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A Two-phase Iterative Algorithm for Improved Approximation by Szasz Operator Using Statistical Perspectives

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Abstract

This paper aims at constructing a two-phase iterative numerical algorithm for the improved approximation of a continuous function by the 'Modified Szasz' operator. The algorithm uses a 'statistical perspective' to more fully expoit the information about the unknown function f. The improvement occurs iteratively. A typical iteration uses the twin statistical concepts of 'Mean Square Error' (MSE) and 'Bias'; the application of the latter concept being preceded by that of the former in the algorithm. At any iteration, the statistical concept of 'MSE' is used in "Phase II", after that of the 'Bias' in "Phase I". The procedure is like a sandwich. The top and bottom slices are the operations of 'Bias-Reduction' in "Phase I" of the algorithm, and the operation of 'MSE-Reduction' in "Phase II" is the stuffing in the sandwich. The improvement acheived by this algorithm is evaluated by means of a simulation study using known functions. The simulation has been confined to three iterations only, for the sake of simplicity.

Keywords: Approximation, Bernstein operator, Simulated empirical study

1. Introduction

Szasz(1950) proposed the following generalization of the well-known Bernstein polynomial approximation operator extending it to an infinite interval.

$$S\left(f;x\right)=\exp\left(-nx\right)\sum_{k=0}^{k=\infty}\frac{(nx)^k}{k!}.f\left(\frac{k}{n}\right)\ \forall f\epsilon C[0,\infty) \tag{1}$$

Heinz-Gerd Lehnhoff (1981) proposed the "Modified Szasz-Mirakjan Operator" as follows:

$$S_n\left(f;x\right)=\frac{\sum_{k=0}^{k=n}T_k\left(x\right)f\left(\frac{k}{n}\right)}{\sum_{k=0}^{k=\infty}T_k\left(x\right)}.\forall x\epsilon C\left[0,1\right]\ \text{and}\ f\epsilon C\left[0,1\right] \tag{2}$$

In which,

$$T_k\left(x\right)=\frac{(nx)^k}{k!}$$

for all non-negative integer values of $'k'$. Motivated by the above, we propose the following modification of the Szasz operator:

$$MS_n\left(f;x\right)=\frac{\sum_{k=0}^{k=n}T_k\left(x\right).f\left(\frac{k}{n}\right)}{\sum_{k=0}^{k=n}T_k\left(x\right)}.\ \forall x\epsilon C\left[0,1\right]\ \text{and}\ f\epsilon C\left[0,1\right] \tag{3}$$

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This modification is a more appropriate one inasmuch as"$MS_n(f; x)$" may be interpreted as the weighted average of the $(n + 1)$ known values of the unknown function $f(x)$, namely '$f(k/n)$';$k = 0(1)n$; "Weights" being "$T_k(x)$". In fact, the $T_k(x)s$ could be interpreted as "proportional to probabilities" ["$T_k(x) \geq 0$"]. Using this interpretation, therefore,

$$MS_n(f; x) = E(f(x)) \tag{4}$$

Incidentally, as we could use a suitable transformation (translation and change of scale) of the variable x, we could assume, without any loss of generality, that we are interested in the approximation of a function in $C[0, 1]$, even if the original function of interest was actually in $C[a, b]$.

2. The Two-Phase Iterative Improvement Algorithm for Modified Szasz Operators $\mathbf{MS_n(f; x)}$

In this section we propose the "Two-Phase Iterative Improvement Algorithm for Modified Szasz Operators $MS_n(f; x)$" using the 'TWIN' statistical perspectives of 'Bias' & 'MSE'. In the statistical sense, $MS_n(f; x)$ is an estimate of the unknown function $f(x)$'. Now, we use our 'estimator' (the modified Szasz Operators $MS_n(f; x)$) to 'estimate' the values of the unknown function $f(x)$ at the knots (k/n),say $Etf(k/n)$, $k = 0(1)n$, and compare these values with the values of the unknown function "$f(x)$", namely '$f(k/n)$'; $k = 0(1)n$. Hence the error at the nth knot ", say $Erf(k/n) \equiv Etf(k/n) - f(k/n), k = 0(1)n$ could be generated enabling us to construct the "Bias Polynomial Function", say

$$ErS_n(f; x) = \frac{\sum_{k=o}^{k=n} T_k(x)\, Erf\left(\frac{k}{n}\right)}{\sum_{k=0}^{k=n} T_k(x)} \tag{5}$$

On the other hand the "Modified Szasz Polynomial" approximation/estimator of the unknown function '$f(x)$' is "$MS_n(f; x)$", as per the equation (3) in the preceding section. This enables us to achieve per our "Phase I" of the iterative algorithm, the "Reduced-Bias Polynomial" approximation / estimator of the unknown function "$f(x)$" just by subtracting the "Estimated Bias Polynomial" per (5) above to get:

$$O_n(f; x) = MS_n(f; x) - ErS_n(f; x) \tag{6}$$

Now, we embark upon the "Phase II" of our proposed 'Iterative Algorithm'. The concept "Minimum Mean Square Error Estimator (MMSEE)"of Searles(1964) is seminal to this phase of our algorithm. As per (1.4), our "Modified Szasz Polynomial" estimator is analogous to the sample-mean '$\bar{x}$'. Searles(1964) considered the class of estimators $k.\bar{x}$, and chose the "Optimal" value say "k_0" by minimizing the MSE ($k\bar{x}$) to lead to the MMSEE ($k_0.\bar{x}$). Similarly, we consider the perturbed 'Polynomial', say b. $O_n(f; x)$, and hence determine the estimated values of the unknown function '$f(x)$' at the knots '(k/n)', say $Etf(k/n), k = 0(1)n$, vis-à-vis known values of the unknown function "$f(x)$", namely '$f(k/n)$'; $k = 0(1)n$. Hence the "Knot-Wise Squared-Error", say$E2rf(k/n) \equiv [Etf(k/n) - f(k/n)]^2$, $k = 0(1)n$ could be generated to lead to the construction of the "Squared-Error Polynomial Function",

$$E2r = \sum_{k=0}^{k=n} E2rf\left(\frac{k}{n}\right) \tag{7}$$

This will be a "Quadratic Polynomial in b", say $Q(b) \equiv A.b^(2) + B.b + C$. To avoid any complex solution to $Q(b) = 0$, we chose $b_0 = -(B/2.A)$ to minimize the value of MSE, leading to a 'Reduced-MSE Polynomial' estimator "$b_0.O_n(f; x)$".

To complete the "FIRST Iteration" we again apply the details of the 'Phase I' to treat our 'Reduced-MSE Polynomial' estimator "$b_0.O_n(f; x)$", to achieve the improved [at Iteration # 1] 'Modified Szasz Polynomial' Operator/Estimator:

$$I[\#1]MSn(f; x) \equiv [Reduced - BiasVersionUsing`PhaseI\prime(Iteration\#1)on``b_0.O_n(f; x)"] \tag{8}$$

Thus operations defining "FIRST Iteration" could well be characterized as a "Sandwich"! The top and bottom slices are the operations of 'Bias-Reduction' per the "Phase I" of our algorithm, and the operation of 'MSE-Reduction' per the "Phase II" is the stuffing in the sandwich. The algorithm is an iterative one amounting to a stacked pile of sandwiches of desired height with the bottom slice of the first iteration serving as the top slice for the second-iteration sandwich, and so on.

At any iteration, the improvements will begin and end with the "Phase I" operation of the 'Bias-Reduction' of the improvement algorithm, sandwiching its "Phase II" operation of 'MSE-Reduction'. As such, at any iteration, we will have two improvement-operations only, namely that of 'Phase II' followed by that of "Phase I" borrowing the last operation of the preceding iteration. Only the "First Iteration" will, therefore be an exception using three improvement-operations Phase I –Phase II – Phase I.

3. The empirical simulation study

To illustrate the gain in efficiency of the "Modified Szasz Operators" by using our proposed "Sandwich-Iterative Algorithm of Improvement of Polynomial Approximation, we have carried out an empirical study. We have taken the cases of $n = 2, 3,$ and $4 (i.e. n + 1 = 3, 4,$ and $5,$ knots) in the empirical study to numerically illustrate the relative gain in efficiency in using the Algorithm vis-à-vis the Original Modified Szasz Polynomial Operator in each example-case of the n-values. Essentially, the empirical study is a simulation one in which we assume that the function to be approximated, namely $f(x)$, is known to us. Once again we have confined ourselves to illustrating the relative gain in efficiency by the Iterative Improvement for the following four functions:

$$f(x) = exp(x), ln(1 + x), sin(1 + x), 5^x$$

.

To illustrate the potential improvement with our proposed Algorithm, we have considered THREE Iterations, and the numerical values of seven quantities - three Percentage Relative Errors ($PREs$) corresponding to Improvement Iteration ($\# = 1, or\ 2, or\ 3$) ($PRE\_I(\#)MS n(f; x)[n]$), Original Modified Szasz Polynomial Operator ($PRE\_MS n(f; x)[n]$), and the three corresponding Percentage Relative Gains (PRG) in using our Iterative Algorithmic Modified Szasz Polynomial Operators in place of the Original Modified Szasz Polynomial Operators $MS n(f; x)[n]$, namely $PRG\_I(\#)MS n(f; x)[n]; \# = 1(1)3$). These quantities are defined as follows. The PRE using (original) Modified Szasz (Polynomial) using n intervals in $[0, 1]$, i.e. $[(k - 1)/n, k/n]; k = 1(1)n$:

$$PRE\_MS_n(f; n)[n] = \frac{\int_0^1 f(x)\,dx - \int_0^1 MS_n(f; x)}{\int_0^1 f(x)\,dx} \times 100$$

The PRE using the Improvement Iteration ($I\#1, or2, or3$) on Bernstein (Polynomial) using n intervals in $[0, 1]$, i.e. $[(k - 1)/n, k/n]; k = 1(1)n$:

$$PRE\_I(\#)\,MS n(f; x)[n] = \frac{\left|\int_0^1 f(x)\,dx - \int_0^1 I(\#)\,MS_n(f; x)\,dx\right|}{\int_0^1 f(x)\,dx} \times 100;\ \text{where}\ \# = 1, 2\ or\ 3$$

The $PREs$ respective to the Original Modified Szasz Polynomial Operator and respective to the First, Second, and the Third Sandwich-Algorithmic Improvement Iteration Polynomials, respectively, for each of for each of the example $\#'s$ of approximation Knots/Intervals. And the Percentage Relative Gains ($PRGs$), defined exactly analogously to PRE, by using the proposed Sandwich-Algorithmic Improvement Iteration: $I\#(e.g. 1, 2\ or 3)$ Polynomials with the n intervals in $[0, 1]$ over using the (Original) Modified Szasz Polynomial Operator for the approximation of the (Targeted) function, f are tabulated in the following four tables (Tables 1 to Table 4) in the Appendix.

4. Conclusion

These seven numerical quantities have been computed using Maple Release 12, for all the four illustrative functions ($exp(x)$, $ln(1 + x)$, $sin(1 + x)$, and 5^x) mentioned in Section 3, and for three values of n ($n = 2, 3, 4$), and are, respectively, tabulated in Tables 1–4 [Appendix]. The $PREs$ for our Algorithmic Sandwich-Iterative Polynomial Approximations are PROGRESSIVELY lower on each subsequent iteration, as compared to that for the Original Modified Szasz Polynomial Approximation, for all the illustrative functions. The $PRGs$ due to the use of our proposed Algorithmic Sandwich-Iterative Polynomial Approximations vis-a-vis the Original Modified Szasz Polynomial Approximation are also PROGRESSIVELY increasing on each subsequent iteration, for all the illustrative functions. Lastly, it is very heartening to note that when we use ($n = 4$) intervals, i.e. FIVE KNOTS for the polynomial approximation, the PRG becomes almost 100% for the third iteration in the function ln(1+x). Otherwise also, the speed of convergence is highly accelerated by the Sandwich-Iterative Algorithmic improvement in the Modified Szasz Polynomial using the Statistical perspective. In fact, the improvement attained by the third iteration is not very great. It could also be noted that this perspective of the Sandwich-Iterative Improvement could be applied to any Polynomial Approximator, other than Modified Szasz Polynomial; more particularly to those belonging to the class of Positive Linear Operators, as they admit to the Probabilistic/Statistical perspective rather more readily!

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APPENDIX

Table 1. (Iterative) Algorithmic ($In\%$) Relative (Absolute) Efficiency/Gain for $f(x) = \exp(x)$.

Items↓	$n \to 2$	3	4
$PRE\_MS_n(f;x)[n]$	9.39739449	7.95647383	6.90804442
PRE_I (1) $MS_n(f;x)[n]$	3.83722333	1.45873986	0.81395291
PRE_I (2) $MS_n(f;x)[n]$	3.81174269	1.42485473	0.77875007
PRE_I (3) $MS_n(f;x)[n]$	3.19966674	1.19674169	0.73548674
PRG_I (1) $MS_n(f;x)[n]$	59.16715709	81.66600061	88.21731791
PRG_I (2) $MS_n(f;x)[n]$	59.43830285	82.09188183	88.72690987
PRG_I (3) $MS_n(f;x)[n]$	65.95155448	84.95889365	89.35318456

Table 2. (Iterative) Algorithmic ($In\%$) Relative (Absolute) Efficiency/Gain for $f(x) = \ln(1+x)$.

Items↓	$n \to 2$	3	4
$PRE\_MS_n(f;x)[n]$	29.15997146	21.87280192	17.61951884
PRE_I (1) $MS_n(f;x)[n]$	3.55783342	1.47804100	0.77459701
PRE_I (2) $MS_n(f;x)[n]$	2.97431401	1.09840220	0.60508025
PRE_I (3) $MS_n(f;x)[n]$	2.45230985	0.81111180	0.49703697
PRG_I (1) $MS_n(f;x)[n]$	87.79891320	93.24256172	95.60375611
PRG_I (2) $MS_n(f;x)[n]$	89.80001056	94.97822801	96.56585259
PRG_I (3) $MS_n(f;x)[n]$	91.59015000	96.29168767	97.17905481

Table 3. (Iterative) Algorithmic ($In\%$) Relative (Absolute) Efficiency/Gain for $f(x) = \sin(1+x)$.

Items↓	$n \to 2$	3	4
$PRE\_MS_n(f;x)[n]$	5.07268191	3.29085233	2.36884255
PRE_I (1) $MS_n(f;x)[n]$	1.70490640	0.65066116	0.33024595
PRE_I (2) $MS_n(f;x)[n]$	1.25881427	0.48450137	0.27943602
PRE_I (3) $MS_n(f;x)[n]$	0.98135020	0.41174432	0.27192705
PRG_I (1) $MS_n(f;x)[n]$	66.39043343	80.22818721	86.05876299
PRG_I (2) $MS_n(f;x)[n]$	75.18444304	85.27732875	88.20368925
PRG_I (3) $MS_n(f;x)[n]$	80.65421371	87.48821619	88.52067832

Table 4. (Iterative) Algorithmic ($In\%$) Relative (Absolute) Efficiency/Gain for $f(x) = 5^x$.

Items↓	$n \to 2$	3	4
$PRE\_MS_n(f;x)[n]$	11.37181326	10.38792657	9.38887683
PRE_I (1) $MS_n(f;x)[n]$	8.86903506	3.76606866	2.10583495
PRE_I (2) $MS_n(f;x)[n]$	8.01359809	3.36536766	2.00297630
PRE_I (3) $MS_n(f;x)[n]$	6.48771809	2.79971107	1.92047301
PRG_I (1) $MS_n(f;x)[n]$	22.00861142	63.74571350	77.57095984
PRG_I (2) $MS_n(f;x)[n]$	29.53104388	67.60308571	78.02744306
PRG_I (3) $MS_n(f;x)[n]$	42.94913267	73.04841289	79.54523158