TRIANGULATED CATEGORIES WITH SEVERAL TRIANGULATIONS

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Abstract. We give a simple algebraic example of a fixed additive category K, with a fixed additive self-equivalence $\Sigma : K \to K$ having arbitrarily many structures of triangulated categories with Σ as suspension.

Introduction. Triangulated categories were introduced in Verdier's PhD thesis [**3**] and by Puppe (without Verdier's key *Octahedron Axiom* about composition). A triangulated category is a *suspended category* (*i.e.* an additive category K with an additive auto-equivalence $\Sigma : K \xrightarrow{\sim} K$ called the suspension) plus a collection $\mathcal{T}$ of triangles which satisfies four well-known axioms, denoted (TR I)-(TR IV) in [**3**, Def. II.1.1.1, p. 93-94]. It is natural to wonder whether these axioms are intrinsical. Or: Can a given suspended category carry two different triangulations? Of course, given a triangulated category $(K, \Sigma, \mathcal{T})$ with triangulation $\mathcal{T}$, we can define the *negative triangulation* $\mathcal{T}^-$ as the class of those triangles (u, v, w)

$$(\Delta) \qquad A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma(A)$$

such that $(-u, -v, -w) \in \mathcal{T}$. Those two triangulations $\mathcal{T}$ and $\mathcal{T}^-$ are different in general, already for $K = \mathrm{K}^{\mathrm{b}}(\mathbb{Z})$ the category of bounded complexes of abelian groups, up to homotopy. Then, we could ask:

Can a suspended category (K, Σ) admit more than two triangulations?

(That is: one triangulation and its negative.) Strangely enough, this question seems to remain unclear, even for a few experts of the subject, see for instance [**2**, Problem 3.4 and Def. 3.2]. The answer to this question is indeed "*yes*", and first in a trivial way: Let $(K, \Sigma, \mathcal{T})$ be a triangulated category such that $\mathcal{T}$ and $\mathcal{T}^-$ are different. Choose an integer $n \geq 1$. Consider the additive category $K^n = K \times \cdots \times K$ with the obvious suspension. Then K^n has at least 2^n different triangulations compatible with its suspension. Well, this certainly sounds like cheating, because we basically only used $\mathcal{T}$ and $\mathcal{T}^-$. So, we would like to build examples, say, with an indecomposable category K. In fact, it is possible to deduce from the results of Sections 16 and 17 of Heller [**1**] that such an example is given by K the usual topological stable homotopy category, although a picky reader might object that Heller does not consider the Octahedron Axiom in *loc. cit.* In this short note, we give a simple algebraic example (see Theorem 7).

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Definition 1. Let (K, Σ) be a suspended category. A *global endomorphism α of* (K, Σ) will be an endomorphism of the identity functor $\mathrm{Id} : K \to K$ which commutes with Σ. In other words, it consists of a collection of endomorphisms $\alpha_A : A \to A$, for all objects $A \in K$, such that for any morphism $f : A \to B$ in K one has $\alpha_B\, f = f\, \alpha_A$, and such that $\alpha_{\Sigma(A)} = \Sigma(\alpha_A)$ for any $A \in K$. A *global automorphism* will be an invertible global endomorphism. A global endomorphism α is *pointwise nilpotent* if for any $A \in K$, there is an $n \in \mathbb{N}$ such that $(\alpha_A)^n = 0$.

Example 2. Let R be a commutative ring and let $b \in R$. Then multiplication by b gives a global endomorphism λ_b of $\mathrm{K}^{\mathrm{b}}(R)$. It is a global automorphism when b is a unit.

Definition 3. Let $(K, \Sigma, \mathcal{T})$ be a triangulated category and let α be a global automorphism of (K, Σ). Define the class $\mathcal{T}_\alpha$ as the collection of those triangles (u, v, w), like in (Δ) above, such that the twisted triangle $(u \cdot \alpha_A\, ,\, v\, ,\, w)$ belongs to $\mathcal{T}$. This condition is equivalent to any of the following: $(\alpha_B \cdot u\, ,\, v\, ,\, w) \in \mathcal{T}$, $(u,\, v \cdot \alpha_B\, ,\, w) \in \mathcal{T}$, $(u\, ,\, \alpha_C \cdot v\, ,\, w) \in \mathcal{T}$, and so on: permuting α with the morphisms, and moving α around; this flexibility follows from Axiom (TR I) and Def. 1.

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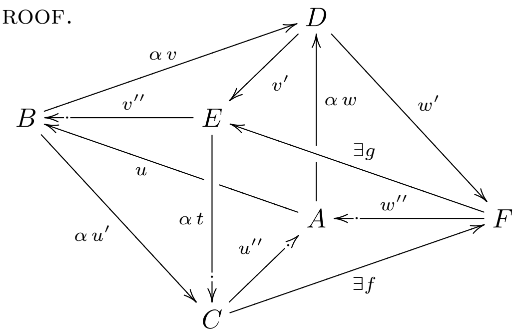
PROPOSITION 4. *Let $(K, \Sigma, \mathcal{T})$ and α be as in Definition 3. Then $(K, \Sigma, \mathcal{T}_\alpha)$ is a triangulated category.*

PROOF.



The proof is straightforward. The Composition Axiom (TR IV), for instance, can be checked by contemplating the diagram on the left (or the reader's favorite picture instead). The arrows with a small dot are of degree 1. Start with a composition $w = v \circ u$. Then choose triangles (u, u', u''), (v, v', v''), (w, w', w'') in $\mathcal{T}_\alpha$. The morphism $t : E \to \Sigma(C)$ is as always defined to be $t := \Sigma(u')v''$. The displayed octahedron is obtained for $\mathcal{T}$ from $\alpha\, w = \alpha\, v \circ u$. It induces the wanted octahedron for $\mathcal{T}_\alpha$ by "removing α". For readability, we have dropped the indices of α, forced by the objects. □

LEMMA 5. *Let R be a commutative ring and $b \in R^\times$ a unit. Assume the existence of $r \in R$ such that: (1) the element r is not a zero divisor and (2) the element r does not divide $1 - b$. Consider the category $K = \mathrm{K}^{\mathrm{b}}(R)$ with its usual triangulation $\mathcal{T}$. Consider the global automorphism λ_b of K (see Example 2). Then the triangulations $\mathcal{T}$ and $\mathcal{T}_{\lambda_b}$ are different.*

PROOF. *Ab absurdo*, assume that $\mathcal{T}_{\lambda_b} = \mathcal{T}$. In the category K, consider the morphism $R \to R$ given by multiplication by r as a morphism of complexes concentrated in degree 0. Let $C(r)$ be its cone with the usual morphisms $i : R \to C(r)$ and $p : C(r) \to \Sigma(R)$. The triangle $(b \cdot r, i, p)$ is then exact. By (TR III), there must exist a morphism $h : C(r) \to C(r)$ which makes the following diagram commute:

$$\begin{array}{ccccccc} R & \xrightarrow{r} & R & \xrightarrow{i} & C(r) & \xrightarrow{p} & \Sigma(R) \\ \| & & \downarrow{\scriptstyle b} & & \exists \downarrow {\scriptstyle h} & & \| \\ R & \xrightarrow{b \cdot r} & R & \xrightarrow{i} & C(r) & \xrightarrow{p} & \Sigma(R). \end{array}$$

The morphism h is characterized by two elements $x, y \in R$ such that $r \cdot x = y \cdot r$ which forces $x = y$ by hypothesis (1). The commutativity (up to homotopy!) of the above diagram implies the existence of $e, f \in R$ such that $b = x + r \cdot e$ and $1 = x + f \cdot r$. This gives $1 - b \in rR$ which contradicts (2). □

EXAMPLE 6. Of course $\mathcal{T}_{-\mathrm{Id}} = \mathcal{T}^-$. The Lemma shows that $\mathcal{T}^- \neq \mathcal{T}$ for $\mathrm{K}^{\mathrm{b}}(\mathbb{Z})$ as claimed above.

THEOREM 7. *There exists a suspended category (K, Σ) which carries infinitely many different triangulations. Moreover, there exists such a (K, Σ) which cannot be decomposed as $(K_1, \Sigma_1) \times (K_2, \Sigma_2)$ with K_1 and K_2 non-zero.*

PROOF. Let S be a commutative domain with infinitely many units. Let $R = S[X]$ be the polynomial ring with coefficients in S. Consider $r = X \in R$. It certainly satisfies conditions (1) and (2) of the above Lemma for any unit $b \in S^\times$ except for $b = 1$. Let us write $\mathcal{T}_b$ for $\mathcal{T}_{\lambda_b}$. It is clear that $(\mathcal{T}_b)_c = \mathcal{T}_{b \cdot c}$ for any $b, c \in S^\times$. Therefore $\mathcal{T}_b = \mathcal{T}_c$ forces $\mathcal{T} = \mathcal{T}_{b^{-1}c}$ and thus $b^{-1}c = 1$ by the Lemma and the above comment. That is: all the triangulations $\mathcal{T}_b$ for $b \in S^\times$ are distinct.

For the "moreover part", assume $(K, \Sigma) = (K_1, \Sigma_1) \times (K_2, \Sigma_2)$ then the projection on the K_1-summand yields a global endomorphism β of K, see Definition 1. At the object $R \in K$, we have necessarily $\beta_R = 0$ or $1 - \beta_R = 0$, since $\mathrm{End}_K(R) \simeq R$ and since R is a domain. Let us say $\beta_R = 0$ for instance. The object R generates K as a triangulated category, this forces β to be pointwise nilpotent (easy induction). But $\beta = \beta^2$ is an idempotent, so we have $\beta = 0$ and thus $K_1 = 0$. Similarly if $1 - \beta_R = 0$, then $K_2 = 0$. □

PROBLEM 8. Is there a suspended category (K, Σ) admitting two triangulations $\mathcal{T}$ and $\mathcal{T}'$ such that $\mathcal{T}' \neq \mathcal{T}_\alpha$ for any global automorphism α of (K, Σ)?

References.

1. Alex Heller, *Stable homotopy categories*, Bull. AMS **74** (1968) pp. 28-63.

2. J.P. May, *The additivity of traces in triangulated categories*, Adv. Math. **163**, no. 1 (2001), pp. 34-73.

3. Jean-Louis Verdier, *Catégories dérivées des catégories abéliennes*, Astérisque **239**, (1996).

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