# An introduction to some aspects of functional analysis, 7: Convergence of operators

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**Abstract**

Here we look at strong and weak operator topologies on spaces of bounded linear mappings, and convergence of sequences of operators with respect to these topologies in particular.

## Contents

# Part I
# The strong operator topology

## 1 Seminorms

Let $V$ be a vector space over the real or complex numbers. A nonnegative real-valued function $N(v)$ on $V$ is said to be a *seminorm* on $V$ if

$$N(t\,v) = |t|\,N(v) \tag{1.1}$$

for every $v \in V$ and $t \in \mathbf{R}$ or $\mathbf{C}$, as appropriate, and

$$N(v + w) \leq N(v) + N(w) \tag{1.2}$$

for every $v, w \in V$. Here $|t|$ denotes the absolute value of a real number $t$, or the modulus of a complex number $t$. If $N(v) > 0$ when $v \neq 0$, then $N(v)$ is a norm on $V$, and

$$d(v, w) = N(v - w) \tag{1.3}$$

defines a metric on $V$.

Let us say that a collection $\mathcal{N}$ of seminorms on $V$ is nice if for each $v \in V$ with $v \neq 0$ there is an $N \in \mathcal{N}$ such that $N(v) > 0$. This implies that $V$ is Hausdorff with respect to the usual topology determined by $\mathcal{N}$, where the open balls associated to elements of $\mathcal{N}$ are open sets, and the collection of all such open balls is a sub-base for the topology. It is well known that $V$ is a topological vector space with respect to this topology, which basically means that the vector space operations are continuous. This includes the case where $\mathcal{N}$ consists of a single norm on $V$.

Similarly, a collection $\Lambda$ of linear functionals on $V$ is said to be nice if for each $v \in V$ with $v \neq 0$ there is a $\lambda \in \Lambda$ such that $\lambda(v) \neq 0$. This is equivalent to saying that the collection $\mathcal{N}(\Lambda)$ of seminorms

$$N_\lambda(v) = |\lambda(v)| \tag{1.4}$$

on $V$ with $\lambda \in \Lambda$ is nice. The topology on $V$ determined by $\mathcal{N}(\Lambda)$ as in the previous paragraph is known as the weak topology on $V$ corresponding to $\Lambda$. This is the same as the weakest topology on $V$ such that each $\lambda \in \Lambda$ is continuous on $V$. It follows that linear combinations of finitely many elements of $\Lambda$ are also continuous on $V$ with respect to this topology, and conversely it is well known that every continuous linear functional on $V$ with respect to this topology can be expressed as a linear combination of finitely many elements of $\Lambda$.

Let $V$ be a topological vector space, and let $V'$ be the dual vector space of continuous linear functionals on $V$. It is well known that $V'$ separates points in $V$ when the topology on $V$ is defined by a nice collection of seminorms on $V$, by the Hahn–Banach theorem. The weak topology on $V$ determined by $\Lambda = V'$ is known as the *weak topology* on $V$. Note that the collection of linear functionals on $V'$ of the form

$$L_v(\lambda) = \lambda(v) \tag{1.5}$$

with $v \in V$ is automatically a nice collection of linear functionals on $V'$. The weak topology on $V'$ associated to this collection of linear functionals is known as the *weak* topology* on $V'$.

Let $V$ be a real or complex vector space again, and let $\mathcal{N}$ be a nice collection of seminorms on $V$. A sequence $\{v_j\}_{j=1}^\infty$ of elements of $V$ converges to $v \in V$ with respect to the topology on $V$ associated to $\mathcal{N}$ if and only if

$$\lim_{j\to\infty} N(v_j - v) = 0 \tag{1.6}$$

for every $N \in \mathcal{N}$. Similarly, if $\Lambda$ is a nice collection of linear functionals on $V$, then a sequence $\{v_j\}_{j=1}^\infty$ of elements of $V$ converges to $v \in V$ with respect to the weak topology on $V$ associated to $\Lambda$ if and only if

$$\lim_{j\to\infty} \lambda(v_j) = \lambda(v) \tag{1.7}$$

for every $\lambda \in \Lambda$.

Let $\mathcal{N}$ be a nice collection of seminorms on $V$ again. If $\mathcal{N}$ has only finitely many elements, then their maximum is a norm on $V$ that determines the same topology on $V$. If $\mathcal{N}$ is countably infinite, then one can show that there is a translation-invariant metric on $V$ that determines the same topology on $V$. Conversely, if there is a countable local base for the topology on $V$ determined by $\mathcal{N}$ at 0, then one can check that there is a subcollection of $\mathcal{N}$ with only finitely or countably many elements that determines the same topology on $V$.

If $V$ is any topological vector space, then a sequence $\{v_j\}_{j=1}^\infty$ of elements of $V$ is said to be a Cauchy sequence in $V$ if for each open set $U \subseteq V$ with $0 \in U$,

$$v_j - v_l \in U \tag{1.8}$$

for all sufficiently large $j$, $l$. Of course, convergent sequences in $V$ are Cauchy sequences. If the topology on $V$ is determined by a nice collection $\mathcal{N}$ of seminorms, then $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence if and only if

$$\lim_{j,l \to \infty} N(v_j - v_l) = 0 \tag{1.9}$$

for every $N \in \mathcal{N}$. If the topology on $V$ is the weak topology associated to a collection $\Lambda$ of linear functionals on $V$, then $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence in $V$ if and only if $\{\lambda(v_j)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$, as appropriate, for each $\lambda \in \Lambda$. This is equivalent to saying that $\{\lambda(v_j)\}_{j=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $\lambda \in \Lambda$, since $\mathbf{R}$ and $\mathbf{C}$ are complete with respect to their standard metrics.

If there is a countable local base for the topology of $V$ at 0, then it is well known that there is a translation-invariant metric on $V$ that determines the same topology on $V$. If $d(\cdot, \cdot)$ is such a metric on $V$, then it is easy to see that a sequence $\{v_j\}_{j=1}^\infty$ of elements of $V$ is a Cauchy sequence in the sense described in the previous paragraph if and only if $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence in the usual sense with respect to $d(\cdot, \cdot)$. A vector space $V$ with a norm $\|v\|_V$ is said to be a Banach space if it is complete with respect to the metric associated to the norm.

# 2 Bounded linear mappings

Let $V$ and $W$ be vector spaces, both real or both complex, and equipped with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. A linear mapping $T$ from $V$ into $W$ is said to be *bounded* if there is a nonnegative real number $C$ such that

$$\|T(v)\|_W \le C\,\|v\|_V \tag{2.1}$$

for every $v \in V$. This implies that

$$\|T(v) - T(v')\|_W = \|T(v - v')\|_W \le C\,\|v - v'\|_V \tag{2.2}$$

for every $v, v' \in V$, so that $T$ is uniformly continuous with respect to the metrics on $V$ and $W$ associated to their norms. Conversely, if a linear mapping $T : V \to W$ is continuous at 0, then there is a $\delta > 0$ such that $\|T(v)\|_W < 1$ for every $v \in V$ with $\|v\|_V < 1$. This implies that $T$ satisfies (2.1), with $C = 1/\delta$.

Let $\mathcal{BL}(V, W)$ be the space of bounded linear mappings from $V$ into $W$. It is easy to see that this is also a vector space over $\mathbf{R}$ or $\mathbf{C}$, as appropriate, with respect to pointwise addition and scalar multiplication. Put

$$\|T\|_{op} = \sup\{\|T(v)\|_W : v \in V,\ \|v\| \le 1\} \tag{2.3}$$

for each $T \in \mathcal{BL}(V, W)$, which is the *operator norm* of $T$. One can check that this does indeed define a norm on $\mathcal{BL}(V, W)$. If $W$ is complete with respect to the metric associated to its norm, then it is well known that $\mathcal{BL}(V, W)$ is also complete with respect to the operator norm. If $W = \mathbf{R}$ or $\mathbf{C}$, as appropriate, then $\mathcal{BL}(V, W)$ reduces to the dual space $V'$ of bounded linear functionals on $V$, and the operator norm is the same as the usual dual norm on $V'$. In particular, $V'$ is always complete with respect to the dual norm.

Suppose that $V_1$, $V_2$, and $V_3$ are vector spaces, all real or all complex, and with norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$, respectively. If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are bounded linear mappings, then the composition $T_2 \circ T_1$ of $T_1$ and $T_2$ is a bounded linear mapping from $V_1$ into $V_3$, and

$$\|T_2 \circ T_1\|_{op,13} \le \|T_1\|_{op,12} \, \|T_2\|_{op,23}. \tag{2.4}$$

Here the subscripts in the operator norms are used to indicate the vector spaces and norms being used. If $V$ is a real or complex vector space with a norm, then it follows that the space $\mathcal{BL}(V) = \mathcal{BL}(V, V)$ of bounded linear mappings from $V$ into itself is an algebra, with composition of mappings as multiplication. Note that the operator norm of the identity mapping $I = I_V$ on $V$ is equal to 1.

Let $V$ and $W$ be as before, and suppose that $E$ is a dense linear subspace of $V$. If $T$ is a bounded linear mapping from $E$ into $W$, and if $W$ is complete with respect to its norm, then it is well known that there is a unique extension of $T$ to a bounded linear mapping from $V$ into $W$. More precisely, the restriction of $T$ to $E$ is uniformly continuous, and a well-known theorem for metric spaces implies that there is a unique extension of $T$ to a uniformly continuous mapping from $V$ into $W$. In this case, it is easy to see that the extension is also a bounded linear mapping from $V$ into $W$.

# 3 The strong operator topology

Let $V$ and $W$ be vector spaces again, both real or both complex, and with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. If $v \in V$, then

$$N_v(T) = \|T(v)\|_W \tag{3.1}$$

defines a seminorm on $\mathcal{BL}(V, W)$, and the collection of these seminorms $N_v$ with $v \in V$ is automatically a nice collection of seminorms on $\mathcal{BL}(V, W)$. The topology on $\mathcal{BL}(V, W)$ determined by this collection of seminorms is known as the *strong operator topology* on $\mathcal{BL}(V, W)$. Of course,

$$\|T(v)\|_W \le \|T\|_{op} \, \|v\|_V \tag{3.2}$$

for every $v \in V$ and $T \in \mathcal{BL}(V, W)$, by the definition of the operator norm. This implies that the strong operator topology on $\mathcal{BL}(V, W)$ is weaker than the topology determined by the operator norm on $\mathcal{BL}(V, W)$, in the sense that every open set in $\mathcal{BL}(V, W)$ with respect to the strong operator topology is also an open set with respect to the topology determined by the metric on $\mathcal{BL}(V, W)$

corresponding to the operator norm. If $W = \mathbf{R}$ or $\mathbf{C}$, as appropriate, then $\mathcal{BL}(V, W)$ is the same as the dual $V'$ of $V$, and the strong operator topology reduces to the weak$^*$ topology on $V'$. However, if $V = \mathbf{R}$ or $\mathbf{C}$, as appropriate, then $\mathcal{BL}(V, W)$ can be identified with $W$, and the strong operator topology corresponds to the topology on $W$ determined by its norm.

A sequence $\{T_j\}_{j=1}^\infty$ of bounded linear mappings from $V$ into $W$ converges to $T \in \mathcal{BL}(V, W)$ with respect to the strong operator topology if and only if $\{T_j(v)\}_{j=1}^\infty$ converges to $T(v)$ with respect to the norm on $W$ for every $v \in V$. In particular, this implies that $\{T_j(v)\}_{j=1}^\infty$ is a bounded sequence in $W$ for every $v \in V$. If $V$ is complete with respect to the metric associated to the norm, then the Banach–Steinhaus theorem implies that the operator norms of the $T_j$'s are uniformly bounded.

Suppose now that $\{T_j\}_{j=1}^\infty$ is a sequence of bounded linear mappings from $V$ into $W$ with uniformly bounded operator norms, so that $\|T_j\|_{op} \le C$ for some $C \ge 0$ and every $j \ge 1$. Suppose also that $\{T_j(v)\}_{j=1}^\infty$ converges in $W$ with respect to the norm on $W$ for each $v \in V$, and let $T(v)$ be the limit of $\{T_j(v)\}_{j=1}^\infty$ in $W$. It is easy to see that $T$ is a linear mapping from $V$ into $W$, and one can also check that $T$ is bounded, with $\|T\|_{op} \le C$. Thus $\{T_j\}_{j=1}^\infty$ converges to $T$ with respect to the strong operator topology on $\mathcal{B}(V, W)$ under these conditions.

Similarly, $\{T_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the strong operator topology if and only if $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in $W$ for every $v \in V$. This also implies that $\{T_j(v)\}_{j=1}^\infty$ is a bounded sequence in $W$ for every $v \in V$, and hence that the operator norms of the $T_j$'s are uniformly bounded when $V$ is complete, by the Banach–Steinhaus theorem. If $W$ is complete, then it follows that $\{T_j(v)\}_{j=1}^\infty$ converges in $W$ for every $v \in V$. If $W$ is complete and the operator norms of the $T_j$'s are uniformly bounded, then $\{T_j\}_{j=1}^\infty$ converges with respect to the strong operator topology, by the remarks in the previous paragraph.

# 4 Shift operators

Let $X$ be a nonempty set, and let $\ell^p(X)$ be the space of $p$-summable real or complex-valued functions $f$ on $X$, for $1 \le p < \infty$. It is well known that this is a Banach space with respect to the usual $\ell^p$ norm

$$\|f\|_p = \|f\|_{\ell^p(X)} = \Big(\sum_{x \in X} |f(x)|^p\Big)^{1/p}. \tag{4.1}$$

Similarly, the space $\ell^\infty(X)$ of bounded real or complex-valued functions $f$ on $X$ is a Banach space with respect to the supremum norm

$$\|f\|_\infty = \|f\|_{\ell^\infty(X)} = \sup_{x \in X} |f(x)|. \tag{4.2}$$

Let $c_0(X)$ be the collection of real or complex-valued functions $f$ on $X$ that vanish at infinity, in the sense that for each $\epsilon > 0$, $|f(x)| > \epsilon$ for only finitely many $x \in X$. One can check that $c_0(X)$ is a closed linear subspace of $\ell^\infty(X)$, and hence also a Banach space with respect to the supremum norm.

In this section, we shall be interested in the case where $X$ is the set $\mathbf{Z}_+$ of positive integers, so that the elements of $\ell^p = \ell^p(\mathbf{Z}_+)$ can be identified with sequences of real or complex numbers. In particular, $c_0 = c_0(\mathbf{Z}_+)$ consists of sequences of real or complex numbers that converge to 0. Consider the shift operator $T$ defined by

$$T(f)(j) = f(j+1) \tag{4.3}$$

for any real or complex-valued function $f$ on $\mathbf{Z}_+$. It is easy to see that $T$ defines a bounded linear operator on $\ell^p$ for each $p$, $1 \le p \le \infty$, with operator norm equal to 1. Note that $T$ also maps $c_0$ into itself, and that the operator norm of $T$ on $c_0$ with respect to the $\ell^\infty$ norm is equal to 1 too.

Let $n$ be a positive integer, and let $T^n$ be the $n$-fold composition of $T$, so that $T^1 = T$ and $T^{n+1} = T \circ T^n$ for each $n$. Equivalently,

$$T^n(f)(j) = f(j+n) \tag{4.4}$$

for each real or complex-valued function $f$ on $\mathbf{Z}_+$. It is easy to see that $T^n$ has operator norm equal to 1 on $\ell^p$ for each $p$, $1 \le p \le \infty$, and for every $n$. Similarly, the operator norm of $T^n$ on $c_0$ with respect to the $\ell^\infty$ norm is equal to 1 for every $n$ as well. However, $\{T^n\}_{n=1}^\infty$ converges to 0 with respect to the strong operator topology on $\ell^p$ when $1 \le p < \infty$, and on $c_0$.

# 5 Multiplication operators

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $b$ be a bounded measurable real or complex-valued function on $X$. Consider the corresponding multiplication operator

$$T_b(f) = b\, f, \tag{5.1}$$

where $f$ is a real or complex-valued measurable function on $X$. This is a bounded linear operator on $L^p(X)$ for each $p$, $1 \le p \le \infty$, with operator norm less than or equal to the $L^\infty$ norm $\|b\|_\infty$ of $b$. In many cases, the operator norm of $T_b$ is equal to $\|b\|_\infty$. More precisely, let $\mathbf{1}_A(x)$ be the characteristic or indicator function of a set $A \subseteq X$, which is equal to 1 when $x \in A$ and to 0 otherwise. In particular, $\mathbf{1}_X$ is the constant function equal to 1 on $X$, and $T_b(\mathbf{1}_X) = b$, which implies that the operator norm of $T_b$ on $L^\infty(X)$ is equal to $\|b\|_\infty$, because $\mathbf{1}_X \in L^\infty(X)$. If $A \subseteq X$ is a measurable set of finite measure, then $\mathbf{1}_A \in L^p(X)$ for each $p$. One can use this to show that the operator norm of $T_b$ on $L^p(X)$ is equal to $\|b\|_\infty$ when every measurable subset of $X$ of positive measure contains a measurable set with positive finite measure. This condition holds when $X$ is $\sigma$-finite, for instance. Of course, if $f \in L^p(X)$ and $p < \infty$, then the set of $x \in X$ where $f(x) \ne 0$ is $\sigma$-finite.

Now let $\{b_j\}_{j=1}^\infty$ be a sequence of bounded measurable functions on $X$, and let $b$ be a bounded measurable function on $X$. Also let

$$T_j(f) = T_{b_j}(f) = b_j\, f \tag{5.2}$$

be the corresponding sequence of multiplication operators, and let $T = T_b$ be as in (5.1). If $\{b_j\}_{j=1}^{\infty}$ converges to $b$ with respect to the $L^\infty$ norm, then $\{T_j\}_{j=1}^{\infty}$ converges to $T$ with respect to the operator norm on $L^p$ for each $p$. The converse also holds in many situations, by the remarks in the previous paragraph. If the $L^\infty$ norms of the $b_j$'s are uniformly bounded, if $\{b_j\}_{j=1}^{\infty}$ converges pointwise almost everywhere on $X$ to $b$, and if $p < \infty$, then $\{T_j\}_{j=1}^{\infty}$ converges to $T$ with respect to the strong operator topology on $\mathcal{BL}(L^p(X))$. This is the same as saying that $\{b_j\, f\}_{j=1}^{\infty}$ converges to $b\, f$ with respect to the $L^p$ norm for every $f$ in $L^p(X)$, which follows from the dominated convergence theorem applied to $|(b_j - b)\, f|^p$. This also works when $\{b_j\}_{j=1}^{\infty}$ converges to $b$ in measure on every measurable sobset of $X$ with finite measure instead of converging pointwise almost everywhere on $X$, by the corresponding version of the dominated convergence theorem. Conversely, suppose that $\{T_j\}_{j=1}^{\infty}$ converges to $T$ with respect to the strong operator topology on $\mathcal{BL}(L^p(X))$, and let $A \subseteq X$ be a measurable set with finite measure. Thus $\mathbf{1}_A \in L^p(X)$, and the convergence of $T_j(\mathbf{1}_A) = b_j\, \mathbf{1}_A$ to $b\, \mathbf{1}_A$ as $j \to \infty$ with respect to the $L^p$ norm implies that $b_j$ converges to $b$ in measure on $A$. If $p = \infty$, then one can apply this to $A = X$, to get that $\{T_j\}_{j=1}^{\infty}$ converges to $T$ with respect to the strong operator topology on $\mathcal{BL}(L^\infty(X))$ if and only if $\{b_j\}_{j=1}^{\infty}$ converges to $b$ with respect to the $L^\infty$ norm.

Suppose that $\mu$ is counting measure on $X$, with all subsets of $X$ being measurable, so that $L^p(X)$ is the same as $\ell^p(X)$. If $T_j = T_{b_j}$ converges to $T = T_b$ with respect to the strong operator topology on $\ell^p(X)$ for any $p$, then $b_j$ converges to $b$ pointwise everywhere on $X$. This is the same as saying that $\{b_j\}_{j=1}^{\infty}$ converges to $b$ in measure on finite subsets of $X$, which are the same as the measurable sets of finite measure in this case. If $b$ is a bounded real or complex-valued function on $X$, then $T = T_b$ also defines a bounded linear operator on $c_0(X)$ with respect to the $\ell^\infty$ norm, with operator norm equal to the $\ell^\infty$ norm of $b$. If $\{b_j\}_{j=1}^{\infty}$ is a sequence of bounded functions on $X$ that have uniformly bounded $\ell^\infty$ norms and which converge pointwise to $b$ everywhere on $X$, then one can check that $T_j = T_{b_j}$ converges to $T = T_b$ with respect to the strong operator topology on $\mathcal{BL}(c_0(X))$. Equivalently, this means that $\{b_j\, f\}_{j=1}^{\infty}$ converges to $b\, f$ with respect to the $\ell^\infty$ norm for every $f \in c_0(X)$. Conversely, if $T_{b_j}$ converges to $T_b$ with respect to the strong operator topology on $\mathcal{BL}(c_0(X))$, then it is easy to see that $\{b_j\}_{j=1}^{\infty}$ converges to $b$ pointwise everywhere on $X$, by taking $f$ to be the indicator function on a set with one element.

As a variant of this, let $X$ be a nonempty topological space, and let $C_b(X)$ be the space of bounded continuous real or complex-valued functions on $X$. It is well known that $C_b(X)$ is a Banach space with respect to the supremum norm. If $b \in C_b(X)$, then the corresponding multiplication operator $T_b$ is a bounded linear operator on $C_b(X)$, with operator norm equal to the supremum norm. This implies that a sequence $\{b_j\}_{j=1}^{\infty}$ of elements of $C_b(X)$ converges to $b$ with respect to the supremum norm if and only if the corresponding multiplication operators $T_{b_j}$ converge to $T_b$ as $j \to \infty$ with respect to the operator norm on $\mathcal{BL}(C_b(X))$. As in the $p = \infty$ case considered earlier, this is also equivalent to the convergence of $T_{b_j}$ to $T_b$ as $j \to \infty$ with respect to the strong operator topology on $\mathcal{BL}(C_b(X))$, because $T_{b_j}(\mathbf{1}_X) = b_j$ and $\mathbf{1}_X \in C_b(X)$. Suppose now that $X$ is a locally compact Hausdorff space which is not compact, and let $C_0(X)$ be the space of continuous real or complex-valued functions $f$ on $X$ that vanish at infinity, in the sense that

$$\{x \in X : |f(x)| \geq \epsilon\} \tag{5.3}$$

is a compact set in $X$ for each $\epsilon > 0$. It is well known that $C_0(X)$ is a closed linear subspace of $C_b(X)$ with respect to the supremum norm. If $b \in C_b(X)$, then the corresponding multiplication operator $T_b$ defines a bounded linear mapping from $C_0(X)$ into itself, with operator norm equal to the supremum norm of $b$. This uses Urysohn's lemma, to get that continuous functions with compact support on $X$ separate points in $X$. Let $\{b_j\}_{j=1}^\infty$ be a sequence of elements of $C_b(X)$ with bounded supremum norms that converges uniformly on compact subsets of $X$ to a function $b$ on $X$, which implies that $b$ is also a bounded continuous function on $X$, since $X$ is locally compact. One can check that $\{b_j\, f\}_{j=1}^\infty$ converges uniformly on $X$ to $b\, f$ as $j \to \infty$ for every $f \in C_0(X)$, which means that the corresponding sequence $\{T_{b_j}\}_{j=1}^\infty$ of multiplication operators converges to $T_b$ with respect to the strong operator topology on $\mathcal{BL}(C_0(X))$. Conversely, if $\{T_{b_j}\}_{j=1}^\infty$ converges to $T_b$ with respect to the strong operator topology on $\mathcal{BL}(C_0(X))$, then $\{b_j\}_{j=1}^\infty$ converges to $b$ uniformly on compact subsets of $X$. This uses Urysohn's lemma again, which implies that there are continuous functions on $X$ with compact support that are equal to 1 on any given compact set in $X$.

# 6 Dense sets

Let $V$ and $W$ be vector spaces, both real or both complex, and with norms $\|v\|_V$ and $\|w\|_W$, respectively. As usual, the linear span of a set $A \subseteq V$ is the set of vectors in $V$ that can be expressed as a linear combination of finitely many elements of $A$, which is the smallest linear subspace of $V$ containing $A$. If the linear span of $A \subseteq V$ is equal to $V$, then the collection of seminorms $N_v(T) = \|T(v)\|_W$ on $\mathcal{BL}(V, W)$ with $v \in A$ is sufficient to determine the strong operator topology on $\mathcal{BL}(V, W)$.

Suppose that $A \subseteq V$ has the property that the linear span of $A$ in $V$ is dense in $V$ with respect to the norm. If $T$ is a bounded linear mapping from $V$ into $W$ such that $T(v) = 0$ for every $v \in A$, then $T(v) = 0$ for every $v$ in the linear span of $A$, and hence $T \equiv 0$ on $V$. Thus the collection of seminorms $N_v(T) = \|T(v)\|_W$ on $\mathcal{BL}(V, W)$ with $v \in A$ is a nice collection of seminorms on $\mathcal{BL}(V, W)$, and the topology determined by this collection of seminorms is contained in the strong operator topology on $\mathcal{BL}(V, W)$. In this case, if $E \subseteq \mathcal{BL}(V, W)$ is bounded with respect to the operator norm, then one can check that the topology induced on $E$ by the strong operator topology on $\mathcal{BL}(V, W)$ is the same as the topology induced on $E$ by the topology determined on $\mathcal{BL}(V, W)$ by the seminorms $N_v(T) = \|T(v)\|_W$ with $v \in A$.

Let $\{T_j\}_{j=1}^\infty$ be a sequence of elements of $\mathcal{BL}(V, W)$, and let $T$ be an element of $\mathcal{BL}(V, W)$. It is easy to see that the set of $v \in V$ such that $\{T_j(v)\}_{j=1}^\infty$

converges to $T(v)$ in $W$ is a linear subspace of $V$. If the operator norms of the $T_j$'s are uniformly bounded, then one can also check that this is a closed linear subspace of $V$. In particular, if $\{T_j(v)\}_{j=1}^\infty$ converges to $T(v)$ for each $v \in A$ for some set $A \subseteq V$ whose linear span is dense in $V$, and if the operator norms of the $T_j$'s are uniformly bounded, then $\{T_j(v)\}_{j=1}^\infty$ converges to $T(v)$ in $W$ for every $v \in V$. This can also be considered as a consequence of the remarks in the previous paragraph.

Similarly, if $\{T_j\}_{j=1}^\infty$ is any sequence of elements of $\mathcal{BL}(V, W)$, then the set of $v \in V$ such that $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in $W$ is a linear subspace of $V$. If the operator norms of the $T_j$'s are uniformly bounded, then one can check that this linear subspace is a closed set in $V$ too. In particular, if $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence for all $v \in A$ for some set $A \subseteq V$ whose linear span is dense in $V$, and if the operator norms of the $T_j$'s are uniformly bounded, then $\{T_j(v)\}_{j=1}^\infty$ is a Cauchy sequence in $W$ for every $v \in V$. If $W$ is complete, then it follows that $\{T_j(v)\}_{j=1}^\infty$ converges in $W$ for every $v \in V$ under these conditions.

# 7 Shift operators, 2

Let $T$ be the shift operator defined by

$$T(f)(j) = f(j+1) \tag{7.1}$$

for any real or complex-valued function $f$ on the set $\mathbf{Z}$ of integers. This defines a one-to-one linear mapping from $\ell^p(\mathbf{Z})$ onto itself for each $p$, which is also an isometry, in the sense that

$$\|T(f)\|_p = \|f\|_p \tag{7.2}$$

for every $f \in \ell^p(\mathbf{Z})$. Note that $T$ maps $c_0(\mathbf{Z})$ onto itself as well. The $n$-fold composition $T^n$ of $T$ is given by

$$T^n(f)(j) = f(j+n) \tag{7.3}$$

for each function $f$ on $\mathbf{Z}$ and every positive integer $n$. This makes sense for every integer $n$ as well, where $T^0$ is the identity mapping, $T^{-1}$ is the inverse of $T$, and so on.

Of course, $T^n$ is also an isometry on $\ell^p(\mathbf{Z})$ for each $n$ and $p$. If $f \in c_0(\mathbf{Z})$, then $T^n(f)(j) \to 0$ as $|n| \to \infty$ for each $j \in \mathbf{Z}$. In particular, this holds for each $f \in \ell^p(\mathbf{Z})$ when $p < \infty$. It follows that $\{T^n\}_{n=1}^\infty$ does not converge with respect to the strong operator topology on $\mathcal{BL}(\ell^p(\mathbf{Z}))$ for any $p$.

Alternatively, consider the average

$$A_n = \frac{1}{n+1} \sum_{l=0}^n T^l \tag{7.4}$$

of $T^l$ for $l = 0, 1, \ldots, n$ for each positive integer $n$, as a linear mapping on the vector space of real or complex-valued functions on $\mathbf{Z}$. It is easy to see that this defines a bounded linear mapping on $\ell^p(\mathbf{Z})$ for each nonnegative integer $n$ and

$p \geq 1$, with operator norm less than or equal to 1, by the triangle inequality. More precisely, one can check that the operator norm of $A_n$ on $\ell^p(\mathbf{Z})$ is equal to 1 for each $n \geq 0$ and $p \geq 1$, and that the operator norm of $A_n$ on $c_0(\mathbf{Z})$ is equal to 1 with respect to the $\ell^\infty$ norm. Of course, $A_n$ maps constant functions on $\mathbf{Z}$ to themselves for each $n$, which implies that the operator norm of $A_n$ on $\ell^\infty(\mathbf{Z})$ is equal to 1 for each $n$. To deal with the other cases, one can consider functions on $\mathbf{Z}$ that are constant on large intervals, and 0 elsewhere.

If $f$ is a function on $\mathbf{Z}$ which is equal to 1 at one point and equal to 0 elsewhere, then one can check that $A_n(f)$ converges to 0 as $n \to \infty$ with respect to the $\ell^p$ norm for each $p > 1$, while the $\ell^1$ norm of $A_n(f)$ is equal to 1 for each $n$. This implies that $\{A_n\}_{n=1}^\infty$ converges to 0 as $n \to \infty$ with respect to the strong operator topology on $\mathcal{BL}(\ell^p(\mathbf{Z}))$ when $1 < p < \infty$, and also with respect to the strong operator topology on $\mathcal{BL}(c_0(\mathbf{Z}))$, by the remarks in the previous section. This also uses the fact that the linear span of these functions $f$ on $\mathbf{Z}$ is dense in $\ell^p(\mathbf{Z})$ when $p < \infty$, and in $c_0(\mathbf{Z})$ with respect to the $\ell^\infty$ norm.

# 8 Other operators

Let $V$ be a real or complex vector space with a norm $\|v\|$, and let $T$ be a bounded linear operator on $V$, with $\|T\|_{op} \leq 1$. Put

$$A_n = \frac{1}{n+1} \sum_{l=0}^{n} T^l \tag{8.1}$$

for each nonnegative integer $n$, where $T^l$ is the $l$th power of $T$ with respect to composition, which is interpreted as being the identity mapping on $V$ when $l = 0$. Thus

$$\|T^l\|_{op} \leq \|T\|_{op}^l \leq 1 \tag{8.2}$$

for each $l$, which implies that

$$\|A_n\|_{op} \leq \frac{1}{n+1} \sum_{l=0}^{n} \|T^l\|_{op} \leq 1 \tag{8.3}$$

for each $n$, by the triangle inequality. If $\|T^l\|_{op} \to 0$ as $l \to \infty$, then it follows that $\|A_n\|_{op} \to 0$ as $n \to \infty$, by an elementary argument.

Observe that

$$(I - T) \sum_{l=0}^{n} T^l = \Big( \sum_{l=0}^{n} T^l \Big) (I - T) = I - T^{n+1} \tag{8.4}$$

for each $n$, by a standard computation. If $I - T$ is an invertible linear mapping on $V$, which is to say that $I - T$ is a one-to-one mapping from $V$ onto itself whose inverse is also a bounded linear mapping on $V$, then we get that

$$\sum_{l=0}^{n} T^l = (I - T^{n+1}) \, (I - T)^{-1} \tag{8.5}$$

for each $n$. This implies that

$$\left\| \sum_{l=0}^{n} T^l \right\|_{op} \le \|I - T^{n+1}\|_{op} \, \|(I-T)^{-1}\|_{op} \le 2 \, \|(I-T)^{-1}\|_{op} \tag{8.6}$$

for each $n$, and hence that

$$\|A_n\| \le \frac{2}{n+1} \, \|(I-T)^{-1}\|_{op} \tag{8.7}$$

for each $n$. Thus $\|A_n\|_{op} \to 0$ as $n \to \infty$ in this case as well.

Similarly, if $v = u - T(u)$ for some $u \in V$, then

$$A_n(v) = \frac{1}{n+1} \, (u - T^{n+1}(u)) \tag{8.8}$$

for each $n$, by (8.4). This implies that $\|A_n(v)\| \to 0$ as $n \to \infty$ for every $v$ in the image of $V$ under $I - V$. The same conclusion holds for all $v$ in the closure of the image of $V$ under $I - V$, because of (8.3). Note that

$$\|A_n(v)\| \le \frac{1}{n+1} \sum_{l=0}^{n} \|T^l(v)\| \tag{8.9}$$

for every $v \in V$ and $n \ge 0$, by the triangle inequality. If $T^l(v) \to 0$ as $l \to \infty$ for some $v \in V$, then it follows that $A_n(v) \to 0$ as $n \to \infty$, by an elementary argument, as before.

Using (8.4) again, we get that

$$A_n(v) - T(A_n(v)) = \frac{1}{n+1} \, (v - T^{n+1}(v)) \tag{8.10}$$

for every $v \in V$ and $n \ge 0$. This implies that

$$\|A_n(v) - T(A_n(v))\| \le \frac{1}{n+1} \, \|v - T^{n+1}(v)\| \le \frac{2}{n+1} \, \|v\| \tag{8.11}$$

for every $v$ and $n$, and hence that

$$\lim_{n \to \infty} (A_n(v) - T(A_n(v))) = 0 \tag{8.12}$$

for every $v \in V$. If $\{A_n(v)\}_{n=1}^{\infty}$ converges to some $w \in V$, then it follows that $T(w) = w$. Of course, if $I - T$ is one-to-one on $V$, then there are no nonzero vectors $w \in V$ such that $T(w) = w$. Otherwise, if $v \in V$ satisfies $T(v) = v$, then it follows that $T^l(v) = v$ for each $l$, so that $A_n(v) = v$ for every $n$.

# 9 Unitary operators

Let $V$ be a real or complex vector space with an inner product $\langle v, w \rangle$. As usual, this leads to a norm $\|v\|$ on $V$, which is the square root of $\langle v, v \rangle$. A linear mapping $T$ from $V$ onto itself is said to be unitary if

$$\langle T(v), T(w) \rangle = \langle v, w \rangle \tag{9.1}$$

for every $v, w \in V$. This implies that

$$\|T(v)\| = \|v\| \tag{9.2}$$

for every $v \in V$, and in particular that the kernel of $T$ is trivial, so that $T$ is one-to-one. Conversely, if a linear mapping $T$ from $V$ into itself satisfies (9.2) for every $v \in V$, then it is well known that $T$ also satisfies (9.1) for every $v, w \in V$, because of polarization identities.

Suppose that $T$ is a unitary linear mapping from $V$ onto itself, and let $Z$ be the image of $V$ under $I - T$. Also let

$$W = Z^\perp = \{w \in V : \langle w, z \rangle = 0 \text{ for every } z \in Z\} \tag{9.3}$$

be the orthogonal complement of $Z$ in $V$, so that $w \in W$ if and only if

$$\langle w, u - T(u) \rangle = 0 \tag{9.4}$$

for every $u \in V$. Because $T$ is unitary,

$$\langle w, T(u) \rangle = \langle T(T^{-1}(w)), T(u) \rangle = \langle T^{-1}(w), u \rangle \tag{9.5}$$

for every $u \in V$, and hence $w \in W$ if and only if

$$\langle w - T^{-1}(w), u \rangle = 0 \tag{9.6}$$

for every $u \in V$. Thus $w \in W$ if and only if $T^{-1}(w) = w$, which is the same as saying that $T(w) = w$.

Suppose that $V$ is also complete with respect to the metric associated to the norm, which is to say that $V$ is a Hilbert space. This can always be arranged by passing to the completion of $V$, and extending $T$ to a continuous linear mapping on the completion, which is unitary as well. Let $\overline{Z}$ be the closure of $Z$ in $V$, which is a closed linear subspace of $V$. It follows from well-known results about Hilbert spaces that every $v \in V$ can be expressed in a unique way as

$$v = w + z, \tag{9.7}$$

where $w \in W$ and $z \in \overline{Z}$. The mapping $P$ from $v \in V$ to the corresponding $w \in W$ is known as the orthogonal projection of $V$ onto $W$.

Let $A_n$ be as in (8.1) for each nonnegative integer $n$. If $z \in \overline{Z}$, then we have seen that $A_n(z) \to 0$ as $n \to \infty$, as in the previous section. Similarly, if $w \in W$, then $A_n(w) = w$ for each $n$. It follows that $\{A_n\}_{n=1}^\infty$ converges to the orthogonal projection $P$ of $V$ onto $W$ with respect to the strong operator topology on $\mathcal{BL}(V)$ under these conditions.

# 10 Measure-preserving transformations

Let $(X, \mathcal{A}, \mu)$ be a measure space, and suppose that $\phi$ is a measure-preserving transformation on $X$. More precisely, this means that $\phi$ is a one-to-one mapping from $X$ onto itself such that $\phi(E)$, $\phi^{-1}(E)$ are measurable for every measurable set $E \subseteq X$, and

$$\mu(\phi(E)) = \mu(E) \tag{10.1}$$

for every measurable set $A \subseteq X$. Put

$$T(f) = T_\phi(f) = f \circ \phi \tag{10.2}$$

for every measurable real or complex-valued function $f$ on $X$, which defines an invertible linear mapping on the vector space of measurable functions on $X$. In particular, $T$ defines an isometric linear mapping from $L^p(X)$ onto itself for each $p$. Note that $L^2(X)$ is a Hilbert space with respect to the standard inner product, and that $T$ defines a unitary mapping on $L^2(X)$.

Let $A_n$ be as in (8.1) for each nonnegative integer $n$, so that $A_n$ defines a bounded linear operator on $L^p(X)$ with operator norm less than or equal to 1 for each $p$, $1 \le p \le \infty$. If $f \in L^2(X)$, then $\{A_n(f)\}_{n=1}^\infty$ converges in $L^2(X)$, as in the previous section. We would like to show that the analogous statement holds in $L^p(X)$ when $1 < p < \infty$, starting with the case where $p > 2$. If $g \in L^2(X) \cap L^\infty(X)$, then $g \in L^p(X)$ for each $p > 2$, and

$$\|g\|_p^p = \int_X |g(x)|^p \, d\mu(x) \le \|g\|_\infty^{p-2} \int_X |g(x)|^2 \, d\mu(x) = \|g\|_\infty^{p-2} \, \|g\|_2^2, \tag{10.3}$$

where $\|g\|_q$ denotes the $L^q$ norm of $g$ for each $q$. Thus

$$\|g\|_p \le \|g\|_\infty^{1-(2/p)} \, \|g\|_2^{2/p}. \tag{10.4}$$

If $\{g_j\}_{j=1}^\infty$ is a sequence of elements of $L^2(X) \cap L^\infty(X)$ that converges with respect to the $L^2$ norm and is uniformly bounded with respect to the $L^\infty$ norm, then one can use (10.4) to show that $\{g_j\}_{j=1}^\infty$ also converges with respect to the $L^p$ norm when $2 < p < \infty$. If $f \in L^2(X) \cap L^\infty(X)$, then $\{A_n(f)\}_{n=1}^\infty$ converges with respect to the $L^2$ norm and is uniformly bounded with respect to the $L^\infty$ norm, and hence converges with respect to the $L^p$ norm when $2 < p < \infty$. This implies that $\{A_n(f)\}_{n=1}^\infty$ converges in $L^p(X)$ for every $f \in L^p(X)$ when $2 < p < \infty$, because $L^2(X) \cap L^\infty(X)$ is dense in $L^p(X)$ when $2 < p < \infty$.

Similarly, if $g \in L^1(X) \cap L^2(X)$, then $g \in L^p(X)$ for each $p \in (1, 2)$, and the $L^p$ norm of $g$ is bounded by a product of positive powers of the $L^1$ and $L^2$ norms of $g$, by Hölder's inequality. If $\{g_j\}_{j=1}^\infty$ is a sequence of elements of $L^1(X) \cap L^2(X)$ that converges with respect to the $L^2$ norm and is uniformly bounded with respect to the $L^1$ norm, then it follows that $\{g_j\}_{j=1}^\infty$ also converges with respect to the $L^p$ norm when $1 < p < \infty$. If $f \in L^1(X) \cap L^2(X)$, then $\{A_n(f)\}_{n=1}^\infty$ converges with respect to the $L^2$ norm and is uniformly bounded with respect to the $L^1$ norm, and hence converges with respect to the $L^p$ norm when $1 < p < 2$. As before, this implies that $\{A_n(f)\}_{n=1}^\infty$ converges in $L^p(X)$ for every $f \in L^p(X)$ when $1 < p < 2$, because $L^1(X) \cap L^2(X)$ is dense in $L^p(X)$ when $1 < p < 2$.

This is all a bit simpler when $\mu(X) < \infty$, in which case $L^\infty(X)$ is a dense linear subspace of $L^p(X)$ for each $p$. In particular, $L^2(X)$ is a dense linear subspace of $L^p(X)$ when $1 \leq p \leq 2$, and the convergence of $\{A_n(f)\}_{n=1}^\infty$ in $L^2(X)$ for each $f \in L^2(X)$ implies the convergence of $\{A_n(f)\}_{n=1}^\infty$ with respect to the $L^p$ norm when $1 \leq p \leq 2$ and $f \in L^2(X)$. This implies that $\{A_n(f)\}_{n=1}^\infty$ converges in $L^p(X)$ for every $f \in L^p(X)$ when $1 \leq p \leq 2$, because $L^2(X)$ is dense in $L^p(X)$ when $p \leq 2$. Note that this includes the case where $p = 1$, which was not covered by the previous argument.

# Part II
# The weak operator topology

## 11 Definitions

Let $V$ and $W$ be vector spaces, both real or both complex, and equipped with norms $\|v\|_V$ and $\|w\|_W$, respectively. If $v \in V$ and $\lambda$ is a bounded linear functional on $W$, then

$$L_{v,\lambda}(T) = \lambda(T(v)) \tag{11.1}$$

defines a bounded linear functional on $\mathcal{BL}(V, W)$ with respect to the operator norm. More precisely,

$$|L_{v,\lambda}(T)| = |\lambda(T(v))| \leq \|\lambda\|_{W'} \, \|T(v)\|_W \leq \|\lambda\|_{W'} \|T\|_{op} \, \|v\|_V \tag{11.2}$$

for every $v \in V$, $\lambda \in W'$, and $T \in \mathcal{BL}(V, W)$, where $\|\lambda\|_{W'}$ denotes the dual norm of $\lambda$ with respect to the norm $\|w\|_W$ on $W$. The weak topology on $\mathcal{BL}(V, W)$ determined by the collection of linear functionals $L_{v,\lambda}$ with $v \in V$ and $\lambda \in W'$ is known as the *weak operator topology* on $\mathcal{BL}(V, W)$. Equivalently, this is the topology on $\mathcal{BL}(V, W)$ determined by the collection of seminorms

$$N_{v,\lambda}(T) = |L_{v,\lambda}(T)| = |\lambda(T(v))|, \tag{11.3}$$

with $v \in V$ and $\lambda \in W'$. Of course, if $T \neq 0$, then $T(v) \neq 0$ for some $v \in V$. The Hahn–Banach theorem implies that $W'$ separates points in $W$, and hence there is a $\lambda \in W'$ such that $\lambda(T(v)) \neq 0$. It follows that these collections of linear functionals and seminorms on $\mathcal{BL}(V, W)$ are nice, so that $\mathcal{BL}(V, W)$ is Hausdorff with respect to the weak operator topology.

As in (11.2),

$$|L_{v,\lambda}(T)| \leq \|\lambda\|_{W'} \, \|T(v)\|_W \tag{11.4}$$

for every $v \in V$, $\lambda \in W'$, and $T \in \mathcal{BL}(V, W)$. This implies that $L_{v,\lambda}$ is a continuous linear functional with respect to the strong operator topology on $\mathcal{BL}(V, W)$ for every $v \in V$ and $\lambda \in W'$. It follows that every open set in $\mathcal{BL}(V, W)$ with respect to the weak operator topology is also an open set with respect to the strong operator topology. If $W = \mathbf{R}$ or $\mathbf{C}$, as appropriate, then $\mathcal{BL}(V, W)$ is the same as the dual $V'$ of $V$, and the weak operator topology is the same as the strong operator topology on $\mathcal{BL}(V, W)$, which corresponds exactly to the weak$^*$ topology on $V'$. However, if $V = \mathbf{R}$ or $\mathbf{C}$, as appropriate, so that $\mathcal{BL}(V, W)$ can be identified with $W$, then the weak operator topology on $\mathcal{BL}(V, W)$ corresponds to the weak topology on $W$, while the strong operator topology on $\mathcal{BL}(V, W)$ corresponds to the topology on $W$ determined by the norm.

Suppose that $W$ is the dual of a vector space $Z$ with a norm $\|z\|_Z$. In this case, one might wish to restrict one's attention to linear functionals on $W$ that correspond to evaluation at elements of $Z$, as for the weak$^*$ topology on $W$ as the dual of $Z$. This leads to a slightly different version of the weak operator topology on $\mathcal{BL}(V, W)$, that may be described as the weak$^*$ operator topology on $\mathcal{BL}(V, W)$. If $V = \mathbf{R}$ or $\mathbf{C}$, as appropriate, then $\mathcal{BL}(V, W)$ can be identified with $W$, and the weak$^*$ operator topology on $\mathcal{BL}(V, W)$ would correspond to the weak$^*$ topology on $W$. Of course, if $Z$ is reflexive, then every bounded linear functional on $W$ is of this form, and the weak$^*$ operator topology reduces to the weak operator topology on $\mathcal{BL}(V, W)$.

Suppose now that the norm on $W$ comes from an inner product $\langle \cdot, \cdot \rangle_W$ on $W$, so that

$$\|w\|_W = \langle w, w \rangle_W^{1/2} \tag{11.5}$$

for every $w \in W$. This implies that

$$\lambda_w(u) = \langle u, w \rangle_W \tag{11.6}$$

is a bounded linear functional on $W$ for every $w \in W$, by the Cauchy–Schwarz inequality. If $W$ is a Hilbert space, which is to say that $W$ is also complete with respect to the metric associated to the norm, then it is well known that every bounded linear functional on $W$ is of the form $\lambda_w$ for some $w \in W$. In this case, the linear functionals (11.1) on $\mathcal{BL}(V, W)$ can be expressed as

$$\widetilde{L}_{v,w}(T) = \langle T(v), w \rangle \tag{11.7}$$

for some $v \in V$ and $w \in W$, and the seminorms (11.3) can be expressed as

$$\widetilde{N}_{v,w}(T) = |\widetilde{L}_{v,w}(T)| = |\langle T(v), w \rangle|. \tag{11.8}$$

# 12 Multiplication operators, 2

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $b$ be a bounded measurable real or complex-valued function on $X$. Also let $T_b(f) = b\, f$ be the corresponding multiplication operator acting on measurable functions on $X$. In particular, this defines a bounded linear operator on $L^p(X)$ for each $p$, $1 \le p \le \infty$. Let $q$ be the exponent conjugate to $p$, so that $1 \le q \le \infty$ and $1/p + 1/q = 1$. If $g \in L^q(X)$, then

$$\lambda_g(f) = \int_X f(x)\, g(x)\, d\mu(x) \tag{12.1}$$

defines a bounded linear functional on $L^p(X)$, by Hölder's inequality. Thus $\lambda_g(T(f))$ is one of the linear functionals on $\mathcal{BL}(L^p(X))$ used to define the weak operator topology. If we apply this to $T_b$, then we get

$$\lambda_g(T_b(f)) = \int_X b(x)\, f(x)\, g(x)\, d\mu(x), \tag{12.2}$$

which can be considered as a linear functional on $L^\infty(X)$, acting on $b$. Of course, $f\,g \in L^1(X)$ under these conditions, by Hölder's inequality, so that (12.2) is one of the usual linear functionals on $L^\infty(X)$ defined by integrating $b$ times an integrable function on $X$. Note that every integrable function on $X$ can be expressed as $f\,g$ for some $f \in L^p(X)$ and $g \in L^q(X)$.

If $X$ is $\sigma$-finite with respect to $\mu$, or if $\mu$ is counting measure on $X$, then it is well known that every bounded linear functional on $L^p(X)$ is of the form (12.1) for some $g \in L^q(X)$ when $1 \le p < \infty$. In this case, the discussion in the previous paragraph implies that the weak$^*$ topology on $L^\infty(X)$ as the dual of $L^1(X)$ corresponds exactly to the topology induced on the set of multiplication operators $T_b$ with $b$ in $L^\infty(X)$ by the weak operator topology on $\mathcal{BL}(L^p(X))$ when $1 \le p < \infty$. There is an analogous statement for $p = \infty$ using the weak$^*$ operator topology on $\mathcal{BL}(L^\infty(X))$ associated to the weak$^*$ topology on $L^\infty(X)$.

Let us now restrict our attention to the case where $\mu$ is counting measure on $X$. Every element of $\ell^1(X)$ determines a bounded linear functional on $c_0(X) \subseteq \ell^\infty(X)$ as before, and it is well known that every bounded linear functional on $c_0(X)$ is of this form. One can show that every summable function on $X$ can be expressed as the product of a summable function on $X$ and a function that vanishes at infinity on $X$. To see this, it is easy to reduce to the case where $X = \mathbf{Z}_+$, because the set of points where a summable function on $X$ is nonzero has only finitely or countably many elements. This case can be handled as in part (b) of Exercise 12 at the end of Chapter 3 in [30].

Using these remarks, one can check that the weak$^*$ topology on $\ell^\infty(X)$ as the dual of $\ell^1(X)$ corresponds exactly to the topology induced on the set of multiplication operators $T_b$ with $b \in \ell^\infty(X)$ by the weak operator topology on $\mathcal{BL}(c_0(X))$. Similarly, the weak$^*$ topology on $\ell^\infty(X)$ corresponds exactly to the topology induced on the set of multiplication operators $T_b$ with $b \in \ell^\infty(X)$ by the weak$^*$ operator topology on $\mathcal{BL}(\ell^1(X))$, where $\ell^1(X)$ is identified with the dual of $c_0(X)$.

# 13 Dual linear mappings

Let $V$ and $W$ be vector spaces, both real or both complex, and equipped with norms $\|v\|_V$ and $\|w\|_W$, respectively. Also let $V'$ and $W'$ be the dual spaces of bounded linear functionals on $V$ and $W$, respectively, with their corresponding dual norms $\|\lambda\|_{V'}$ and $\|\mu\|_{W'}$. If $T$ is a bounded linear mapping from $V$ into $W$, then there is an associated dual linear mapping $T'$ from $W'$ into $V'$, which sends $\mu \in W'$ to the linear functional $T'(\mu)$ on $V$ defined by

$$T'(\mu) = \mu \circ T. \tag{13.1}$$

Thus

$$(13.2)\quad |(T'(\mu))(v)| = |\mu(T(v))| \le \|\mu\|_{W'}\,\|T(v)\|_W \le \|\mu\|_{W'}\,\|T\|_{op,VW}\,\|v\|_V$$

for every $v \in V$ and $\mu \in W'$, where $\|T\|_{op,VW}$ is the operator norm of $T$ as a bounded linear mapping from $V$ into $W$. This implies that

$$(13.3)\qquad \|T'(\mu)\|_{W'} \le \|T\|_{op,VW}\,\|\mu\|_{W'}$$

for every $\mu \in W'$, so that $T'$ is a bounded linear mapping from $W'$ into $V'$, with operator norm less than or equal to $\|T\|_{op,VW}$. Similarly,

$$(13.4)\quad |\mu(T(v))| = |(T'(\mu))(v)| \le \|T'(\mu)\|_{V'}\,\|v\|_V \le \|T'\|_{op,W'V'}\,\|\mu\|_{W'}\,\|v\|_V$$

for every $v \in V$ and $\mu \in W'$, where $\|T'\|_{op,W'V'}$ is the operator norm of $T'$ as a bounded linear mapping from $W'$ into $V'$. Using this and the Hahn–Banach theorem, one gets that

$$(13.5)\qquad \|T(v)\|_W \le \|T'\|_{op,W'V'}\,\|v\|_V$$

for every $v \in V$, and hence that

$$(13.6)\qquad \|T'\|_{op,W'V'} = \|T\|_{op,VW}.$$

Of course,

$$(13.7)\qquad T \mapsto T'$$

defines a linear mapping from $\mathcal{BL}(V, W)$ into $\mathcal{BL}(W', V')$. One can check that the weak operator topology on $\mathcal{BL}(V, W)$ corresponds exactly to the topology induced on the image of $\mathcal{BL}(V, W)$ in $\mathcal{BL}(W', V')$ under (13.7) by the weak$^*$ operator topology on $\mathcal{BL}(W', V')$. If $V$ is reflexive, then the weak and weak$^*$ operator topologies on $\mathcal{BL}(W', V')$ are the same.

Let $V''$ and $W''$ be the dual spaces of bounded linear functionals on $V'$ and $W'$, respectively, with their corresponding dual norms $\|\cdot\|_{V''}$ and $\|\cdot\|_{W''}$. It is well known that

$$(13.8)\qquad L_v(\lambda) = \lambda(v)$$

defines an element of $V''$ for every $v \in V$, and that

$$(13.9)\qquad v \mapsto L_v$$

is an isometric linear mapping from $V$ into $V''$. Let $T$ be a bounded linear mapping from $V$ into $W$, and let $T'$ be the dual linear mapping from $W'$ into $V'$, as in the previous paragraph. Repeating the process, we get a bounded linear mapping $T''$ from $V''$ into $W''$ such that

$$(13.10)\qquad \|T''\|_{op,V''W''} = \|T'\|_{op,W'V'} = \|T\|_{op,VW},$$

where $\|T''\|_{op,V''W''}$ is the corresponding operator norm of $T''$. It is easy to see that the restriction of $T''$ to the image of $V$ in $V''$ under (13.9) corresponds exactly to $T$ in the obvious way. Of course, if $V$ and $W$ are reflexive, then we can identify $V$, $W$ with $V$", $W$", and hence we can identify $\mathcal{BL}(V,W)$ with $\mathcal{BL}(V'',W'')$. In this case, $T''$ corresponds exactly to $T$ with respect to these identifications.

Suppose now that $V$ and $W$ are Hilbert spaces, with inner products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, respectively. If $T$ is a bounded linear mapping from $V$ into $W$, then it is well known that there is a unique adjoint linear mapping $T^*$ from $W$ into $V$ such that

$$\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V \tag{13.11}$$

for every $v \in V$ and $w \in W$. In the real case, one can use the inner products to identify $V$ and $W$ with their dual spaces, and $T^*$ is basically the same as the dual linear mapping $T'$ defined earlier. This does not quite work in the complex case, because of complex conjugation. In particular,

$$T \mapsto T^* \tag{13.12}$$

is a linear mapping from $\mathcal{BL}(V,W)$ into $\mathcal{BL}(W,V)$ in the real case, and a conjugate-linear mapping in the complex case. In both cases,

$$\|T^*\|_{op,WV} = \|T\|_{op,VW} \tag{13.13}$$

and

$$(T^*)^* = T \tag{13.14}$$

for every $T \in \mathcal{BL}(V,W)$, where the subscripts in (13.13) indicate which operator norm is being used, as before. In particular, (13.14) implies that (13.12) maps $\mathcal{BL}(V,W)$ onto $\mathcal{BL}(W,V)$. It is easy to see that (13.12) is also a homeomorphism from $\mathcal{BL}(V,W)$ onto $\mathcal{BL}(W,V)$ with respect to the corresponding weak operator topologies.

Let $V_1$, $V_2$, and $V_3$ be vector spaces, all real or all complex, and with norms $\|\cdot\|_{V_1}$, $\|\cdot\|_{V_2}$, and $\|\cdot\|_{V_3}$, respectively. If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are bounded linear mappings, then their composition $T_2 \circ T_1$ is a bounded linear mapping from $V_1$ into $V_3$. One can check that the dual of $T_2 \circ T_1$ is equal to $T_1' \circ T_2'$, as a bounded linear mapping from $V_3'$ into $V_1'$. Similarly, if $V_1$, $V_2$, and $V_3$ are Hilbert spaces, then the adjoint of $T_2 \circ T_1$ is equal to $T_1^* \circ T_2^*$ as a bounded linear mapping from $V_3$ into $V_1$.

# 14 Shift operators, 3

Let $f$ be a real or complex-valued function on $\mathbf{Z}$, and put

$$T(f)(j) = f(j+1) \tag{14.1}$$

for each $j \in \mathbf{Z}$, as in Section 7. Suppose that $1 \le p, q \le \infty$ are conjugate exponents, so that $1/p + 1/q = 1$, and let $f \in \ell^p(\mathbf{Z})$ and $g \in \ell^q(\mathbf{Z})$ be given.

Note that

$$\begin{aligned}(14.2)\qquad \sum_{j=-\infty}^{\infty} |T^n(f)(j)|\,|g(j)| &= \sum_{j=-\infty}^{\infty} |f(j+n)|\,|g(j)| \\ &\le \|T^n(f)\|_p\,\|g\|_q = \|f\|_p\,\|g\|_q\end{aligned}$$

for each $n$, by Hölder's inequality. If $1 < p, q < \infty$, then one can check that

$$(14.3)\qquad \sum_{j=-\infty}^{\infty} T^n(f)(j)\,g(j) = \sum_{j=-\infty}^{\infty} f(j+n)\,g(j) \to 0$$

as $|n| \to \infty$. More precisely, if $g(j) \ne 0$ for only finitely many $j$, then this follows from the fact that $f \in c_0(\mathbf{Z})$ when $p < \infty$. One can then get (14.3) for every $f \in \ell^p(\mathbf{Z})$ and $g \in \ell^q(\mathbf{Z})$ using the fact that functions with finite support on $\mathbf{Z}$ are dense in $\ell^q(\mathbf{Z})$ when $q < \infty$. This implies that $T^n \to 0$ as $n \to \infty$ with respect to the weak operator topology on $\mathcal{BL}(\ell^p(\mathbf{Z}))$ when $1 < p < \infty$. Similarly, (14.3) holds when $f \in c_0(\mathbf{Z})$ and $g \in \ell^1(\mathbf{Z})$, which implies that $T^n \to 0$ with respect to the weak operator topology on $\mathcal{BL}(c_0(\mathbf{Z}))$. This also works when $f \in \ell^1(\mathbf{Z})$ and $g \in c_0(\mathbf{Z})$, which implies that $T^n \to 0$ as $n \to \infty$ with respect to the weak$^*$ operator topology on $\mathcal{BL}(\ell^1(\mathbf{Z}))$, where $\ell^1(\mathbf{Z})$ is identified with the dual of $c_0(\mathbf{Z})$. However, this does not work when $f \in \ell^1(\mathbf{Z})$ and $g \in \ell^\infty(\mathbf{Z})$, as one can see by taking $g$ to be a nonzero constant function on $\mathbf{Z}$. Thus $\{T^n\}_{n=1}^\infty$ does not converge to 0 with respect to the weak operator topology on $\mathcal{BL}(\ell^1(\mathbf{Z}))$.

If $f \in \ell^p(\mathbf{Z})$ and $g \in \ell^q(\mathbf{Z})$, where $1 \le p, q \le \infty$ are conjugate exponents, then

$$\begin{aligned}(14.4)\qquad \sum_{j=-\infty}^{\infty} (T(f))(j)\,g(j) &= \sum_{j=-\infty}^{\infty} f(j+1)\,g(j) \\ &= \sum_{j=-\infty}^{\infty} f(j)\,g(j-1) \\ &= \sum_{j=-\infty}^{\infty} f(j)\,(T^{-1}(g))(j).\end{aligned}$$

This shows that the dual of $T$ on $\ell^p(\mathbf{Z})$ can be identified with $T^{-1}$ on $\ell^q(\mathbf{Z})$ when $1 \le p < \infty$, and similarly the dual of $T$ on $c_0(\mathbf{Z})$ can be identified with $T^{-1}$ on $\ell^1(\mathbf{Z})$. This also shows that the adjoint of $T$ on $\ell^2(\mathbf{Z})$ as a Hilbert space with respect to the standard inner product can be identified with $T^{-1}$ in the real case. The same conclusion holds in the complex case, because

$$(14.5)\qquad \sum_{j=-\infty}^{\infty} (T(f))(j)\,\overline{g(j)} = \sum_{j=-\infty}^{\infty} f(j)\,\overline{(T^{-1}(g))(j)}$$

for all complex-valued square-summable functions $f$, $g$ on $\mathbf{Z}$, as in (14.4).

Now let $f$ be a real or complex-valued function on $\mathbf{Z}_+$, and put

$$A(f)(j) = f(j-1) \tag{14.6}$$

when $j \ge 2$, and $A(f)(1) = 0$. If we extend functions on $\mathbf{Z}_+$ to functions on $\mathbf{Z}$ by setting them equal to 0 when $j \le 0$, then $A(f)$ corresponds exactly to $T^{-1}(f)$, where $T$ is as in the previous paragraphs. In particular, $A^n \to 0$ as $n \to \infty$ with respect to the weak operator topology on $\mathcal{BL}(\ell^p(\mathbf{Z}_+))$ when $1 < p < \infty$, and with respect to the weak operator topology on $\mathcal{BL}(c_0(\mathbf{Z}_+))$, for the same reasons as before. This also works with respect to the weak$^*$ operator topology on $\mathcal{BL}(\ell^1(\mathbf{Z}_+))$, but not with respect to the weak operator topology on $\mathcal{BL}(\ell^1(\mathbf{Z}_+))$.

Similarly, put

$$B(f)(j) = f(j+1) \tag{14.7}$$

for each function $f$ on $\mathbf{Z}_+$, which is the same as the operator considered in Section 4. Let $1 \le p, q \le \infty$ be conjugate exponents again, and note that $A$ and $B$ are bounded linear operators on $\ell^p(\mathbf{Z}_+)$ for each $p$, and that they both map $c_0(\mathbf{Z}_+)$ into itself. If $f \in \ell^p(\mathbf{Z}_+)$ and $g \in \ell^q(\mathbf{Z}_+)$, then

$$\begin{aligned} \sum_{j=1}^{\infty} A(f)(j)\, g(j) &= \sum_{j=2}^{\infty} f(j-1)\, g(j) \\ &= \sum_{j=1}^{\infty} f(j)\, g(j+1) = \sum_{j=1}^{\infty} f(j)\, B(g)(j), \end{aligned} \tag{14.8}$$

where the convergence of the sums follows from Hölder's inequality. Let us identify the dual of $\ell^p(\mathbf{Z}_+)$ with $\ell^q(\mathbf{Z}_+)$ in the usual way when $1 \le p < \infty$, and the dual of $c_0(\mathbf{Z}_+)$ with $\ell^1(\mathbf{Z}_+)$. It follows from (14.8) that the dual of $A$ on $\ell^p(\mathbf{Z}_+)$ can be identified with $B$ acting on $\ell^q(\mathbf{Z}_+)$ when $1 \le p < \infty$, and that the dual of $A$ on $c_0(\mathbf{Z}_+)$ can be identified with $B$ acting on $\ell^1(\mathbf{Z}_+)$. Similarly, the dual of $B$ on $\ell^p(\mathbf{Z}_+)$ can be identified with $A$ acting on $\ell^q(\mathbf{Z}_+)$ when $1 \le p < \infty$, and the dual of $B$ on $c_0(\mathbf{Z}_+)$ can be identified with $A$ acting on $\ell^1(\mathbf{Z}_+)$. This also implies that $A$ and $B$ are adjoints of each other on $\ell^2(\mathbf{Z}_+)$, as a Hilbert space with respect to the usual inner product in the real case. One can get the same conclusion in the complex case using the fact that

$$\sum_{j=1}^{\infty} A(f)(j)\, \overline{g(j)} = \sum_{j=1}^{\infty} f(j)\, \overline{B(g)(j)}, \tag{14.9}$$

for all complex-valued square-summable functions $f$, $g$ on $\mathbf{Z}_+$, as in (14.8).

# 15 Uniform boundedness

Let $Z$ be a vector space over the real or complex numbers equipped with a norm $\|z\|_Z$, and let $Z'$ be the dual space of bounded linear functionals on $Z$, equipped with the dual norm $\|\lambda\|_{Z'}$. Also let $E$ be a subset of $Z'$ which is bounded pointwise on $Z$, in the sense that for each $z \in Z$, the real or complex numbers of the form $\lambda(z)$ with $\lambda \in E$ are bounded. If $Z$ is complete with respect to the metric associated to the norm, then the Banach–Steinhaus theorem implies that $E$ is bounded in $Z'$ with respect to the dual norm.

Now let $W$ be a real or complex vector space with a norm $\|w\|_W$, and let $W'$ be the dual of $W$, with the dual norm. Suppose that $E \subseteq W$ is weakly bounded, in the sense that for each $\mu \in W'$, the set of real or complex numbers of the form $\mu(w)$ with $w \in E$ is bounded. Under these conditions, it is also well known that $E$ has to be bounded with respect to the norm on $W$. This can be derived from the remarks in the previous paragraph, applied to $Z = W'$, and using the standard isometric embedding of $W$ in the dual $W''$ of $W'$. Note that $W'$ is always complete with respect to the dual norm.

Suppose that $V$ and $W$ are vector spaces, both real or both complex, and equipped with norms $\|v\|_V$ and $\|w\|_W$, respectively. Let $E$ be a subset of $\mathcal{BL}(V, W)$ with the property that for each $v \in V$ and $\mu \in W'$, the real or complex numbers of the form $\mu(T(v))$ with $T \in E$ are bounded. This implies that for each $v \in V$, the set of vectors in $W$ of the form $T(v)$ with $T \in E$ is bounded, by the remarks in the previous paragraph. If $V$ is complete, then it follows that $E$ is bounded with respect to the operator norm on $\mathcal{BL}(V, W)$, by the Banach–Steinhaus theorem. There are analogous statements for the case where $W$ is the dual of a Banach space $Z$, and one restricts one's attention to bounded linear functionals on $W$ corresponding to evaluation at elements of $Z$.

In particular, if $\{T_j\}_{j=1}^\infty$ is a sequence of bounded linear mappings from $V$ into $W$ that converges with respect to the weak operator topology on $\mathcal{BL}(V, W)$, then $\{\mu(T_j(v))\}_{j=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$ and $\mu \in W'$, and hence is bounded. This implies that the operator norms of the $T_j$'s are bounded when $V$ is complete, as in the preceding paragraph. Similarly, if $W$ is the dual of a Banach space $Z$, and if $\{T_j\}_{j=1}^\infty$ converges with respect to the weak$^*$ operator topology on $\mathcal{BL}(V, W)$, then the operator norms of the $T_j$'s are bounded.

# 16 Continuous linear functionals

Let $V$ and $W$ be vector spaces, both real or both complex, and equipped with norms $\|v\|_V$ and $\|w\|_W$, respectively. Suppose that $L$ is a linear functional on $\mathcal{BL}(V, W)$ which is continuous with respect to the strong operator topology. This implies that there are finitely many vectors $v_1, \ldots, v_n$ in $V$ and a nonnegative real number $C$ such that

$$|L(T)| \le C \max_{1 \le j \le n} \|T(v_j)\|_W \tag{16.1}$$

for every $T \in \mathcal{BL}(V, W)$. More precisely, this can be derived from the fact that $|L(T)| < 1$ for all $T$ in an open set in $\mathcal{BL}(V, W)$ with respect to the strong operator topology that contains 0.

Let $W^n$ be the set of $n$-tuples $(w_1, \ldots, w_n)$ with coordinates in $W$. This is a vector space with respect to coordinatewise addition and scalar multiplication, which is the same as the direct sum of $n$ copies of $W$. Of course,

$$\|(w_1, \ldots, w_n)\|_{W^n} = \max_{1 \le j \le n} \|w_j\|_W \tag{16.2}$$

defines a norm on $W^n$, for which the corresponding topology is the same as the product topology associated to the topology on $W$ determined by the norm $\|w\|_W$. Observe that

$$T \mapsto (T(v_1), \ldots, T(v_n)) \tag{16.3}$$

defines a continuous linear mapping from $\mathcal{BL}(V, W)$ into $W^n$, with respect to the strong operator topology on $\mathcal{BL}(V, W)$ and the topology determined by the norm on $W^n$. Moreover, $L(T) = 0$ when $T(v_j) = 0$ for each $j$, by (16.1), so that $L(T)$ actually depends only on $T(v_1), \ldots, T(v_n)$.

We may as well ask also that $v_1, \ldots, v_n$ be linearly independent in $V$, since otherwise we can drop some of the $v_j$'s until this occurs. The condition (16.1) will still hold, but perhaps with a different constant $C$. Using the Hahn–Banach theorem and the linear independence of the $v_j$'s, any $n$-tuple of vectors in $W$ can occur as $T(v_1), \ldots, T(v_n)$ for some $T \in \mathcal{BL}(V, W)$, so that (16.3) maps $\mathcal{BL}(V, W)$ onto $W^n$. This implies that $L$ can be expressed as the composition of a bounded linear functional on $W^n$ with (16.3). Alternatively, one can argue directly that $L$ is the composition of a bounded linear functional on a subspace of $W^n$ with (16.3), and then use the Hahn–Banach theorem to extend the bounded linear functional on a subspace of $W^n$ to all of $W^n$.

Of course, every bounded linear functional on $W^n$ can be expressed as a sum of bounded linear functionals on $W$ applied to each of the $n$ coordinates. This implies that $L$ can be expressed as

$$L(T) = \sum_{j=1}^{n} \lambda_j(T(v_j)) \tag{16.4}$$

for some bounded linear functionals $\lambda_1, \ldots, \lambda_n$ on $W$ and every $T \in \mathcal{BL}(V, W)$. Conversely, if a linear functional $L$ on $\mathcal{BL}(V, W)$ is of this form for some finite collection of vectors $v_1, \ldots, v_n \in V$ and bounded linear functionals $\lambda_1, \ldots, \lambda_n$ on $W$, then $L$ is obviously continuous with respect to the strong operator topology. In fact, $L$ is also continuous with respect to the weak operator topology on $\mathcal{BL}(V, W)$ under these conditions.

## 17 Bilinear functionals

Let $V$ and $Z$ be vector spaces, both real or both complex, and equipped with norms $\|v\|_V$ and $\|z\|_Z$, respectively. Also let $b(v, z)$ be a real or complex-valued function of $v \in V$ and $z \in Z$, as appropriate. If $b(v, z)$ is a linear function of $v$ for each $z \in Z$, and a linear function of $z$ for each $v \in V$, then $b(v, z)$ is said to be bilinear. A bilinear functional $b(v, z)$ on $V \times Z$ is said to be bounded if

$$|b(v, z)| \le C \, \|v\|_V \, \|z\|_Z \tag{17.1}$$

for some nonnegative real number $C$ and every $v \in V$ and $z \in Z$. It is easy to see that a bounded bilinear functional $b(v,z)$ on $V \times Z$ is continuous with respect to the product topology on $V \times Z$ associated to the topologies on $V$ and $Z$ determined by their norms. Conversely, if $b(v,z)$ is continuous at 0 with respect to the product topology on $V \times Z$, then there are positive real numbers $r$, $t$ such that

$$|b(v,z)| < 1 \tag{17.2}$$

for every $v \in V$ and $z \in Z$ that satisfy $\|v\|_V < r$ and $\|z\|_Z < t$. Using this, one can check that (17.1) holds with $C = 1/r\,t$.

Suppose that $b(v,z)$ is a bounded bilinear functional on $V \times Z$, and put $b_v(z) = b(v,z)$ for each $v \in V$ and $z \in Z$. Thus $b_v(z)$ is a bounded linear functional on $Z$ for each $v \in V$, and

$$v \mapsto b_v \tag{17.3}$$

defines a bounded linear mapping from $V$ into the dual $Z'$ of $Z$, with respect to the dual norm $\|\cdot\|_{Z'}$ on $Z$. More precisely, if $b(v,z)$ satisfies (17.1), then the operator norm of (17.3) is less than or equal to $C$. Conversely, every bounded linear mapping from $V$ into $Z'$ determines a bounded bilinear functional $b(v,z)$ on $V \times Z$ in this way.

Suppose for the moment that $V$ is a real vector space, and let $(W, \langle \cdot, \cdot \rangle_W)$ be a real inner product space. Also let $\|w\|_W$ be the corresponding norm on $W$, which is the square root of $\langle w, w \rangle_W$. If $T$ is a bounded linear mapping from $V$ into $W$, then

$$b(v,w) = \langle T(v), w \rangle_W \tag{17.4}$$

is a bounded bilinear functional on $V \times W$. Conversely, if $W$ is a Hilbert space, then every bounded bilinear functional on $V \times W$ can be expressed as (17.4) for some bounded linear mapping $T$ from $V$ into $W$. This uses the Riesz representation theorem, which says that every bounded linear functional on $W$ can be expressed in terms of the inner product with a unique element of $W$.

If $V$ and $W$ are complex and $T$ is a bounded linear mapping from $V$ into $W$, then (17.4) is linear in $v$ and conjugate-linear in $w$. Of course, (17.4) still satisfies the boundedness condition (17.1), with $C$ equal to the operator norm of $T$. Conversely, if $b(v,w)$ is a complex-valued function on $V \times W$ which is linear in $v$, conjugate-linear in $w$, and bounded in the sense of (17.1), then $b(v,w)$ is of the form (17.4) for some bounded linear mapping from $V$ into $W$. As before, this can be derived from the representation of bounded linear functionals on $W$ in terms of the inner product.

## 18 Compactness

Let $V$ and $Z$ be vector spaces again, both real or both complex, and equipped with norms $\|v\|_V$ and $\|z\|_Z$, respectively. As in the previous section, the space $\mathcal{BL}(V, Z')$ of bounded linear mappings from $V$ into the dual $Z'$ of $Z$ can be identified with the space of bounded bilinear functionals $b(v,z)$ on $V \times Z$. Note that the space of bounded bilinear functionals on $V \times Z$ is a vector space with respect to pointwise addition and scalar multiplication of functions. The weak$^*$ operator topology on $\mathcal{BL}(V, Z)$ corresponds to the weak topology on the space of bounded bilinear functionals on $V \times Z$ determined by the collection of linear functionals of the form

$$\widehat{L}_{v,z}(b) = b(v, z), \tag{18.1}$$

with $v \in V$ and $z \in Z$.

Let $B$ be the closed unit ball in $\mathcal{BL}(V, Z')$ with respect to the operator norm. This can be identified with the collection of bounded bilinear functionals $b(v, z)$ on $V \times Z$ that satisfy (17.1) with $C = 1$, as before. It is easy to see that this is a closed set in $\mathcal{BL}(V, Z')$ with respect to the weak$^*$ operator topology, which is the same as saying that the corresponding collection of bounded bilinear functionals is a closed set with respect to the weak topology mentioned in the previous paragraph. One can also show that $B$ is compact with respect to this topology. This is analogous to the Banach–Alaoglu theorem, which says that the closed unit ball in the dual of a vector space with a norm is compact with respect to the weak$^*$ topology.

As usual, there are some simplifications when $V$ and $Z$ are separable. Let $A_V$ and $A_Z$ be subsets of $V$ and $Z$ with only finitely or countably many elements whose linear spans are dense in $V$ and $Z$, respectively. In this case, one can get the same topology on $B$ using linear functionals of the form (18.1) with $v \in A_V$ and $z \in A_Z$. This implies that the induced topology on $B$ is determined by a metric, so that compactness of $B$ is equivalent to sequential compactness. Let $\{b_j(v, z)\}_{j=1}^\infty$ be a sequence of bounded bilinear functionals on $V \times Z$ that satisfy (17.1) with $C = 1$. We would like to show that there is a subsequence $\{b_{j_l}(v, z)\}_{l=1}^\infty$ that converges for each $v \in V$ and $z \in Z$ to a bounded bilinear functional $b(v, z)$ on $V \times Z$ that also satisfies (17.1) with $C = 1$. Of course, $\{b_j(v, z)\}_{j=1}^\infty$ is a bounded sequence of real or complex numbers for each $v \in V$ and $z \in Z$, and hence there is a subsequence depending on $v$ and $z$ that converges in $\mathbf{R}$ or $\mathbf{C}$, as appropriate. By standard Cantor diagonalization arguments, there is an increasing sequence $\{j_l\}_{l=1}^\infty$ of positive integers such that $\{b_{j_l}(v, z)\}_{l=1}^\infty$ converges in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in A_V$ and $z \in A_Z$. Using bilinearity, it follows that $\{b_{j_l}(v, z)\}_{l=1}^\infty$ converges for every $v$, $z$ in the linear spans of $A_V$, $A_Z$ in $V$, $Z$, respectively. Because the linear spans of $A_V$, $A_Z$ are dense in $V$, $Z$, one can use the uniform boundedness of the $b_j$'s to get that $\{b_{j_l}(v, z)\}_{l=1}^\infty$ is a Cauchy sequence in $\mathbf{R}$ or $\mathbf{C}$ for every $v \in V$ and $z \in Z$. This implies that $\{b_{j_l}(v, z)\}_{l=1}^\infty$ converges to a real or complex number $b(v, z)$ for every $v \in V$ and $z \in Z$. It is easy to see that $b(v, z)$ is a bilinear functional on $V \times Z$, since $b_j(v, z)$ is bilinear for each $j$. Similarly, $b(v, z)$ satisfies (17.1) with $C = 1$, because of the corresponding property of $b_j(v, z)$ for each $j$.

Let $V$ and $W$ be vector spaces, both real or both complex, and equipped with norms. If $W$ is reflexive, then the closed unit ball in $\mathcal{BL}(V, W)$ is compact with respect to the weak operator topology. This follows from the previous discussion applied to $Z = W'$, and identifying $W$ with its second dual. In the context of Hilbert spaces, this would typically be reformulated in terms of the inner product. In particular, for complex Hilbert spaces, one can also use slightly different bilinearity conditions, as in the previous section.

# 19 Other operators, 2

Let $V$ be a real or complex vector space with a norm $\|v\|_V$, and let $T$ be a bounded linear operator on $V$, with operator norm less than or equal to 1. Put

$$A_n = \frac{1}{n+1} \sum_{l=0}^{n} T^l \tag{19.1}$$

for each nonnegative integer $n$, so that $A_n$ also has operator norm less than or equal to 1, as in Section 8. Also let $V'$ be the dual space of bounded linear functionals on $V$, with the corresponding dual norm $\|\lambda\|_{V'}$, and let $T'$ be the dual linear mapping on $V'$ associated to $T$ as in Section 13. Thus $T'$ is a bounded linear operator on $V'$, with operator norm equal to the operator norm of $T$ on $V$. Note that

$$(T^l)' = (T')^l \tag{19.2}$$

for each nonnegative integer $l$, and hence that

$$A_n' = \frac{1}{n+1} \sum_{l=0}^{n} (T')^l \tag{19.3}$$

for every nonnegative integer $n$. As before, the operator norm of $A_n'$ on $V'$ is equal to the operator norm of $A_n$ on $V$ for each $n$, which is less than or equal to 1. Remember that the closed unit ball in $\mathcal{BL}(V')$ is compact with respect to the weak$^*$ operator topology, as in the previous section. If $V$ is reflexive, then $V$ can be identified with the dual of $V'$, and $T$ can be identified with the dual of $T'$, so that the remarks in this section can be applied directly to $V$ and $T$.

If $V$ and $V'$ are separable, then there is a subsequence of $\{A_n'\}_{n=1}^\infty$ that converges with respect to the weak$^*$ operator topology on $\mathcal{BL}(V')$ to a bounded linear operator $R$ on $V'$, with operator norm less than or equal to 1, as in the previous section. Otherwise, put

$$E_k = \{A_n' : n \ge k\} \tag{19.4}$$

for each nonnegative integer $k$, and let $\overline{E_k}$ be the closure of $E_k$ with respect to the weak$^*$ operator topology on $\mathcal{BL}(V')$. Note that $E_{k+1} \subseteq E_k$ for each $k$, which implies that $\overline{E_{k+1}} \subseteq \overline{E_k}$. Of course, $\overline{E_k} \ne \emptyset$ for each $k$, and hence the compactness of the closed unit ball in $\mathcal{BL}(V')$ with respect to the weak$^*$ operator topology implies that

$$\bigcap_{k=0}^{\infty} \overline{E_k} \ne \emptyset, \tag{19.5}$$

by standard arguments. Let $R$ be an element of this intersection, which is a bounded linear operator on $V'$ with operator norm less than or equal to 1.

Put

$$E_k(\lambda) = \{A'_n(\lambda) : n \geq k\} \tag{19.6}$$

for each $\lambda \in V'$, and let $\overline{E_k(\lambda)}$ be the closure of $E_k(\lambda)$ with respect to the weak$^*$ topology on $V'$. It is easy to see that

$$R(\lambda) \in \bigcap_{k=0}^{\infty} \overline{E_k(\lambda)} \tag{19.7}$$

for each $\lambda \in V'$, using the definition of the weak$^*$ operator topology on $\mathcal{BL}(V')$. If $\{A'_n(\lambda)\}_{n=0}^{\infty}$ converges to some $\mu \in V'$ with respect to the weak$^*$ topology on $V'$, then $\mu$ is the only element of $\bigcap_{k=0}^{\infty} \overline{E_k(\lambda)}$, by standard arguments. Thus $R(\lambda) = \mu$ under these conditions.

Let $I'$ be the identity operator on $V'$, which is the dual of the identity operator $I$ on $V$. If $\lambda \in V'$ is in the closure of $(I' - T')(V')$ with respect to the dual norm on $V'$, then $\{A'_n(\lambda)\}_{n=0}^{\infty}$ converges to 0 with respect to the dual norm on $V'$, as in Section 8. This implies that $R(\lambda) = 0$, by the remarks in the previous paragraph. Similarly, if $\lambda \in V'$ satisfies $T'(\lambda) = \lambda$, then $A'_n(\lambda) = \lambda$ for every $n \geq 0$, and hence $R(\lambda) = \lambda$. If $\lambda$ is any element of $V'$, then

$$A'_n(\lambda) - T'(A'_n(\lambda)) \to 0 \tag{19.8}$$

as $n \to \infty$ with respect to the dual norm on $V'$, as in Section 8. Using this, one can show that

$$T'(R(\lambda)) = R(\lambda) \tag{19.9}$$

for every $\lambda \in V'$. More precisely, this also uses the fact that $T'$ is continuous with respect to the weak$^*$ topology on $V'$. Alternatively, one can check directly that $R$ commutes with $T'$, because $T'$ commutes with $A'_n$ for each $n$.

# 20 Composition operators

Let $X$ be a nonempty topological space, and let $C_b(X)$ be the space of bounded continuous real or complex-valued functions on $X$, with the supremum norm. Also let $\phi$ be a homeomorphism from $X$ onto itself, and put

$$T(f) = T_\phi(f) = f \circ \phi \tag{20.1}$$

for every $f \in C_b(X)$. This defines an isometric linear mapping from $C_b(X)$ onto itself under these conditions. Note that

$$T^l(f) = f \circ \phi^l \tag{20.2}$$

for each $f \in C_b(X)$ and $l \in \mathbf{Z}_+$, where $\phi^l$ is the $l$-fold composition of $\phi$. This also works when $l \leq 0$, with the usual conventions.

Suppose for the moment that $X$ is a compact Hausdorff topological space, so that continuous functions on $X$ are automatically bounded, and $C_b(X)$ is the same as the space $C(X)$ of all continuous real or complex-valued functions on $X$. Also let $C(X)'$ be the dual space of bounded linear functionals on $C(X)$ with respect to the supremum norm. The elements of $C(X)'$ can be identified with real or complex-valued regular Borel measures on $X$, by a version of the Riesz representation theorem. Remember that $\lambda \in C(X)'$ is said to be nonnegative if $\lambda(f) \geq 0$ for every nonnegative real-valued continuous function $f$ on $X$, in which case $\lambda$ corresponds to a nonnegative real-valued regular Borel measure on $X$, and the dual norm of $\lambda$ is equal to $\lambda(\mathbf{1}_X)$. Note that the set of nonnegative elements of $C(X)'$ is closed with respect to the weak$^*$ topology on $C(X)'$.

Let $T'$ be the dual linear mapping on $C(X)'$ associated to $T$, which is an isometric linear mapping from $C(X)'$ onto itself, with respect to the dual norm. The elements $\lambda$ of $C(X)'$ that satisfy

$$T'(\lambda) = \lambda \tag{20.3}$$

correspond exactly to real or complex-valued regular Borel measures on $X$ that are invariant under $\phi$. Note that

$$(T'(\lambda))(\mathbf{1}_X) = \lambda(T(\mathbf{1}_X)) = \lambda(\mathbf{1}_X) \tag{20.4}$$

for every $\lambda \in C(X)'$, and that $T'(\lambda)$ is nonnegative when $\lambda$ is nonnegative. Let $A_n'$ be as in (19.3), which is the dual mapping associated to $A_n$ in (19.1) for each $n$. The operator norm of $A_n'$ on $C(X)'$ is less than or equal to 1 for each $n$, and one can argue as in the previous section to get the existence of limiting operators $R$ with respect to the weak$^*$ operator topology on $\mathcal{BL}(C(X)')$.

If $\lambda \in C(X)'$ is nonnegative, then $A_n'(\lambda)$ is nonnegative for each $n$, and it is easy to see that $R(\lambda)$ is nonnegative too. Similarly, $(A_n'(\lambda))(\mathbf{1}_X) = \lambda(\mathbf{1}_X)$ for each $\lambda \in C(X)'$ and $n \geq 0$, which implies that

$$(R(\lambda))(\mathbf{1}_X) = \lambda(\mathbf{1}_X). \tag{20.5}$$

In particular, $R(\lambda) \neq 0$ when $\lambda(\mathbf{1}_X) \neq 0$. If $\lambda \in C(X)'$ satisfies (20.3), then $A_n'(\lambda) = \lambda$ for each $n$, and hence $R(\lambda) = \lambda$. We also have that (19.9) holds for every $\lambda \in C(X)'$, as in the previous section.

Suppose instead that $X$ is a locally compact Hausdorff topological space which is not compact, and let $C_0(X)$ be the space of continuous real or complex-valued functions on $X$ that vanish at infinity. Remember that $C_0(X)$ is a closed linear subspace of $C_b(X)$ with respect to the supremum norm, and let $C_0(X)'$ be the dual space of bounded linear functionals on $C_0(X)$. As before, the elements of $C_0(X)$ can be identified with real or complex-valued regular Borel measures on $X$, by a version of the Riesz representation theorem. If $\phi$ is a homeomorphism from $X$ onto itself, then $T = T_\phi$ maps $C_0(X)$ onto itself, since continuous mappings send compact sets to compact sets. If $X$ is an infinite set equipped with the discrete topology, then $C_0(X)$ is the same as $c_0(X)$, whose dual can be identified with $\ell^1(X)$.

At any rate, if $X$ is not compact, then $\mathbf{1}_X$ is not an element of $C_0(X)$, which is the main difference between this case and the previous one. Of course, one can always consider the one-point compactification of $X$, for which the corresponding space of continuous functions can be identified with the linear span of $C_0(X)$ and constant functions on $X$. This is the same as the space of continuous functions on $X$ with a limit at infinity, which is also a closed linear subspace of $C_b(X)$. The dual space of bounded linear functionals can then be identified with the linear span of $C_0(X)'$ and the linear functional associated to the limit at infinity. If $\phi$ is a homeomorphism from $X$ onto itself, then $\phi$ has a natural extension to a homeomorphism on the one-point compactification of $X$, which sends the point at infinity to itself.

Let $X$ be an infinite set equipped with the discrete topology, so that $C_b(X) = \ell^\infty(X)$ can be identified with the dual of $\ell^1(X)$. If $\phi$ is a one-to-one mapping from $X$ onto itself, then the composition operator $T = T_\phi$ on $\ell^\infty(X)$ can be identified with the dual of the composition operator on $\ell^1(X)$ associated to $\phi^{-1}$. This permits us to apply the remarks in the previous section directly to $T_\phi$ on $\ell^\infty(X)$, to get a bounded linear operator $A$ on $\ell^\infty(X)$ which can be approximated by the operators $A_n$ in (19.1) for arbitrarily large $n$ with respect to the weak$^*$ operator topology on $\mathcal{BL}(\ell^\infty(X))$. Of course, constant functions on $\mathbf{Z}$ are invariant under $T$, and hence $A$ maps constant functions to themselves too. The analogue of (19.9) in this context implies that

$$T(A(f)) = A(f) \tag{20.6}$$

for every $f \in \ell^\infty(X)$.

Let us now take $X = \mathbf{Z}$ and $\phi(j) = j + 1$. In this case, constant functions on $\mathbf{Z}$ are the only functions that are invariant under $T$, so that $A(f)$ is a constant function on $\mathbf{Z}$ for every $f \in \ell^\infty(\mathbf{Z})$. Thus $A$ can be expressed as

$$A(f) = \lambda(f)\,\mathbf{1}_{\mathbf{Z}} \tag{20.7}$$

for some bounded linear functional $\lambda$ on $\ell^\infty(\mathbf{Z})$ such that $\lambda(\mathbf{1}_{\mathbf{Z}}) = 1$. If $f$ is a nonnegative real-valued bounded function on $\mathbf{Z}$, then $T^l(f) \geq 0$ for each $l$, $A_n(f) \geq 0$ for each $n$, and hence $A(f) \geq 0$. This implies that $\lambda(f) \geq 0$, so that $\lambda$ is a nonnegative linear functional on $\ell^\infty(\mathbf{Z})$. We also have that

$$A_n(f - T(f)) \to 0 \tag{20.8}$$

as $n \to \infty$ with respect to the supremum norm on $\ell^\infty(\mathbf{Z})$ for every $f \in \ell^\infty(\mathbf{Z})$, as in Section 8. It follows that $A(f - T(f)) = 0$ for every $f \in \ell^\infty(\mathbf{Z})$, and hence that $\lambda(f - T(f)) = 0$ for every $f \in \ell^\infty(\mathbf{Z})$. Equivalently,

$$\lambda(T(f)) = \lambda(f) \tag{20.9}$$

for every $f \in \ell^\infty(\mathbf{Z})$, which is to say that $\lambda$ is invariant under translations. If $f \in c_0(\mathbf{Z})$, then we have seen that $\|A_n(f)\|_\infty \to 0$ as $n \to \infty$, so that $\lambda(f) = 0$.

# 21 Continuity properties

Let $V$, $W$, and $Z$ be vector spaces, all real or all complex, and equipped with norms $\|v\|_V$, $\|w\|_W$, and $\|z\|_Z$, respectively. Let $B$ be a bounded linear mapping from $V$ into $W$, and consider

$$R \mapsto R \circ B \tag{21.1}$$

as a linear mapping from $\mathcal{BL}(W, Z)$ into $\mathcal{BL}(V, Z)$. It is easy to see that (21.1) is a bounded linear mapping from $\mathcal{BL}(W, Z)$ into $\mathcal{BL}(V, Z)$ with respect to the corresponding operator norms. One can also check that (21.1) is continuous with respect to the strong operator topologies on $\mathcal{BL}(W, Z)$ and $\mathcal{BL}(V, Z)$, and with respect to the weak operator topologies on $\mathcal{BL}(W, Z)$ and $\mathcal{BL}(V, Z)$. If $Z$ is the dual of another real or complex vector space with a norm, then (21.1) is continuous with respect to the corresponding weak$^*$ operator topologies on $\mathcal{BL}(W, Z)$ and $\mathcal{BL}(V, Z)$.

Now let $A$ be a bounded linear mapping from $W$ into $Z$, and consider

$$T \mapsto A \circ T \tag{21.2}$$

as a linear mapping from $\mathcal{BL}(V, W)$ into $\mathcal{BL}(V, Z)$. As before, (21.2) is a bounded linear mapping from $\mathcal{BL}(V, W)$ into $\mathcal{BL}(V, Z)$, with respect to the corresponding operator norms. Similarly, (21.2) is continuous with respect to the strong operator topologies on $\mathcal{BL}(V, W)$ and $\mathcal{BL}(V, Z)$, and with respect to the weak operator topologies on $\mathcal{BL}(V, W)$ and $\mathcal{BL}(V, Z)$. If $W$ and $Z$ are both duals of other real or complex vector spaces with norms, then the weak$^*$ operator topologies can be defined on $\mathcal{BL}(V, W)$ and $\mathcal{BL}(V, Z)$ in the usual way too. If $A$ is the dual of a bounded linear mapping from the pre-dual of $Z$ into the pre-dual of $W$, then (21.2) is continuous with respect to the corresponding weak$^*$ operator topologies on $\mathcal{BL}(V, W)$ and $\mathcal{BL}(V, Z)$ as well.

If $A$, $B$ are bounded linear mappings from $W$ into $Z$ and from $V$ into $W$, respectively, then $A \circ B$ is a bounded linear mapping from $V$ into $Z$, and we can consider

$$(A, B) \mapsto A \circ B \tag{21.3}$$

as a mapping from $\mathcal{BL}(W, Z) \times \mathcal{BL}(V, W)$ into $\mathcal{BL}(V, Z)$. In order to look at the behavior of this mapping at a point $(A_0, B_0)$, it is helpful to observe that

$$A \circ B - A_0 \circ B_0 = A \circ (B - B_0) + (A - A_0) \circ B_0. \tag{21.4}$$

In particular,

$$\begin{aligned} \|A \circ B - A_0 \circ B_0\|_{op,VZ} \quad \le \quad & \|A\|_{op,WZ} \, \|B - B_0\|_{op,VW} \\ & + \|A - A_0\|_{op,WZ} \, \|B_0\|_{op,VW}, \end{aligned} \tag{21.5}$$

where the subscripts indicate which operator norm is being used. It follows easily from this that (21.3) is continuous with respect to the topologies on $\mathcal{BL}(V, W)$, $\mathcal{BL}(W, Z)$, and $\mathcal{BL}(V, Z)$ determined by the corresponding operator norms, and using the associated product topology on $\mathcal{BL}(W, Z) \times \mathcal{BL}(V, W)$.

Similarly,

$$\|(A \circ B - A_0 \circ B_0)(v)\|_Z \leq \|A\|_{op,WZ} \, \|(B - B_0)(v)\|_W + \|(A - A_0)(B_0(v))\|_Z \tag{21.6}$$

for every $v \in V$. Suppose that $\mathcal{E}$ is a bounded subset of $\mathcal{BL}(W, Z)$ with respect to the operator norm, and consider (21.3) as a mapping from $\mathcal{E} \times \mathcal{BL}(V, W)$ into $\mathcal{BL}(V, Z)$. It is easy to see that this mapping is continuous with respect to the strong operator topologies on $\mathcal{BL}(V, W)$ and $\mathcal{BL}(V, Z)$, and using the topology induced on $\mathcal{E}$ by the strong operator topology on $\mathcal{BL}(W, Z)$. Of course, one also uses the corresponding product topology on $\mathcal{E} \times \mathcal{BL}(V, W)$. Note that $A \circ B$ tends to 0 with respect to the strong operator topology on $\mathcal{BL}(V, Z)$ when $B$ approaches 0 with respect to the strong operator topology on $\mathcal{BL}(V, W)$ and the operator norm of $A$ remains bounded, which corresponds to the case where $B_0 = 0$.

If $v \in V$ and $\lambda$ is a bounded linear functional on $Z$, then

$$|\lambda((A \circ B - A_0 \circ B_0)(v))| \leq \|\lambda\|_{Z'} \, \|A\|_{op,WZ} \, \|(B - B_0)(v)\|_W + |\lambda((A - A_0)(B_0(v))|, \tag{21.7}$$

where $\|\lambda\|_{Z'}$ denotes the dual norm of $\lambda$ corresponding to the norm $\|z\|_Z$ on $Z$. Let $\mathcal{E}$ be a bounded subset of $\mathcal{BL}(W, Z)$ with respect to the operator norm again, equipped now with the topology induced by the weak operator topology on $\mathcal{BL}(W, Z)$. Also let $\mathcal{BL}(V, W)$ be equipped with the strong operator topology, and let $\mathcal{E} \times \mathcal{BL}(V, W)$ be equipped with the corresponding product topology. If $\mathcal{BL}(V, Z)$ is equipped with the weak operator topology, then one can check that (21.3) is continuous as a mapping from $\mathcal{E} \times \mathcal{BL}(V, W)$ into $\mathcal{BL}(V, Z)$, using (21.7). There is an analogous statement for the weak$^*$ operator topologies instead of the weak operator topologies on $\mathcal{BL}(V, Z)$ and $\mathcal{BL}(W, Z)$ when $Z$ is a dual space.

Suppose that $V = W = Z = \ell^p(\mathbf{Z}_+)$ for some $p$, $1 < p < \infty$, and let $A$ and $B$ be the shift operators discussed in Section 14. Thus $A^n$ and $B^n$ have operator norm equal to 1 on $\ell^p(\mathbf{Z}_+)$ for each $n \in \mathbf{Z}_+$, $A^n \to 0$ as $n \to \infty$ with respect to the weak operator topology on $\mathcal{BL}(\ell^p(\mathbf{Z}_+))$, and $B^n \to 0$ as $n \to \infty$ with respect to the strong operator topology on $\mathcal{BL}(\ell^p(\mathbf{Z}_+))$. However, $B^n \circ A^n$ is the identity operator on $\ell^p(\mathbf{Z}_+)$ for each $n \geq 1$, and hence does not converge to 0 as $n \to \infty$ with respect to the weak operator topology on $\mathcal{BL}(\ell^p(\mathbf{Z}_+))$. The analogous statements for these operators acting on $c_0(\mathbf{Z}_+)$ also hold. There are analogous statements for these operators acting on $\ell^1(\mathbf{Z}_+)$ as well, except that $A^n \to 0$ as $n \to \infty$ with respect to the weak$^*$ operator topology on $\mathcal{BL}(\ell^1(\mathbf{Z}_+))$, where $\ell^1(\mathbf{Z}_+)$ is identified with the dual of $c_0(\mathbf{Z}_+)$. Of course, $A^n \circ B^n \to 0$ as $n \to \infty$ with respect to the strong operator topology on $\mathcal{BL}(\ell^p(\mathbf{Z}_+))$ when $1 \leq p < \infty$, and on $\mathcal{BL}(c_0(\mathbf{Z}_+))$. This simply uses the fact that $B^n \to 0$ as $n \to \infty$ with respect to the strong operator topology, while the operator norm of $A^n$ is bounded.

Let $V$ and $W$ be vector spaces again, both real or both complex, and with norms $\|v\|_V$ and $\|w\|_W$, respectively. Also let $A$, $A_0$ be bounded linear operators from $V$ onto $W$ which are invertible, in the sense that their inverses are bounded linear mappings from $W$ onto $V$. Observe that

$$\begin{aligned} A^{-1} - A_0^{-1} &= A^{-1} \circ A_0 \circ A_0^{-1} - A^{-1} \circ A \circ A_0^{-1} \\ &= A^{-1} \circ (A_0 - A) \circ A_0^{-1}, \end{aligned} \tag{21.8}$$

and hence that

$$\|A^{-1} - A_0^{-1}\|_{op,WV} \le \|A^{-1}\|_{op,WV} \, \|A_0 - A\|_{op,VW} \, \|A_0\|_{op,WV}. \tag{21.9}$$

If $A$ is sufficiently close to $A_0$ with respect to the operator norm on $\mathcal{BL}(V, W)$, then one can show that the operator norm of $A^{-1}$ is uniformly bounded, using the triangle inequality to estimate $\|A(v)\|_W$ from below in terms of $\|A_0(v)\|_W$. It follows from this and (21.9) that $A \mapsto A^{-1}$ is continuous as a mapping from the set of invertible elements of $\mathcal{BL}(V, W)$ into $\mathcal{BL}(W, V)$ with respect to the operator norm.

If $w \in W$, then

$$\|(A^{-1} - A_0^{-1})(w)\|_V \le \|A^{-1}\|_{op,WV} \, \|((A_0 - A)(A_0^{-1}(w)))\|_W \tag{21.10}$$

for all invertible bounded linear mappings $A$, $A_0$ from $V$ into $W$, by (21.8). Let $\mathcal{E}$ be a collection of bounded linear mappings from $V$ into $W$ which are invertible, and whose inverses have uniformly bounded operator norms. It follows from (21.10) that $A \mapsto A^{-1}$ is continuous as a mapping from $\mathcal{E}$ into $\mathcal{BL}(W, V)$, where $\mathcal{E}$ is equipped with the topology induced by the strong operator topology on $\mathcal{BL}(V, W)$, and where $\mathcal{BL}(W, V)$ is equipped with the strong operator topology as well.

# References


[1] W. Arveson, *An Invitation to $C^*$-Algebras*, Springer-Verlag, 1976.

[2] W. Arveson, *A Short Course on Spectral Theory*, Springer-Verlag, 2002.

[3] G. Bachman and L. Narici, *Functional Analysis*, Dover, 2000.

[4] L. Baggett, *Functional Analysis*, Dekker, 1992.

[5] R. Bass, *Real Analysis for Graduate Students: Measure and Integration Theory*, 2011. http://homepages.uconn.edu/~rib02005/real.html.

[6] R. Bhatia, *Notes on Functional Analysis*, Hindustan Book Agency, 2009.

[7] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011.

[8] J. Cerdà, *Linear Functional Analysis*, American Mathematical Society, Real Sociedad Matemática Española, 2010.

[9] J. Conway, *A Course in Functional Analysis*, 2nd edition, Springer-Verlag, 1990.

[10] J. Conway, *A Course in Operator Theory*, American Mathematical Society, 2000.

[11] J. Conway, *A Course in Abstract Analysis*, American Mathematical Society, 2012.

[12] R. Douglas, *Banach Algebra Techniques in Operator Theory*, 2nd edition, Springer-Verlag, 1998.

[13] Y. Eidelman, V. Milman, and A. Tsolomitis, *Functional Analysis*, American Mathematical Society, 2004.

[14] G. Folland, *Real Analysis*, 2nd edition, Wiley, 1999.

[15] G. Folland, *A Guide to Advanced Real Analysis*, Mathematical Association of America, 2009.

[16] P. Halmos, *Measure Theory*, Van Nostrand, 1950.

[17] P. Halmos, *Lectures on Ergodic Theory*, Chelsea, 1960.

[18] P. Halmos, *A Hilbert Space Problem Book*, 2nd edition, Springer-Verlag, 1982.

[19] P. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, 2nd edition, AMS Chelsea, 1998.

[20] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1975.

[21] S. Kantorovitz, *Introduction to Modern Analysis*, Oxford University Press, 2003.

[22] S. Kesavan, *Functional Analysis*, Hindustan Book Agency, 2009.

[23] A. Knapp, *Basic Real Analysis*, Birkhäuser, 2005.

[24] A. Knapp, *Advanced Real Analysis*, Birkhäuser, 2005.

[25] S. Lang, *Real and Functional Analysis*, 3rd edition, Springer-Verlag, 1993.

[26] P. Lax, *Functional Analysis*, Wiley, 2002.

[27] B. MacCluer, *Elementary Functional Analysis*, Springer, 2009.

[28] T. Morrison, *Functional Analysis*, Wiley, 2001.

[29] D. Promislow, *A First Course in Functional Analysis*, Wiley, 2008.

[30] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.

[31] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.

[32] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, 1991.

[33] B. Rynne and M. Youngson, *Linear Functional Analysis*, 2nd edition, Springer-Verlag, 2008.

[34] K. Saxe, *Beginning Functional Analysis*, Springer-Verlag, 2002.

[35] M. Schechter, *Principles of Functional Analysis*, 2nd edition, American Mathematical Society, 2002.

[36] E. Stein and R. Shakarchi, *Real Analysis*, Princeton University Press, 2005.

[37] E. Stein and R. Shakarchi, *Functional Analysis*, Princeton University Press, 2011.

[38] C. Swartz, *An Introduction to Functional Analysis*, Dekker, 1992.

[39] C. Swartz, *Elementary Functional Analysis*, World Scientific, 2009.

[40] K. Yosida, *Functional Analysis*, Springer-Verlag, 1995.

[41] R. Zimmer, *Essential Results of Functional Analysis*, University of Chicago Press, 1990.