \upharpoonright M_1$ does not fork over M_0 and p does not fork over M_1.
(2) Set local character: if $M \in K^{(\chi_0')^+\text{-sat}}$ and $p \in \text{gS}(M)$, there exists $M_0 \in K^{(\chi_0')^+\text{-sat}}$ of size $(\chi_0')^+$ such that $M_0 \leq M$ and p does not fork over M_0.
(3) Transitivity: Assume $M_0 \leq M_1 \leq M_2$ are all $(\chi_0')^+$-saturated and $p \in \text{gS}(M_2)$. If p does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0, then p does not fork over M_0.
(4) Uniqueness: If $M_0 \leq M$ are all $(\chi_0')^+$-saturated and $p, q \in \text{gS}(M)$ do not fork over M_0, then $p \upharpoonright M_0 = q \upharpoonright M_0$ implies $p = q$.
(5) Local extension over saturated models: If $M_0 \leq M$ are both saturated, $\|M_0\| = \|M\| \geq (\chi_0')^+$, $p \in \text{gS}(M_0)$, there exists $q \in \text{gS}(M)$ such that q extends p and does not fork over M_0.

Proof. Use [Vasb, Theorem 7.5]. The generator used is the one given by Proposition 7.4.(2) there. □

Note that the extension property need not hold in general. However if the class is superstable we have:

Fact 5.22. If K is χ_0'-superstable, then:

(1) ([Vasb, Theorem 8.9] or [Vasa, Theorem 7.1]) Forking has:
  (a) The extension property: If $M_0 \leq M$ are $(\chi_0')^+$-saturated and $p \in \text{gS}(M_0)$, then there exists $q \in \text{gS}(M)$ extending p and not forking over M_0.
  (b) The chain local character property: If $\langle M_i : i < \delta \rangle$ is an increasing chain of $(\chi_0')^+$-saturated models and $p \in$

gS($\bigcup_{i<\delta} M_i$), then there exists $i < \delta$ such that p does not fork over M_i.

(2) [BV, Lemma 6.9] For any $\lambda > (\chi_0')^+$, $K^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(K^{\lambda\text{-sat}}) = \lambda$. Moreover, forking induces a good $(\geq (\chi_0')^{++})$-frame with underlying class $K^{(\chi_0')^{++}\text{-sat}}$.

For notational convenience, we "increase" χ_0:

Notation 5.23. Let $\chi_1 := (\chi_0')^{++}$. Let $\chi_1' := \gamma(\chi_1)$.

We obtain a characterization of forking that adds to those proven in [Vasb]. A form of it already appears in [She09a, Observation IV.4.6]. Again, we define more cardinal parameters:

Notation 5.24. Let $\chi_2 := \beth_\omega(\chi_0)$.

Remark 5.25. We have that $\chi_0 < \chi_0' < \chi_1 < \chi_1' < \chi_2 < h(\text{LS}(K))$.

Lemma 5.26. Let M_0, M be χ_2-saturated with $M_0 \leq M$. Let $p \in \text{gS}(M)$. The following are equivalent:

(1) p does not fork over M_0.
(2) $p \upharpoonright M_0$ has a nonforking extension to $\text{gS}(M)$ and there exists $M_0' \leq M_0$ with $\|M_0'\| < \chi_2$ such that p does not syntactically split over M_0'.
(3) $p \upharpoonright M_0$ has a nonforking extension to $\text{gS}(M)$ and there exists $\mu \in [\chi_0^+, \chi_2)$ and $\mathbf{I}$ a $(\mu, \mu, \gamma(\mu)^+)$-Morley sequence for p, with all the witnesses inside M_0, such that $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$.

Remark 5.27. In this section, we will end up assuming that K is χ_0'-superstable, hence (Fact 5.22) forking has the extension property so the first part of (2) and (3) always hold. However in Section 6 we will use Lemma 5.26 in the strictly stable case.

We need more definitions and facts before giving the proof:

Fact 5.28 (V.A.1.12 in [She09b])**.** If $p \in \text{gS}(M)$ and M is χ_0^+-saturated, there exists $A \subseteq |M|$ of size at most χ_0 such that p does not syntactically split over A.

Definition 5.29 (Definition 5.9 in [BV])**.** A sequence $\mathbf{I}$ is *μ-based on A* if for any B, $\text{Av}_\mu(\mathbf{I}/B)$ does not syntactically split over A (when the average exists).

Fact 5.30 (Claim IV.1.23.(2) in [She09a] or see Lemma 5.10 in [BV])**.** Let $\mathbf{I}$ be a sequence and let $\mathbf{J} \subseteq \mathbf{I}$ have size at least μ. Then $\mathbf{I}$ is μ-based on $\mathbf{J}$.

Fact 5.31 (Lemma 5.20 in [BV])**.** Let $\mathbf{I}$ be a μ^+-Morley sequence over A (for some type). If $\mathbf{I}$ is μ-convergent, then $\mathbf{I}$ is μ-based on A.

Fact 5.32. Let $M_0 \leq M$ be both $(\chi_1')^+$-saturated. Let $\mu := \|M_0\|$. Let $p \in \text{gS}(M)$ and let $\mathbf{I}$ be a $(\mu^+, \mu^+, \gamma(\mu))$-Morley sequence for p over M_0 with all the witnesses inside M. Then:

(1) If p does not syntactically split over M_0, then $\text{Av}(\mathbf{I}/M) = p$.
(2) If p does not fork over M_0, then $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$.

Proof. For syntactic splitting, this is [BV, Lemma 5.25]. The Lemma is actually more general and the proof of [BV, Lemma 6.9] shows that this also works for forking. $\square$

Proof of Lemma 5.26. Before starting, note that if $\mu < \chi_2$, then K is stable in $2^{\mu+\chi_0} < \chi_2$ by Fact 5.19. Thus there are unboundedly many stability cardinals below χ_2, so we have "enough space" to build Morley sequences.

- (1) implies (2): By Fact 5.28, we can find $M_0' \leq M_0$ such that $p \restriction M_0$ does not syntactically split over M_0' and $\|M_0'\| \leq \chi_1$. Taking M_0' bigger, we can assume M_0' is χ_1-saturated and $p \restriction M_0$ does not fork over M_0'. Thus by transitivity p does not fork over M_0'. Let $\mathbf{I}$ be a (χ_1')+-Morley sequence for $p \restriction M_0$ over M_0' inside M_0. By Fact 5.16, $\mathbf{I}$ is χ_1'-convergent. By Fact 5.31, $\mathbf{I}$ is χ_1'-based on M_0'. Note also that $\mathbf{I}$ is a $(\chi_1')^+$-Morley sequence for p over M_0' and by Fact 5.32, $\text{Av}_{\chi_1'}(\mathbf{I}/M_0) = p$ so as $\mathbf{I}$ is χ_1'-based on M_0', p does not syntactically split over M_0'.
- (2) implies (3): As in the proof of (1) implies 2) (except chi_1 could be bigger. Here, the splitting-like notion is just syntactic splitting).
- (3) implies (2): By Fact 5.16, $\mathbf{I}$ is $\gamma(\mu)$-convergent. Pick any $\mathbf{J} \subseteq \mathbf{I}$ of length $\gamma(\mu)$ and use Fact 5.30 to find $M_0' \leq M_0$ of size $\gamma(\mu)$ such that $\mathbf{J}$ is $\gamma(\mu)$-based on M_0'. Since also $\mathbf{J}$ is $\gamma(\mu)$-convergent, we have that $\mathbf{I}$ is $\gamma(\mu)$-based on M_0'. Thus $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$ does not syntactically split over M_0'.
- (2) implies (1): Without loss of generality, we can choose M_0' to be such that $p \restriction M_0$ also does not fork over M_0'. Let $\mu := \|M_0'\| + \chi_0$. Build a $(\mu^+, \mu^+, \gamma(\mu))$-Morley sequence $\mathbf{I}$ for p over M_0' inside M_0. If q is the nonforking extension of $p \restriction M_0$ to M, then $\mathbf{I}$ is also a Morley sequence for q over M_0' so by the proof of (1) implies (2) we must have $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M) = q$, but also

$\mathrm{Av}_{\gamma(\mu)}(\mathbf{I}/M) = p$, since p does not syntactically split over M_0' (Fact 5.32). Thus $p = q$.

□

The next result is a version of [She90, Theorem III.3.10] in our context. It is implicit in the proof of [BV, Theorem 5.27].

Lemma 5.33. Let $M \in K^{\chi_2\text{-sat}}$. Let $\lambda \geq \chi_2$ be such that K is stable in unboundedly many $\mu < \lambda$. The following are equivalent.

(1) M is λ-saturated.
(2) If $q \in \mathrm{gS}(M)$ is not algebraic and does not syntactically split over $A \subseteq |M|$ with $|A| < \chi_2$, there exists a $((|A| + \chi_0)^+, (|A| + \chi_0)^+, \lambda)$-Morley sequence for p over A inside M.

We use one more fact in the proof, telling us when the average is realized by an element of the sequence.

Fact 5.34 (Lemma 5.6 in [BV])**.** Let $\mathbf{I}$ be a sequence and let μ be a cardinal such that $\mathbf{I} \geq \mu$. Let $A \subseteq \mathfrak{C}$ and let $p \in \mathrm{gS}(A)$ be such that $\mathrm{Av}_\mu(\mathbf{I}/A) = p$. If $|\mathbf{I}| > \mu + |\mathrm{gS}(A)|$, then there exists $b \in \mathbf{I}$ realizing p.

Proof of Lemma 5.33. (1) implies (2) is trivial using saturation. Now assume (2). Let $p \in \mathrm{gS}(B)$, $|B| < \lambda$, $B \subseteq |M|$. We show that p is realized in M. Let $q \in \mathrm{gS}(M)$ extend p. If q is algebraic, we are done so assume it is not. Let $A \subseteq |M|$ have size $(\chi_1')^+$ such that q does not fork over A. By Lemma 5.26, we can increase A if necessary so that q does not syntactically split over A and $\mu := |A| \geq \chi_0$. Now by (2), there exists a (μ^+, μ^+, λ)-Morley sequence $\mathbf{I}$ for q over A inside M. Now by Fact 5.32, $\mathrm{Av}_{\gamma(\mu)}(\mathbf{I}/M) = q$. Thus $\mathrm{Av}_{\gamma(\mu)}(\mathbf{I}/B) = p$. By Fact 5.34 and the hypothesis of stability in unboundedly many cardinals below λ, p is realized by an element of $\mathbf{I}$ and hence by an element of M. □

From now on we assume superstability:

Hypothesis 5.35.

(1) K is χ_0'-superstable.
(2) Eventually, we also assume Hypothesis 5.37.

The advantage of considering Morley sequences indexed by arbitrary linear orders (as opposed to just well-orderings) is that they are closed under unions:

Lemma 5.36. Let $\langle I_\alpha : \alpha < \delta \rangle$ be an increasing (with respect to substructure) sequence of linear orders and let $I_\delta := \bigcup_{\alpha<\delta} I_\alpha$. Let M_0, M be χ_2-saturated such that $M_0 \leq M$. Let μ_0, μ_1, μ_2 be such that $\chi_2 < \mu_0 \leq \mu_1 \leq \mu_2$, $p \in \text{gS}(M)$ and for $\alpha < \delta$, let $\mathbf{I}_\alpha := \langle a_i : i \in I_\alpha \rangle$ together with $\langle N_i^\alpha : i \in I_\alpha \rangle$ be (μ_0, μ_1, μ_2)-Morley for p over M_0, with $N_i^\alpha \leq N_i^\beta \leq M$ for all $\alpha \leq \beta < \delta$ and $i \in I_\alpha$. For $i \in I_\alpha$, let $N_i^\delta := \bigcup_{\beta \in [\alpha, \delta)} N_i^\beta$. Let $\mathbf{I}_\delta := \langle a_i : i \in I_\delta \rangle$.

If p does not fork over M_0, then $\mathbf{I}_\delta \frown \langle N_i^\delta : i \in I_\delta \rangle$ is (μ_0, μ_1, μ_2)-Morley for p over M_0.

Proof. By Lemma 5.26, p does not syntactically split over M_0. Therefore the only problematic clauses in Definition 5.11 are (4) and (7). Let's check (4): let $i \in I_\delta$. By hypothesis, $\bar{a}_i$ realizes $p \upharpoonright N_i^\alpha$ for all sufficiently high $\alpha < \delta$. By local character of forking, there exists $\alpha < \delta$ such that $\text{gtp}(\bar{a}_i / N_i^\delta)$ does not fork over N_i^α. Since $\text{gtp}(\bar{a}_i / N_i^\delta) \upharpoonright N_i^\alpha = p \upharpoonright N_i^\alpha$ and p does not fork over $M_0 \leq N_i^\alpha$, we must have by uniqueness that $p \upharpoonright N_i^\delta = \text{gtp}(\bar{a}_i / N_i^\delta)$. The proof of (7) is similar. $\square$

For convenience, we make χ_2 even bigger:

Hypothesis 5.37. Let $\chi := \gamma(\chi_2)$. A Morley sequence means a $(\chi_2^+, \chi_2^+, \chi)$-Morley sequence.

Remark 5.38. By Remark 5.25, we still have $\chi < h(\text{LS}(K))$.

We are finally in a position to prove solvability (in fact even uniform solvability). We will use condition (3) in Proposition 5.4.

Definition 5.39. We define a class of models K' and a binary relation $\leq_{K'}$ on K' as follows.

- K' is a class of $L' := L(K')$-structures, where:

$$L' := L(K) \cup \{N_0, N, F, R \mid i < \chi\}$$

  and for all $i < \chi$:
  - N_0, R, are binary relations symbols.
  - N is a tertiary relation symbol.
  - F is a binary function symbol.
- An L'-structure M is in K' if and only if:
  (1) $M \upharpoonright L(K) \in K^{\chi\text{-sat}}$.
  (2) R^M is a linear ordering on $|M|$. We write I for this linear ordering.

(3) For[13] all $a \in |M|$ and all $i \in I$, $N^M(a,i) \leq M \restriction L(K)$ (where we see $N^M(a,i)$ as an $L(K)$-structure. In particular, $N^M(a,i) \in K$), $N_0^M(a) \leq N^M(a,i)$, $N_0^M(a)$ is saturated of size χ_2.
(4) There exists a map $a \mapsto p_a$ from $|M|$ onto the non-algebraic Galois types over $M \restriction L(K)$ such that for all $a \in |M|$:
   (a) p_a does not fork[14] over $N_0^M(a)$.
   (b) $\langle F^M(a,i) : i \in I\rangle \frown \langle N^M(a,i) : i \in I\rangle$ is a Morley sequence for p_a over $N_0^M(a)$.

- $M \leq_{K'} M'$ if and only if:
  (1) $M \subseteq M'$.
  (2) $M \restriction L(K) \leq M' \restriction L(K)$.
  (3) For all $a \in |M|$, $N_0^M(a) = N_0^{M'}(a)$.

Before checking that K' is an AEC, we show that this accomplishes what we want:

Lemma 5.40. Let $\lambda \geq \chi$.

(1) If $M \in K_\lambda$ is saturated, then there exists an expansion M' of M to L' such that $M' \in K'$.
(2) If $M' \in K'$ has size λ, then $M' \restriction L(K)$ is saturated.

Proof.

(1) Let $R^{M'}$ be a well-ordering of $|M|$ of type λ. Identify $|M|$ with λ. By stability, we can fix a bijection $p \mapsto a_p$ from $\text{gS}(M)$ onto $|M|$. For each $p \in \text{gS}(M)$ which is not algebraic, fix $N_p \leq M$ χ-saturated such that p does not fork over N_p and $\|N_p\| = \chi_2$. Then use saturation to build $\langle a_p^i : i < \lambda\rangle \frown \langle N_p^i : i < \lambda\rangle$ Morley for p over N_p (inside M). Let $N_0^{M'}(a_p) := N_p$, $N^{M'}(a_p, i) := N_p^i$, $F^{M'}(a,i) := a_p^i$. For p algebraic, pick $p_0 \in \text{gS}(M)$ nonalgebraic and let $N_0^{M'}(a_p) := N_0^{M'}(a_{p_0})$, $N^{M'}(a_{p_0}) := N^{M'}(a_{p_0})$, $F^{M'}(a_p) := F^{M'}(a_{p_0})$.
(2) By Lemma 5.33.

□

Lemma 5.41. $(K', \leq_{K'})$ is an AEC with $\text{LS}(K') = \chi$.

[13]For a binary relation Q we write $Q(a)$ for $\{b \mid Q(a,b)\}$, similarly for a tertiary relation.

[14]Note that by Lemma 5.26 this also implies that it does not syntactically split over $N_0^M(a)$.

Proof. It is straightforward to check that K' is an abstract class with coherence. Moreover:

- $\underline{K' \text{ satisfies the chain axioms}}$: Let $\langle M_i : i < \delta \rangle$ be increasing in K'. Let $M_\delta := \bigcup_{i<\delta} M_i$.
  - $M_0 \leq_{K'} M_\delta$, and if $N \geq_{K'} M_i$ for all $i < \delta$, then $N \geq_{K'} M_\delta$: Straightforward.
  - $M_\delta \in K'$: $M_\delta \upharpoonright L(K)$ is χ-saturated by Fact 5.22. Moreover, R^{M_δ} is clearly a linear ordering of M_δ. Write I_i for the linear ordering (M_i, R_i). Condition 3 in the definition of K' is also easily checked. We now check Condition 4.
    Let $a \in |M_\delta|$. Fix $i < \delta$ such that $a \in |M_i|$. Without loss of generality, $i = 0$. By hypothesis, for each $i < \delta$, there exists $p_a^i \in \text{gS}(M_i \upharpoonright L(K))$ not algebraic such that $\langle F^{M_i}(a,j) \mid j \in I_i \rangle \frown \langle N^{M_i}(a,j) \mid j \in I_i \rangle$ is a Morley sequence for p_a^i over $N_0^{M_i}(a) = N_0^{M_0}(a)$. Clearly, $p_a^i \upharpoonright N_0^{M_0}(a) = p_a^0 \upharpoonright N_0^{M_0}(a)$ for all $i < \delta$. Moreover by assumption p_a^i does not fork over $N_0^{M_0}$. Thus for all $i < j < \delta$, $p_a^j \upharpoonright M_i = p_a^i \upharpoonright M_i$. By extension and uniqueness, there exists $p_a \in \text{gS}(M_\delta \upharpoonright L(K))$ that does not fork over $N_0^{M_0}(a)$ and we have $p_a \upharpoonright M_i = p_a^i$ for all $i < \delta$. Now by Lemma 5.36, $\langle F^{M_\delta}(a,j) \mid j \in I_\delta \rangle \frown \langle N^{M_\delta}(a,j) \mid j \in I_\delta \rangle$ is a Morley sequence for p_a over $N_0^{M_0}(a)$.
    Moreover, the map $a \mapsto p_a$ is onto the nonalgebraic Galois types over $M_\delta \upharpoonright L(K)$: let $p \in \text{gS}(M_\delta \upharpoonright L(K))$ be nonalgebraic. Then there exists $i < \delta$ such that p does not fork over M_i. Let $a \in |M_i|$ be such that $\langle F^{M_i}(a,j) \mid j \in I_i \rangle \frown \langle N^{M_i}(a,j) \mid j \in I_i \rangle$ is a Morley sequence for $p \upharpoonright M_i$ over $N_0^{M_i}(a)$. It is easy to check it is also a Morley sequence for p over $N_0^{M_i}(a)$. By uniqueness of the nonforking extension, we get that the extended Morley sequence is also Morley for p, as desired.
- $\underline{\text{LS}(K') = \chi}$: An easy closure argument.

$\square$

Theorem 5.42. K is uniformly (χ, χ)-solvable.

Proof. By Lemma 5.41, K' is an AEC with $\text{LS}(K') = \chi$. Now combine Lemma 5.40 and Proposition 5.4. Note that saturated models of size at least χ_0 are superlimit by Lemma 5.22, and K has arbitrarily large saturated models by superstability. $\square$

For the next theorems, we drop our hypotheses.

Theorem 5.43. If K is $\mathrm{LS}(K)$-superstable with amalgamation and $(< \mathrm{LS}(K))$-tame, then there exists $\theta < h(\mathrm{LS}(K))$ such that K is uniformly (θ, θ)-solvable.

Proof. By Fact 2.10, Remark 5.38, and Theorem 5.42 . □

Theorem 5.44. Let K be a stable tame AEC with amalgamation. The following are equivalent.

(1) For all high-enough λ, K is λ-superstable.
(2) There exists θ such that K is uniformly (θ, θ)-solvable.
(3) There exists θ such that for all high-enough λ, K is (λ, θ)-solvable.

Proof. (1) implies (2) is Theorem 5.43, (2) implies (3) follows directly from the definition of solvability, and (3) implies (1) is because (3) implies that there is a superlimit in all high-enough λ and by Theorem 4.8, this implies superstability. □

6. Superstability below the Hanf number

In this section, we show that we can require "$\mathrm{LS}(K) \leq \mu_\ell < h(\mathrm{LS}(K))$" in Theorem 1.2 (provided the class is $(< \mathrm{LS}(K))$-tame). While this improves on some bounds e.g. in Section 4, the arguments are harder.

Theorem 6.1. Let K be a $(< \mathrm{LS}(K))$-tame AEC with amalgamation, joint embedding, and arbitrarily large models. Assume K is stable. Then the following are equivalent:

(1) There exists $\mu_1 < h(\mathrm{LS}(K))$ such that for every $\lambda \geq \mu_1$, for all $\delta < \lambda^+$, for all increasing continuous $\langle M_i : i \leq \delta \rangle$ in K_λ and all $p \in \mathrm{gS}(M_\delta)$, if M_{i+1} is universal over M_i for all $i < \delta$, then there exists $i < \delta$ such that p does not λ-split over M_i.
(2) There exists $\mu_2 < h(\mathrm{LS}(K))$ such that for every $\lambda \geq \mu_2$, for some $\kappa \leq \lambda$, there is a good λ-frame on $K_\lambda^{\kappa\text{-sat}}$.
(3) There exists $\mu_3 < h(\mathrm{LS}(K))$ such that for every $\lambda \geq \mu_3$, K has uniqueness of limit models in cardinality λ.
(4) There exists $\mu_4 < h(\mathrm{LS}(K))$ such that for every $\lambda \geq \mu_4$, K has a superlimit model of cardinality λ.
(5) There exists $\mu_5 < h(\mathrm{LS}(K))$ such that for every $\lambda \geq \mu_5$, the union of a chain of λ-saturated models is λ-saturated.
(6) There exists $\mu_6 < h(\mathrm{LS}(K))$ such that for all $\lambda \geq \mu_6$, K is (λ, μ_6)-solvable.

Proof. By Theorem 5.43 (and being slightly more careful with the bounds, i.e. using [BV, Remark 6.12]), (1) implies (6). Moreover (6) implies (4) by definition. Thus it is enough to show the equivalence of (1)-(5).

We now revisit the proof of Theorem 4.8. Now the proofs of (1) implies (2)-(5) there show that we can take $\mu_2, \dots, \mu_5 < h(\text{LS}(K))$. Similarly for (2) implies (3) and (5) implies (4). Now we show that both (3) and (4) imply (5^-), which is the following statement:

(5^-) For unboundedly many cardinals $\lambda < h(\text{LS}(K))$, the union of a chain of λ-saturated models is λ-saturated.

If (3), then Lemma 3.10 gives (5^-). If (4), then let $\chi < h(\text{LS}(K))$ be as given by Fact 3.11. Without loss of generality, $\chi \geq \mu_4$. Let λ be a successor cardinal such that $\lambda = \lambda^{<\chi}$ (note that there are unboundedly many such λ below $h(\text{LS}(K))$). By Fact 5.19, K is stable in λ and in unboundedly many $\mu < \lambda$ (namely in its predecessor). By regularity, K must have a saturated model of size λ. By Lemma 3.8, chains of λ-saturated models in K_λ have a λ-saturated union. Using that λ is a successor, it is straightforward to check that this implies that the union of a chain of λ-saturated models is λ-saturated. Hence we get (5^-).

It remains to show that (5^-) implies (1). We follow the beginning of Section 5 (before Hypothesis 5.35). Let $\chi_0, \chi_0', \chi_1, \chi_1', \chi_2$ be as there. Take $\lambda \in (\chi_2, h(\text{LS}(K)))$ such that (5^-) holds. Let $\lambda' := (\lambda^{\chi_0})^+$. Note that (by Fact 5.19) K is stable in both λ^{χ_0} and λ'.

Now let $\delta < (\lambda')^+$ and let $\langle M_i : i < \delta \rangle$ be an increasing chain of saturated models in $K_{\lambda'}$. Let $M_\delta := \bigcup_{i<\delta} M_i$. Let $p \in \text{gS}(M_\delta)$. As in the proof of Theorem 4.8, it is enough to show that there exists $i < \delta$ such that p does not fork over M_i. Without loss of generality, δ is regular. If $\delta \geq \chi_2$, by set local character there exists M_0' of size χ_1 such that p does not fork over M_0' and $M_0' \leq M_\delta$, so pick $i < \delta$ such that $M_0' \leq M_i$ and use base monotonicity.

Now assume $\delta < \chi_2$. By definition of λ, we have that M_δ is λ-saturated. We also have that p does not fork over M_δ so by Lemma 5.26, there exists $\mu \in [\chi_0^+, \chi_2)$ and $\mathbf{I}$ a $(\mu, \mu, \gamma(\mu)^+)$-Morley sequence for p with all the witnesses inside M_δ such that $\text{Av}_{\gamma(\mu)}(\mathbf{I}/M_\delta) = p$. Since M_δ is λ-saturated, we can increase $\mathbf{I}$ if necessary to assume $|\mathbf{I}| \geq \lambda$. Write $\mathbf{I}_i := |M_i| \cap \mathbf{I}$. Since $\delta < \chi_2 < \lambda$, there must exists $i < \delta$ such that $|\mathbf{I}_i| \geq \chi_2$. Note that $\mathbf{I}_i$ is a (μ, μ, χ_2)-Morley sequence for p. Because $\mathbf{I}$ is $\gamma(\mu)$-convergent and $|\mathbf{I}_i| \geq \chi_2 > \gamma(\mu)$, $\text{Av}_{\gamma(\mu)}(\mathbf{I}_i/M_\delta) = p$. Letting $M' \geq M_\delta$ be a saturated model of size λ' and using local extension over saturated models (Fact 5.21), $p \upharpoonright M_i$ has a nonforking extension to $\mathrm{gS}(M')$ and hence to $\mathrm{gS}(M_\delta)$. By Lemma 5.26, p does not fork over M_i, as desired[15]. □

The proof does not tell us if there is a Hanf number for superstability, namely:

Question 6.2. Let K be a $(< \mathrm{LS}(K))$-tame AEC with amalgamation which is λ-superstable for some $\lambda \geq h(\mathrm{LS}(K))$. Is K μ-superstable for some $\mu < h(\mathrm{LS}(K))$?

We end by improving Fact 2.10.(2). Recall that this tells us that (in tame AECs with amalgamation) superstability follows from categoricity in a high-enough cardinal. We give an improvement that does not use tameness and improves the bound on the categoricity cardinal. Even though all the ingredients are contained in [SV99], this has not appeared in print before.

Theorem 6.3 (The ZFC Shelah-Villaveces theorem)**.** Let K be an AEC with arbitrarily large models and amalgamation in $\mathrm{LS}(K)$[16]. Let $\lambda > \mathrm{LS}(K)$ be such that $K_{<\lambda}$ has no maximal models. If K is categorical in λ, then K is $\mathrm{LS}(K)$-superstable.

Proof. Set $\mu := \mathrm{LS}(K)$. In the proof of [SV99, Theorem 2.2.1], in (c), ask that $\sigma = \chi$, where χ is the least cardinal such that $2^\chi > \mu$. The proof that (c) cannot happen goes through, and the rest only uses amalgamation in μ. □

Corollary 6.4. Let K be an AEC with amalgamation in $\mathrm{LS}(K)$. If K is categorical in a $\lambda \geq h(\mathrm{LS}(K))$, then there exists $\lambda_0 < h(\mathrm{LS}(K))$ such that K is μ-superstable in all $\mu \in [\lambda_0, \lambda)$ where K_μ has amalgamation.

Proof. Combine Theorem 6.3 with [Vasb, Proposition 10.13] (the argument uses only amalgamation in $\mathrm{LS}(K)$). □

We can use the ZFC Shelah-Villaveces theorem to prove the following interesting result, showing that the solvability spectrum satisfies an analog of Shelah's categoricity conjecture in tame AECs (Shelah conjectures that this should hold in general, see Question 4.4 in the introduction to [She09a]). For notational purpose, we introduce one more definition:

[15] A similar argument already appears in the proof [She09a, Theorem IV.4.10]

[16] In [SV99], this is replaced by GCH.

Definition 6.5. K is *λ-solvable* if it is (λ, θ)-solvable for some $\theta < h(\mathrm{LS}(K))$.

Theorem 6.6. Let K be a $(< \mathrm{LS}(K))$-tame AEC with amalgamation. If K is λ-solvable for some $\lambda \geq h(\mathrm{LS}(K))$, then there exists $\theta < h(\mathrm{LS}(K))$ such that:

(1) K is θ-superstable.
(2) K is (μ, θ)-solvable for all $\mu \geq \theta$.

The idea of the proof is simple: we show that solvability in one cardinal implies superstability and then use Theorem 5.43 to get solvability in the other cardinals. To see that solvability implies superstability, we examine the proof of the ZFC Shelah-Villaveces theorem. There, as in many arguments involving categoricity and EM models, the full power of categoricity in λ is not used. Rather, all that is used is that there is a unique (up to isomorphism) EM model of size λ, and that every model in $K_{\leq \lambda}$ embeds into an EM model. Solvability in λ implies these two conditions (because the superlimit model is unique by Proposition 3.7 and universal by definition).

Proof of Theorem 6.6. Let $\theta_0 < h(\mathrm{LS}(K))$ be such that K is (λ, θ_0)-solvable. First observe that K_λ has joint embedding, as any superlimit model is universal. Therefore (e.g. by [Vasb, Proposition 10.13]), there exists $\chi_0 < h(\mathrm{LS}(K))$ such that $K_{\geq \chi_0}$ has joint embedding and no maximal models. Without loss of generality, $\chi_0 = \theta_0$. By the standard argument (see for example [Bal09, Theorem 8.21]), $K_{\geq \theta_0}$ is stable in all $\mu \in [\theta_0, \lambda)$. By the proof of Theorem 6.3, K is θ_0-superstable, and thus by Fact 2.10.(1) θ-superstable for any $\theta \geq \theta_0$. By Theorem 5.43 (we have to be slightly more careful with the bound on χ, see [BV, Remark 6.12]), there exists $\theta < h(\mathrm{LS}(K))$ with $\theta \geq \theta_0$ such that K is uniformly (θ, θ)-solvable, hence by definition (μ, θ)-solvable for all $\mu \geq \theta$. □

Remark 6.7. Since we required the starting solvability parameter θ_0 to be below $h(\mathrm{LS}(K))$, this does not quite answer Question 6.2.

7. Future work

While we managed to prove that some analogs of the conditions in Fact 1.1 are equivalent, much remains to be done. For example, even in tame AECs with amalgamation, we do not know whether stability on a tail of cardinals or having a saturated model on a tail of cardinals should imply superstability (although superstability certainly implies these). This would be a useful tool to check that specific examples are superstable.

Another direction would be to make precise what the analog to (5) and (6) in 1.1 should be in tame AECs. One possible definition for (6) would be:

Definition 7.1. Let λ and μ be cardinal numbers. We say that K has the *(λ, μ)-tree property* provided there exists $\{p_n(\mathbf{x}; \mathbf{y}_n) \mid n < \omega\}$ Galois-types such that $|\text{dom}(p_n)| < \mu$ and there exists $\{M_\eta \mid \eta \in {}^{\leq\omega}\lambda\}$ such that for all $n < \omega, \nu \in {}^{n}\lambda$ and every $\eta \in {}^{\omega}\lambda$:

$$\langle M_\eta, M_\nu \rangle \models p_n \iff \nu \text{ is an initial segment of } \eta.$$

We say that K has the *tree property* if it has it for all high-enough μ and all high-enough λ (where the "high-enough" quantifier on λ can depend on μ).

We can ask whether superstability implies that K does not have the tree property, or at least obtain many models from the tree property as in [GS86]. This is conjectured in [She99] (see the remark after Claim 5.5 there).

As for the D-rank in (5), perhaps a simpler analog would be the U-rank defined in terms of $(< \kappa)$-satisfiability in [BG, Definition 7.2] (another candidate for a rank is Lieberman's R-rank, see [Lie13]).

Definition 7.2. Let K be a $(< \text{LS}(K))$-tame AEC with amalgamation. Let $\kappa > \text{LS}(K)$ be least such that $\kappa = \beth_\kappa$ (for concreteness). We define a map U with domain a type over κ-saturated models and codomain an ordinal or ∞ inductively by, for $p \in \text{gS}(M)$:

(1) Always, $U[p] \geq 0$.
(2) For α limit, $U[p] \geq \alpha$ if and only if $U[p] \geq \beta$ for all $\beta < \alpha$.
(3) $U[p] \geq \beta + 1$ if and only if there exists a κ-saturated $M' \geq M$ with $\|M'\| = \|M\|$ and an extension $q \in \text{gS}(M')$ of p such that q is not $(< \kappa)$-satisfiable over M and $U[q] \geq \beta$.
(4) $U[p] = \alpha$ if and only if $U[p] \geq \alpha$ and $U[p] \not\geq \alpha + 1$.
(5) $U[p] = \infty$ if and only if $U[p] \geq \alpha$ for all ordinals α.

By [BG, Theorem 7.9], superstability implies that the U-rank is bounded but we do not know how to prove the converse. Perhaps it is possible to show that $U = \infty$ implies the tree property.

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