Depth

Abstract: Depth functions represent a recently emerging powerful methodology in nonparametric multivariate inference. They provide multivariate notions of order statistics and generate quantile contours, outlyingness functions, and sign and rank functions. There are wide possibilities for constructing such functions, but these narrow considerably when criteria such as affine equivariance or invariance, computational ease, asymptotic behavior, and robustness are applied. We review basic definitions, leading examples, key properties, and selected applications.

Keywords: Multivariate; Nonparametric; Depth functions; Quantiles; Outlyingness functions; Signs and ranks

Introduction

Univariate nonparametric analysis relies heavily on *signs and ranks*, *order statistics*, *quantiles*, and *outlyingness functions*. These have natural and straightforward formulations, based on the natural linear order in the real line $\mathbb{R}$. However, in passing from univariate to *multivariate* statistical analysis, especially for the purpose of *nonparametric* approaches, various issues and special considerations come into play.

Let us first observe that in $\mathbb{R}^d$ for $d \geq 2$ there is no natural order and therefore no straightforward extension of the above concepts. It also should be noted that simple coordinatewise methods fail to capture properly the relevant geometry. For example, whereas the median of a univariate data set represents a notion of "center", the d-vector of coordinatewise medians can lie outside the convex hull of the data. Also, although the vector of means provides a useful alternative tool, it does not represent "center" in the sense of quantile-based inference.

Depth functions $D(\boldsymbol{x})$ constructively solve this "problem" by introducing a notion of "center" as the maximal-depth point and providing a center-outward ordering of points $\boldsymbol{x}$ in $\mathbb{R}^d$. Nested contours of points of equal depth play many useful roles. Many interesting approaches toward construction of suitable depth functions have been put forth, beginning with the seminal paper of Tukey [25].

Visualization of data depth contours in $\mathbb{R}^d$ is not feasible except for very low d. Hence the understanding of how a particular depth function is performing is based essentially on its conceptual mathematical formulation, its theoretical properties, and its computational complexity with increasing dimension d.

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Besides the fundamental roles of defining "center" in $\mathbb{R}^d$ and providing measures of centrality of points $\boldsymbol{x}$ in $\mathbb{R}^d$, depth functions also offer effective new ways to extract relevant information from multivariate data. They generate associated outlyingness functions, quantile functions, and sign and rank functions, as well as descriptive measures for dispersion, skewness, and kurtosis. Some important features of higher dimensional distributions can be captured via one-dimensional curves based on associated depth contours.

In the following, we formally define depth functions and associated outlyingness, quantile, sign, and rank functions. Several important depth functions are discussed in detail. Desirable properties of depth functions are described, and some illustrative techniques of application of depth functions are mentioned.

The present exposition is merely a sketch and cannot encompass the full range and richness of depth and related functions and their applications. Further broad perspectives can be found in [12], [29], [17], and [22]. The use of depth functions for descriptive statistics in hydrology is described in [2] and vnn044. A timely search of the literature will yield additional "depths" and a great variety of other applications.

Together with related outlyingness, quantile, and rank functions, depth functions are developing into a powerful new methodology in nonparametric multivariate analysis, encompassing exploratory data analysis, regression, classification, clustering, functional data analysis, and other contexts of application.

Depth and Outlyingness Functions

Definitions and Examples

Depth Functions Associated with a given distribution F on $\mathbb{R}^d$, a *depth function* $D(\boldsymbol{x}, F)$ is constructed to provide an F-based *center-outward ordering* of points $\boldsymbol{x}$ in $\mathbb{R}^d$. *High* depth corresponds to "centrality", and thus the point or set of points $\boldsymbol{x}$ globally maximizing $D(\boldsymbol{x}, F)$ represents a notion of "center".

The distinction between depth and density contours should be noted. For a probability distribution F on $\mathbb{R}^d$, the associated probability density function $f(\boldsymbol{x})$ characterizes probability mass in a neighborhood of $\boldsymbol{x}$ (a local characteristic). A depth function $D(\boldsymbol{x}, F)$, on the other hand, measures the *centrality* of $\boldsymbol{x}$ in the given probability distribution (a global characteristic). For example, for a uniform distribution on the unit square in $\mathbb{R}^2$, the density function is constant with no contours of equal density, whereas typical depth functions peak at a center, decrease in directions away from the center, and provide useful nested contours of equal-depth points (see **Figures 1** and **2**). In some cases of F and $D(\cdot, F)$, for example ellipsoidal F and typical $D(\cdot, F)$, the associated families of density and depth contours actually coincide, but even then their labels and their interpretations are different.

Outlyingness Functions An *outlyingness function* $O(\boldsymbol{x}, F)$ provides an F-based *center-outward ordering* of points $\boldsymbol{x}$ in $\mathbb{R}^d$ with high values corresponding to "outlyingness" and $O(\boldsymbol{x}, F)$ increasing outwardly from the "center" (minimal outlyingness point).

Depth functions D and outlyingness functions O correspond to each other through one-to-one relations, for example $D = a + bO$ or $D = 1/(1 + O)$. The two types of function generate each other (inversely) and merely represent two points of view about the same entity.

For a data set $\mathbb{X}_n = \{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\}$ in $\mathbb{R}^d$, the depth and outlyingness functions $D(\boldsymbol{x}, F)$ and $O(\boldsymbol{x}, F)$ induce sample versions $D(\boldsymbol{x}, \mathbb{X}_n)$ and $O(\boldsymbol{x}, \mathbb{X}_n)$ and an associated center-outward *ranking* of the data points.

Below we examine the most important five depth/outlyingness functions. Both conceptually and structurally, they are quite different in design, and their sample versions present quite different theoretical and computational challenges.

Example: Halfspace (Tukey) Depth Although outlyingness functions have been in use for a long time, notably using the Mahalanobis distance, for example, the first instance of orienting specifically to *centrality* and without restriction to ellipsoidal contours is the *halfspace depth* (Tukey [25]): for $\boldsymbol{x} \in \mathbb{R}^d$,

$$D_{\mathrm{H}}(\boldsymbol{x}, F) = \inf\{F(H) : \boldsymbol{x} \in H \text{ closed halfspace}\},$$

the minimal probability attached to any closed halfspace with $\boldsymbol{x}$ on the boundary. In particular, the *sample* halfspace depth of $\boldsymbol{x}$ is the minimum fraction of data points in any closed halfspace containing $\boldsymbol{x}$. See **Figure 1** for illustration showing also the difference from density contours. The halfspace depth has been studied in detail by Donoho and Gasko [4], Nolan [18], and Massé [15], [16]. Extended versions are formulated by Zhang [27]. Some new perspectives are provided by Kong and Mizera [9], Hallin, Paindaveine, and Šiman [8], Dang and Serfling [3], Serfling and Zuo [24], and Dutta, Ghosh, and Chaudhuri [6].

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Example: Simplicial Depth The wide potential scope of depth functions became clear with the introduction of an important second one, the *simplicial depth* (Liu [10]): for $\boldsymbol{x} \in \mathbb{R}^d$,

$$D_{\mathrm{S}}(\boldsymbol{x}, F) = P(\boldsymbol{x} \in S[\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{d+1}]),$$

where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{d+1}$ represent independent observations from F and $S[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}]$ denotes the simplex in $\mathbb{R}^d$ with vertices $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}$, that is, the set of points in $\mathbb{R}^d$ that are convex combinations of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}$. For a data set in $\mathbb{R}^2$, the sample simplicial depth of a point $\boldsymbol{x}$ is obtained by considering all triangles formed with three data points as vertices and taking the fraction of them that cover $\boldsymbol{x}$.

Example: Projection Depth and Outlyingness With $\mu(\cdot)$ and $\sigma(\cdot)$ any univariate location and scale measures, the *projection depth* is defined by

$$D_{\mathrm{P}}(\boldsymbol{x}, F) = \sup_{\|\boldsymbol{u}\| = 1} \left| \frac{\boldsymbol{u}'\boldsymbol{x} - \mu(F_{\boldsymbol{u}'\boldsymbol{X}})}{\sigma(F_{\boldsymbol{u}'\boldsymbol{X}})} \right|, \ \boldsymbol{x} \in \mathbb{R}^d,$$

and imposes a heavy computational burden. The associated *projection outlyingness* is $O_{\mathrm{P}}(\boldsymbol{x}, F) = D_{\mathrm{P}}(\boldsymbol{x}, F)/(1 + D_{\mathrm{P}}(\boldsymbol{x}, F))$, $\boldsymbol{x} \in \mathbb{R}^d$. See Liu [11], Zuo and Serfling [29], and Zuo [28].

Example: Mahalanobis Distance Outlyingness Perhaps the oldest notion of outlyingness in $\mathbb{R}^d$, $d \geq 2$, is that based on the distance introduced by Mahalanobis [14]. For location and scatter measures $\boldsymbol{m}(F)$ and nonsingular $\boldsymbol{V}(F)$, and with $\| \cdot \|$ the Euclidean norm, the corresponding Mahalanobis distance $\mathrm{MD}(\boldsymbol{x}, F)$ $= \|\boldsymbol{V}(F)^{-1/2}(\boldsymbol{x} - \boldsymbol{m}(F))\|$, $\boldsymbol{x} \in \mathbb{R}^d$, is widely used as an outlyingness function, taking values in $[0, \infty)$. Equivalently, $O_{\mathrm{MD}}(\boldsymbol{x}, F) = \mathrm{MD}(\boldsymbol{x}, F)/(1 + \mathrm{MD}(\boldsymbol{x}, F))$ takes values in $[0, 1]$

Example: Spatial Depth The *spatial outlyingness* is given by $O_{\mathrm{SP}}(\boldsymbol{x}, F) = \|E\boldsymbol{S}(\boldsymbol{x} - \mathbf{X})\|$, where $\mathbf{X}$ has distribution F and

$$\boldsymbol{S}(\boldsymbol{x}) = \begin{cases} \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \text{if } \boldsymbol{x} \neq \mathbf{0}, \\ \mathbf{0}, \ \text{ if } \boldsymbol{x} = \mathbf{0}, \end{cases}$$

the vector (or "spatial") *sign function* in $\mathbb{R}^d$. The corresponding *spatial depth* $D_{\mathrm{SP}}(\boldsymbol{x}, F) = 1 - \|E\boldsymbol{S}(\boldsymbol{x} - \mathbf{X})\|$ was introduced by Vardi and Zhang [26]. See **Figure 2** for illustration of spatial depth and contours, showing greater smoothness in comparison with halfspace depth. Whereas the preceding examples of depth and outlyingness functions are affine invariant, the spatial ones are only orthogonally invariant. This is corrected by transformation-retransformation sample versions (see Oja [19] and Serfling [23] for background and discussion).

Desirable Properties

Some basic properties are desired of any depth function. For example, *affine invariance* requires that $D(\boldsymbol{x}, F)$ be independent of the coordinate system. When F is *symmetric* about $\boldsymbol{\theta}$ in some sense, $D(\boldsymbol{x}, F)$ should also be symmetric about $\boldsymbol{\theta}$ as well as maximal at this point. Also desirable is that $D(\boldsymbol{x}, F)$ *decrease along each ray* outward from the deepest point. Further, it is important that sample versions $D(\boldsymbol{x}, \mathbb{X}_n)$ possess not only the above properties but also robustness and favorable convergence properties.

Contours, Central Regions, and Volume Functionals

Not only the underlying pointwise functions $D(\boldsymbol{x}, F)$ and $O(\boldsymbol{x}, F)$, but also the associated contours, or equivalence classes of points of equal depth, play special roles. Linked with the contours are useful central regions $\{\boldsymbol{x} : D(\boldsymbol{x}, F) \geq \alpha\}$, $\alpha > 0$. Thus, for example, the univariate boxplot may be extended for F on $\mathbb{R}^d$ by using central regions to describe a "middle half" or "middle 75%" or "middle 90%" of the population. The central region $C_F(p)$ having probability weight p induces a corresponding *volume functional*: $v_F(p) =$ $v$olume of $C_F(p)$. A convenient two-dimensional plot of $v_F(p)$ versus p characterizes the expansion of the central regions of F with increasing probability weight, and two or more distributions in any dimension can be compared by comparing their "scale curves" in a single plot. See Liu, Parelius, and Singh [12] for broad discussion of these and other applications. Ideally, central regions are *affine equivariant* and *nested*.

Extended Notions of Depth Function

The above formulation of depth functions and the illustrative examples are oriented to *location inference* for data consisting of points in $\mathbb{R}^d$. For other inference and/or data settings, modified notions of depth have been introduced, some quite complicated. Two of the more straightforward examples are "regression depth" for fitting of regression hyperplanes [20] and "band depth" for functional data [13].

Quantile and Rank Functions

Associated with any depth function are not only an outlyingness function as discussed above, but also a quantile function and a rank function. Here we formulate the latter functions in their own right. In fact, one can start with any of these four types of function and generate the others. In any application, the choice of one of these as starting point depends on the particular context and emphasis.

Quantile Functions For a distribution F on $\mathbb{R}^d$, an associated *quantile function* is indexed by $\boldsymbol{u}$ in the unit ball $\mathbb{B}^d(\mathbf{0})$ in $\mathbb{R}^d$, attaches to each point $\boldsymbol{x}$ a *quantile representation* $\boldsymbol{Q}(\boldsymbol{u}, F)$, and generates *nested* contours $\{\boldsymbol{Q}(\boldsymbol{u}, F) : \|\boldsymbol{u}\| = c\}$, $0 \leq c < 1$. For $\boldsymbol{u} = \mathbf{0}$, the most central point $\boldsymbol{Q}(\mathbf{0}, F)$ is interpreted as a *d-dimensional median* $\boldsymbol{M}_F$. For $\boldsymbol{u} \neq \mathbf{0}$, the index $\boldsymbol{u}$ represents *direction* in some sense, for example, direction to $\boldsymbol{Q}(\boldsymbol{u}, F)$ from $\boldsymbol{M}_F$, or *expected* direction to $\boldsymbol{Q}(\boldsymbol{u}, F)$ from random $\boldsymbol{X} \sim F$. The magnitude $\|\boldsymbol{u}\|$ represents an *outlyingness parameter*, higher values corresponding to more extreme points. The contours for $\|\boldsymbol{u}\| = c$ thus represent equivalence classes of points of equal outlyingness. (But in general c need not be the enclosed probability weight.) Any depth function $D(\boldsymbol{x}, F)$ generates an associated quantile function as a labeling of its contours:

Rank Functions The quantile function $\boldsymbol{Q}(\boldsymbol{u}, F)$, $\boldsymbol{u} \in \mathbb{B}^d(\mathbf{0})$, has an *inverse*, given at each point $\boldsymbol{x} \in \mathbb{R}^d$ by the point $\boldsymbol{u}$ in $\mathbb{B}^d(\mathbf{0})$ for which $\boldsymbol{x}$ has $\boldsymbol{Q}(\boldsymbol{u}, F)$ as its quantile representation, i.e., by the solution $\boldsymbol{u}$ of the equation $\boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{u}, F)$. (Under the typical condition that the nested contours of $\boldsymbol{Q}(\cdot, F)$ do not intersect for different $\boldsymbol{u}$, the solution $\boldsymbol{u}$ is unique. In the univariate case, it suffices that F is strictly increasing.) The solutions $\boldsymbol{u}$ taken over different $\boldsymbol{x}$ define a *centered rank function* $\boldsymbol{R}(\boldsymbol{x}, F)$, $\boldsymbol{x} \in \mathbb{R}^d$, which takes values in $\mathbb{B}^d(\mathbf{0})$, with the origin assigned as rank of the multivariate median $\boldsymbol{Q}(\mathbf{0}, F)$. Thus $\boldsymbol{R}(\boldsymbol{x}, F) = \mathbf{0}$ for $\boldsymbol{x} = \boldsymbol{Q}(\mathbf{0}, F)$, and $\boldsymbol{R}(\boldsymbol{x}, F)$ gives a "directional rank" in $\mathbb{B}^d(\mathbf{0})$ for other $\boldsymbol{x}$. For testing a multivariate location hypothesis on the $\boldsymbol{Q}$-based median $\boldsymbol{Q}(\mathbf{0}, F)$ in $\mathbb{R}^d$, i.e., $H_0 : \boldsymbol{Q}(\mathbf{0}, F) = \boldsymbol{\theta}_0$, a generalization of the univariate sign test statistic is provided by the test statistic $\boldsymbol{R}(\boldsymbol{\theta}_0, \mathbb{X}_n)$. Further, as noted above, the magnitude of the rank function (the index of the quantile) defines an outlyingness function: $O(\boldsymbol{x}, F) = \|\boldsymbol{R}(\boldsymbol{x}, F)\|$. In turn, $O(\boldsymbol{x}, F)$ generates a depth function $D(\boldsymbol{x}, F)$.

Example: Depth-Induced Quantile and Rank Functions For $D(\boldsymbol{x}, F)$ possessing nested contours enclosing the "median" $\boldsymbol{M}_F$ and bounding central regions $\{\boldsymbol{x} : D(\boldsymbol{x}, F) \geq \alpha\}$, $\alpha > 0$, the depth contours induce $\boldsymbol{Q}(\boldsymbol{u}, F)$, $\boldsymbol{u} \in \mathbb{B}^d(\boldsymbol{0})$, with each $\boldsymbol{x} \in \mathbb{R}^d$ given a quantile representation, as follows. For $\boldsymbol{x} = \boldsymbol{M}_F$, denote it by $\boldsymbol{Q}(\boldsymbol{0}, F)$. For $\boldsymbol{x} \neq \boldsymbol{M}_F$, denote it by $\boldsymbol{Q}(\boldsymbol{u}, F)$ with $\boldsymbol{u} = p\boldsymbol{v}$, where p is the probability weight of the central region with $\boldsymbol{x}$ on its boundary and $\boldsymbol{v}$ is the unit vector toward $\boldsymbol{x}$ from $\boldsymbol{M}_F$. In this case, $\boldsymbol{u} = \boldsymbol{R}(\boldsymbol{x}, F)$ indicates direction toward $\boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{u}, F)$ from $\boldsymbol{M}_F$, and the outlyingness parameter $\|\boldsymbol{u}\| = \|\boldsymbol{R}(\boldsymbol{x}, F)\|$ is the probability weight of the central region with $\boldsymbol{Q}(\boldsymbol{u}, F)$ on its boundary.

Example: The Spatial Quantile and Rank Functions For univariate Z with $E|Z| < \infty$, the pth quantile for $0 < p < 1$ may be characterized as any value θ minimizing $E\{|Z - \theta| + (2p - 1)(Z - \theta)\}$ (Ferguson [7, p. 51]). Equivalently, transforming to the interval $(-1, 1)$ via $u = 2p - 1$, and defining $\Phi(u, t) = |t| + ut$, $-1 < u < 1$, we obtain θ by minimizing $E\{\Phi(u, Z - \theta) - \Phi(u, Z)\}$, where subtraction of $\Phi(u, Z)$ eliminates the need of a moment assumption on Z. As a multivariate extension, d-dimensional "spatial" or "geometric" quantiles were introduced by Dudley and Koltchinskii [5] and Chaudhuri [1]. Following the latter, for random vector $\boldsymbol{X}$ having cdf F on $\mathbb{R}^d$, and for $\boldsymbol{u}$ in $\mathbb{B}^d(\boldsymbol{0})$, the *$\boldsymbol{u}$th spatial quantile* $\boldsymbol{Q}_{\mathrm{SP}}(\boldsymbol{u}, F)$ is given by $\boldsymbol{\theta}$ minimizing $E\{\Phi(\boldsymbol{u}, \boldsymbol{X} - \boldsymbol{\theta}) - \Phi(\boldsymbol{u}, \boldsymbol{X})\}$, where $\Phi(\boldsymbol{u}, \boldsymbol{t}) = \|\boldsymbol{t}\| + \boldsymbol{u}'\boldsymbol{t}$. In particular, $\boldsymbol{Q}_{\mathrm{SP}}(\boldsymbol{0}, F)$ is the well-known *spatial median*. Equivalently, in terms of the *spatial sign function* (or *unit vector function*) $\boldsymbol{S}(\boldsymbol{x})$ used above, the quantile $\boldsymbol{Q}_{\mathrm{SP}}(\boldsymbol{u}, F)$ may be represented as the solution $\boldsymbol{x}$ of the equation $E\{\boldsymbol{S}(\boldsymbol{x} - \boldsymbol{X})\} = \boldsymbol{u}$. For fixed $\boldsymbol{x}$, solving this same equation for $\boldsymbol{u}$ yields the *spatial centered rank function*, $\boldsymbol{R}_{\mathrm{SP}}(\boldsymbol{x}, F) = E\{\boldsymbol{S}(\boldsymbol{x} - \boldsymbol{X})\}$, the *expected direction* to $\boldsymbol{x}$ from a random point $\boldsymbol{X} \sim F$. (Compare with the interpretation of the *depth-induced* $\boldsymbol{R}(\boldsymbol{x}, F)$ as *direction from the median*, as seen above.) Möttönen and Oja (1995) introduce $\boldsymbol{R}_{\mathrm{SP}}(\boldsymbol{x}, F)$ in developing rank-based multivariate hypothesis testing methods, and Oja [19] provides an extensive methodology based on spatial ranks. Transformation-retransformation spatial quantiles and ranks are affine equivariant. We see that $\|\boldsymbol{R}_{\mathrm{SP}}(\boldsymbol{x}, F)\|$ is the spatial outlyingness treated above and generates the spatial depth treated above. Serfling [21] treats further properties of $\boldsymbol{Q}_{\mathrm{SP}}(\boldsymbol{u}, F)$ and introduces related nonparametric multivariate descriptive measures for location, spread, skewness, and kurtosis.

Acknowledgements

The author greatly appreciates stimulating discussions with G. L. Thompson, Marc Hallin, Hannu Oja, and Ron Randles, among many others, and the pleasure of working on these topics with doctoral students Yijun Zuo, Jin Wang, Xin Dang, Weihua Zhou, and Satyaki Mazumder. The support of National Science Foundation Grant DMS-0805786 and National Security Agency Grant H98230-08-1-0106 is gratefully acknowledged.

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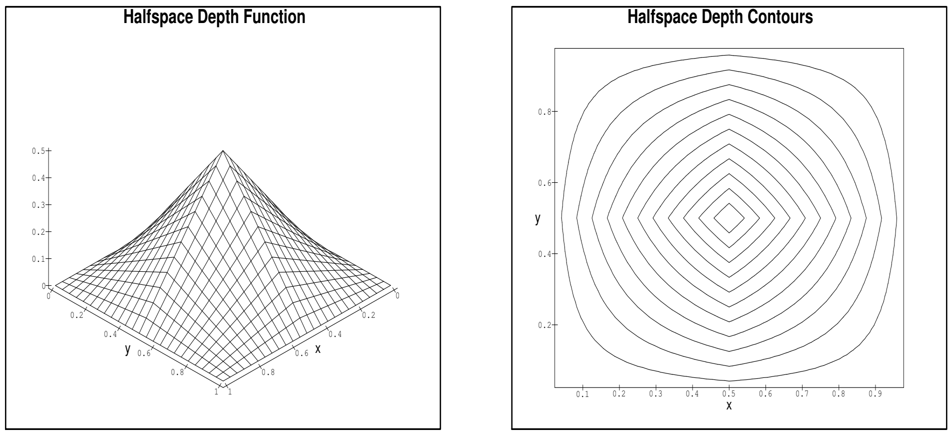

Figure 1 Halfspace Depth, F Uniform on Unit Square

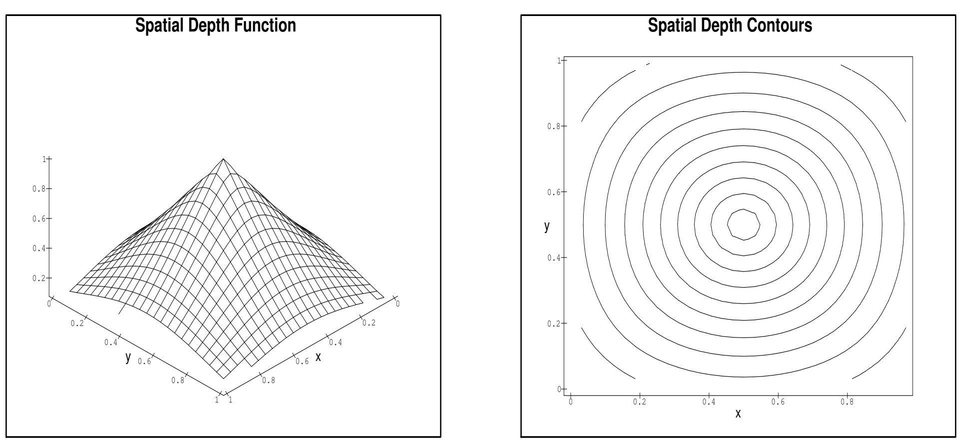

Figure 2 Spatial Depth, F Uniform on Unit Square