The fundamental theorem of projective geometry

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Abstract

We prove the fundamental theorem of projective geometry. In addition to the usual statement, we also prove a variant in the presence of a symplectic form.

1 Introduction

Let K be a field. The fundamental theorem of projective geometry says that an abstract automorphism of the set of lines in K^n which preserves "incidence relations" must have a simple algebraic form. The most natural way of describing these incidence relations is via the associated Tits building.

Building. The *Tits building* of K^n, denoted $\mathcal{T}_n(K)$, is the poset of nonzero proper subspaces of K^n. The following are two fundamental examples of automorphisms of $\mathcal{T}_n(K)$.

- The group $\text{GL}_n(K)$ acts linearly on K^n. This descends to an action of $\text{PGL}_n(K)$ on $\mathcal{T}_n(K)$.
- Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of K^n and let $\tau : K \to K$ be a field automorphism. The map $K^n \to K^n$ defined via the formula

$$\sum_{i=1}^{n} c_i \vec{v}_i \mapsto \sum_{i=1}^{n} \tau(c_i) \vec{v}_i$$

  induces an automorphism of $\mathcal{T}_n(K)$.

It turns out that every automorphism of $\mathcal{T}_n(K)$ is a sort of combination of the above types of automorphisms.

Semilinear automorphisms. A *semilinear transformation* of K^n is a set map $f : K^n \to K^n$ for which there exists a field automorphism $\tau : K \to K$ such that

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = \tau(c_1)f(\vec{v}_1) + \tau(c_2)f(\vec{v}_2) \qquad (c_1, c_2 \in K, \vec{v}_1, \vec{v}_2 \in K^n).$$

A semilinear transformation $f : K^n \to K^n$ is a *semilinear automorphism* if it is bijective. Let $\Gamma\text{L}_n(K)$ be the group of semilinear automorphisms of K^n. This contains a normal subgroup isomorphic to K^*, namely the set of all scalar matrices. The quotient of $\Gamma\text{L}_n(K)$ by this normal subgroup is denoted $\text{P}\Gamma\text{L}_n(K)$.

The fundamental theorem. The group $\mathrm{P\Gamma L}_n(K)$ clearly acts on $\mathcal{T}_n(K)$. The following theorem will be proved in §2.

Theorem 1 (Fundamental theorem of projective geometry)**.** *If K is a field and $n \geq 3$, then* $\mathrm{Aut}(\mathcal{T}_n(K)) = P\Gamma L_n(K)$*.*

This theorem has its origins in 19^{th} century work of von Staudt [4]. I do not know a precise reference for the above modern version of it, but on [2, p. 52] it is attributed to Kamke. The proof we give is adapted from [1, Chapter II.10]. Another excellent source that contains a lot of other related results is [3].

Remark. In the classical literature, an automorphism of $\mathcal{T}_n(K)$ is called a *collineation*.

Remark. Theorem 1 is false for $n = 2$ since $\mathcal{T}_2(K)$ is a discrete poset.

Remark. The map $\Gamma\mathrm{L}_n(K) \to \mathrm{Aut}(K)$ that takes $f \in \Gamma\mathrm{L}_n(K)$ to the $\tau \in \mathrm{Aut}(K)$ associated to f is surjective and has kernel $\mathrm{GL}_n(K)$. We thus have a short exact sequence

$$1 \longrightarrow \mathrm{GL}_n(K) \longrightarrow \Gamma\mathrm{L}_n(K) \longrightarrow \mathrm{Aut}(K) \longrightarrow 1.$$

Fixing a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for K^n, we obtain a splitting $\mathrm{Aut}(K) \to \Gamma\mathrm{L}_n(K)$ of this short exact sequence that takes $\tau \in \mathrm{Aut}(K)$ to the map $K^n \to K^n$ defined via the formula

$$\sum_{i=1}^{n} c_i \vec{v}_i \mapsto \sum_{i=1}^{n} \tau(c_i) \vec{v}_i.$$

It follows that $\Gamma\mathrm{L}_n(K) = \mathrm{GL}_n(K) \rtimes \mathrm{Aut}(K)$.

Symplectic building. There is also a natural symplectic analogue of the fundamental theorem of projective geometry. Recall that a *symplectic form* on K^n is an alternating bilinear form $\omega(\cdot, \cdot)$ such that the map

$$\begin{aligned} K^n &\to (K^n)^* \\ \vec{v} &\mapsto (\vec{w} \mapsto \omega(\vec{v}, \vec{w})) \end{aligned}$$

is an isomorphism. If a symplectic form on K^n exists, then $n = 2g$ for some $g \geq 1$. Moreover, all symplectic forms on K^{2g} are equivalent. If ω is a symplectic form on K^{2g}, then a subspace $V \subset K^{2g}$ is *isotropic* if $\omega(\vec{v}, \vec{w}) = 0$ for all $\vec{v}, \vec{w} \in K^{2g}$. Isotropic subspace of K^{2g} are at most g-dimensional. Define $\mathcal{TP}_{2g}(K)$ to be the poset of nonzero isotropic subspaces of K^{2g}.

Symplectic semilinear. The symplectic group $\mathrm{Sp}_{2g}(K)$ acts on $\mathcal{TP}_{2g}(K)$, but in fact the automorphism group is much larger. Define

$$\Gamma\mathrm{P}_{2g}(K) = \{f \in \Gamma\mathrm{L}_{2g}(K) \mid \omega(f(\vec{v}), f(\vec{w})) = 0 \text{ if and only if } \omega(\vec{v}, \vec{w}) = 0\}.$$

The group $\Gamma\mathrm{P}_{2g}(K)$ contains a normal subgroup isomorphic to K^* consisting of scalar matrices; let $\mathrm{P\Gamma P}_{2g}(K)$ be the quotient.

Fundamental theorem, symplectic. The group $\mathrm{P}\Gamma\mathrm{P}_{2g}(K)$ clearly acts on $\mathcal{TP}_{2g}(K)$. The following theorem will be proved in §3.

Theorem 2 (Fundamental theorem of symplectic projective geometry)**.** *If K is a field and $g \geq 2$, then $\mathrm{Aut}(\mathcal{TP}_{2g}(K)) = P\Gamma P_{2g}(K)$.*

Remark. Theorem 2 is false for $g = 1$ since in that case $\mathcal{TP}_{2g}(K)$ is a discrete poset.

2 Proof of fundamental theorem of projective geometry

In this section, we prove Theorem 1. It is enough to prove that each element of $\mathrm{Aut}(\mathcal{T}_n(K))$ is induced by some element of $\Gamma\mathrm{L}_n(F)$. To simplify some of our arguments, we add the subspaces 0 and K^n to $\mathcal{T}_n(K)$; of course, any automorphism of $\mathcal{T}_n(K)$ must fix 0 and K^n. Fixing some $F \in \mathrm{Aut}(\mathcal{T}_n(K))$, we begin with the following observation.

Claim 1. *Let $\vec{a}_1, \ldots, \vec{a}_p \in K^n$ be nonzero vectors. For $1 \leq i \leq p$, let $\vec{b}_i \in K^n$ be such that $F(\langle \vec{a}_i \rangle) = \langle \vec{b}_i \rangle$. Then*

$$F(\langle \vec{a}_1, \ldots, \vec{a}_p \rangle) = \langle \vec{b}_1, \ldots, \vec{b}_p \rangle.$$

Proof of claim. The subspace $\langle \vec{a}_1, \ldots, \vec{a}_p \rangle$ is the minimal subspace of K^n containing each $\langle \vec{a}_i \rangle$. Since F is an automorphism of the poset $\mathcal{T}_n(K)$, we see that $F(\langle \vec{a}_1, \ldots, \vec{a}_p \rangle)$ is the minimal subspace of K^n containing each $F(\langle \vec{a}_i \rangle) = \langle \vec{b}_i \rangle$. The claim follows. □

Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for K^n. To construct the desired $f \in \Gamma\mathrm{L}_n(K)$, we must construct a field automorphism $\tau : K \to K$ and a basis $\{\vec{w}_1, \ldots, \vec{w}_n\}$ for K^n; we can then define $f : K^n \to K^n$ via the formula

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \tau(c_1)\vec{w}_1 + \cdots + \tau(c_n)\vec{w}_n \qquad (c_1, \ldots, c_n \in K). \quad (2.1)$$

We start with the basis $\{\vec{w}_1, \ldots, \vec{w}_n\}$. First, let $\vec{w}_1 \in K^n$ be any vector such that $F(\langle \vec{v}_1 \rangle) = \langle \vec{w}_1 \rangle$. The choice of $\vec{w}_1$ will be our only arbitrary choice; everything else will be determined by it (as it must since we are proving that the automorphism group of $\mathcal{T}_n(K)$ is the *projective* version of the group of semilinear automorphisms). We now construct $\{\vec{w}_2, \ldots, \vec{w}_n\}$.

Claim 2. *For $2 \leq i \leq n$, there exists a unique $\vec{w}_i \in K^n$ such that*

$$F(\langle \vec{v}_i \rangle) = \langle \vec{w}_i \rangle \quad \text{and} \quad F(\langle \vec{v}_1 + \vec{v}_i \rangle) = \langle \vec{w}_1 + \vec{w}_i \rangle.$$

Moreover, the set $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is a basis for K^n.

Proof of claim. Pick $\vec{u}_i \in K^n$ such that $F(\langle \vec{v}_i \rangle) = \langle \vec{u}_i \rangle$. Using Claim 1, we then have

$$F(\langle \vec{v}_1 + \vec{v}_i \rangle) \subset F(\langle \vec{v}_1, \vec{v}_i \rangle) = \langle \vec{w}_1, \vec{u}_i \rangle.$$

Since $F(\langle \vec{v}_1 + \vec{v}_i \rangle) \neq \langle \vec{u}_i \rangle$, it follows that there exists a unique $\lambda_i \in K$ such that $F(\langle \vec{v}_1 + \vec{v}_i \rangle) = \langle \vec{w}_1 + \lambda_i \vec{u}_i \rangle$. The desired vector is thus $\vec{w}_i := \lambda_i \vec{u}_i$. To see that $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is a basis for K^n, observe that we can use Claim 1 to deduce that

$$K^n = F(K^n) = F(\langle \vec{v}_1, \ldots, \vec{v}_n \rangle) = \langle \vec{w}_1, \ldots, \vec{w}_n \rangle. \qquad \square$$

The construction of the field automorphism $\tau : K \to K$ will take several steps. The next two claims construct it as a set map.

Claim 3. *For $2 \leq i \leq n$, there exists a unique set map $\tau_i : K \to K$ such that*

$$F(\langle \vec{v}_1 + c\vec{v}_i \rangle) = \langle \vec{w}_1 + \tau_i(c)\vec{w}_i \rangle \qquad (c \in K).$$

Proof of claim. We define τ_i as follows (this construction is very similar to that in Claim 2). Consider $c \in K$. We can apply Claim 1 to see that

$$F(\langle \vec{v}_1 + c\vec{v}_i \rangle) \subset F(\langle \vec{v}_1, \vec{v}_i \rangle) = \langle \vec{w}_1, \vec{w}_i \rangle.$$

Since $F(\langle \vec{v}_1 + c\vec{v}_i \rangle) \neq \langle \vec{w}_i \rangle$, we see that there exists a unique $\tau_i(c) \in K$ such that $F(\langle \vec{v}_1 + c\vec{v}_i \rangle) = \langle \vec{w}_1 + \tau_i(c)\vec{w}_i \rangle$. $\square$

We remark that the uniqueness of τ_i implies that $\tau_i(0) = 0$ and $\tau_i(1) = 1$.

Claim 4. *For distinct $2 \leq i, j \leq n$, we have $\tau_i = \tau_j$.*

Proof of claim. Consider a nonzero $c \in K$. We have

$$\langle \vec{v}_i - \vec{v}_j \rangle \subset \langle \vec{v}_i, \vec{v}_j \rangle \quad \text{and} \quad \langle \vec{v}_i - \vec{v}_j \rangle \subset \langle \vec{v}_1 + c\vec{v}_i, \vec{v}_1 + c\vec{v}_j \rangle.$$

Applying Claim 1 twice, we see that

$$F(\langle \vec{v}_i - \vec{v}_j \rangle) \subset \langle \vec{w}_i, \vec{w}_j \rangle \quad \text{and} \quad F(\langle \vec{v}_i - \vec{v}_j \rangle) \subset \langle \vec{w}_1 + \tau_i(c)\vec{w}_i, \vec{w}_1 + \tau_j(c)\vec{w}_j \rangle.$$

We have

$$\langle \vec{w}_i, \vec{w}_j \rangle \cap \langle \vec{w}_1 + \tau_i(c)\vec{w}_i, \vec{w}_1 + \tau_j(c)\vec{w}_j \rangle = \langle \tau_i(c)\vec{w}_i - \tau_j(c)\vec{w}_j \rangle,$$

so we deduce that

$$F(\langle \vec{v}_i - \vec{v}_j \rangle) = \langle \tau_i(c)\vec{w}_i - \tau_j(c)\vec{w}_j \rangle.$$

The left hand side does not depend on c, so despite its appearance the right hand side must also be independent of c. In particular, we have

$$\langle \vec{w}_i - \vec{w}_j \rangle = \langle \tau_i(1)\vec{w}_i - \tau_j(1)\vec{w}_j \rangle = \langle \tau_i(c)\vec{w}_i - \tau_j(c)\vec{w}_j \rangle.$$

The only way this equality can hold is if $\tau_i(c) = \tau_j(c)$, as desired. $\square$

Let $\tau : K \to K$ be the set map $\tau_2 = \tau_3 = \cdots = \tau_n$. We will prove that τ is an automorphism of K below in Claims 7-9. First, however, we will prove two claims whose main purpose will be to show that the element $f \in \Gamma\mathrm{L}_n(K)$ constructed via (2.1) actually induces F (but which will also be used to prove that τ is an automorphism).

Claim 5. *For $c_2, \ldots, c_n \in K$, we have*

$$F(\langle \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \rangle) = \langle \vec{w}_1 + \tau(c_2)\vec{w}_2 + \cdots + \tau(c_n)\vec{w}_n \rangle.$$

Proof of claim. We will prove that

$$F(\langle \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \rangle) = \langle \vec{w}_1 + \tau(c_2)\vec{w}_2 + \cdots + \tau(c_p)\vec{w}_p \rangle$$

for all $2 \leq p \leq n$ by induction on p. The base case $p = 2$ is the defining property of τ, so assume that $2 < p \leq n$ and that the above equation holds for smaller values of p. Applying Claim 1 and our inductive hypothesis, we see that

$$\begin{aligned} F(\langle \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \rangle) \subset & F(\langle \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{p-1}\vec{v}_{p-1}, \vec{v}_p \rangle) \\ = & \langle \vec{w}_1 + \tau(c_2)\vec{w}_2 + \cdots + \tau(c_{p-1})\vec{w}_{p-1}, \vec{w}_p \rangle. \end{aligned}$$

Moreover, $F(\langle \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \rangle)$ is not $\langle \vec{w}_p \rangle$, so we deduce that there exists some $d \in K$ such that

$$F(\langle \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \rangle) = \langle \vec{w}_1 + \tau(c_2)\vec{w}_2 + \cdots + \tau(c_{p-1})\vec{w}_{p-1} + d\vec{w}_p \rangle.$$

We want to prove that $d = \tau(c_p)$. Applying Claim 1 and the defining property of τ, we see that

$$\begin{aligned} F(\langle \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \rangle) \subset & F(\langle \vec{v}_1 + c_p\vec{v}_p, \vec{v}_2, \ldots, \vec{v}_{p-1} \rangle) \\ = & \langle \vec{w}_1 + \tau(c_p)\vec{w}_p, \vec{w}_2, \ldots, \vec{w}_{p-1} \rangle. \end{aligned}$$

Comparing this with the previous displayed equation, we see that the only possibility is that $d = \tau(c_p)$, as desired. □

Claim 6. *For $c_2, \ldots, c_n \in K$, we have*

$$F(\langle c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \rangle) = \langle \tau(c_2)\vec{w}_2 + \cdots + \tau(c_n)\vec{w}_n \rangle.$$

Proof of claim. By Claim 1, we have

$$F(\langle c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \rangle) \subset F(\langle \vec{v}_2, \ldots, \vec{v}_n \rangle) = \langle \vec{w}_2, \ldots, \vec{w}_n \rangle.$$

Also, combining Claim 1 with Claim 5 we have

$$\begin{aligned} F(\langle c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \rangle) \subset & F(\langle \vec{v}_1, \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \rangle) \\ = & \langle \vec{w}_1, \vec{w}_1 + \tau(c_2)\vec{w}_2 + \cdots + \tau(c_n)\vec{w}_n \rangle. \end{aligned}$$

The only way both of these equations can hold is if

$$F(\langle c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \rangle) = \langle \tau(c_2)\vec{w}_2 + \cdots + \tau(c_n)\vec{w}_n \rangle,$$

as claimed. □

The next three claims prove that τ is an automorphism of K. We remark that the proofs of Claims 7 and 8 are where we use the assumption $n \geq 3$.

Claim 7. *For $c, d \in K$ we have $\tau(c+d) = \tau(c) + \tau(d)$.*

Proof of claim. By Claim 5, we have

$$F(\langle \vec{v}_1 + (c+d)\vec{v}_2 + \vec{v}_3 \rangle) = \langle \vec{w}_1 + \tau(c+d)\vec{w}_2 + \vec{w}_3 \rangle.$$

Combining Claim 1 with Claims 5 and 6, we have

$$\begin{aligned} F(\langle \vec{v}_1 + (c+d)\vec{v}_2 + \vec{v}_3 \rangle) \subset & F(\langle \vec{v}_1 + c\vec{v}_2, d\vec{v}_2 + \vec{v}_3 \rangle) \\ = & \langle \vec{w}_1 + \tau(c)\vec{w}_2, \tau(d)\vec{w}_2 + \vec{w}_3 \rangle. \end{aligned}$$

Combining these two equations, we get that

$$\langle \vec{w}_1 + \tau(c+d)\vec{w}_2 + \vec{w}_3 \rangle \subset \langle \vec{w}_1 + \tau(c)\vec{w}_2, \tau(d)\vec{w}_2 + \vec{w}_3 \rangle.$$

The only way this can hold is if $\tau(c+d) = \tau(c) + \tau(d)$, as claimed. □

Claim 8. *For $c, d \in K$ we have $\tau(cd) = \tau(c)\tau(d)$.*

Proof of claim. By Claim 5, we have

$$F(\langle \vec{v}_1 + cd\vec{v}_2 + c\vec{v}_3 \rangle) = \langle \vec{w}_1 + \tau(cd)\vec{w}_2 + \tau(c)\vec{w}_3 \rangle.$$

Combining Claim 1 with Claim 6, we have

$$F(\langle \vec{v}_1 + cd\vec{v}_2 + c\vec{v}_3 \rangle) \subset F(\langle \vec{v}_1, d\vec{v}_2 + \vec{v}_3 \rangle) = \langle \vec{w}_1, \tau(d)\vec{w}_2 + \vec{w}_3 \rangle.$$

Combining these two equations, we get that

$$\langle \vec{w}_1 + \tau(cd)\vec{w}_2 + \tau(c)\vec{w}_3 \rangle \subset \langle \vec{w}_1, \tau(d)\vec{w}_2 + \vec{w}_3 \rangle.$$

The only way this can hold is if $\tau(cd) = \tau(c)\tau(d)$, as claimed. □

Claim 9. *The map $\tau : K \to K$ is an automorphism of K.*

Proof of claim. We know that τ is a set map satisfying $\tau(0) = 0$ and $\tau(1) = 1$. Claims 7 and 8 imply that τ is a ring homomorphism. Since K is a field, τ must be injective. We must prove that τ is surjective. Consider $\bar{c} \in K$. Since F is an automorphism of the set of lines in K^n, there exists some line $L \subset K^n$ such that $F(L) = \langle \vec{w}_1 + \bar{c}\vec{w}_2 \rangle$. Every line in K^n is either of the form $\langle \vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \rangle$ or $\langle c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \rangle$ for some $c_2, \ldots, c_n \in K$. Examining Claims 5-6, we see that in fact $L = \langle \vec{v}_1 + c\vec{v}_2 \rangle$ for some $c \in K$ satisfying $\tau(c) = \bar{c}$, as desired. □

We now have constructed our basis $\{\vec{w}_1, \ldots, \vec{w}_n\}$ for K^n and our automorphism $\tau : K \to K$, so we can define $f \in \Gamma\mathrm{L}_n(K)$ via the formula

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \tau(c_1)\vec{w}_1 + \cdots + \tau(c_n)\vec{w}_n \qquad (c_1, \ldots, c_n \in K).$$

Claim 10. *The semilinear automorphism $f : K^n \to K^n$ induces the automorphism $F \in \mathrm{Aut}(\mathcal{T}_n(K))$.*

Proof of claim. Consider $V \in \mathcal{T}_n(K)$. We can write $V = \langle \vec{x}_1, \ldots, \vec{x}_p \rangle$, where each x_i is either of the form $\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ or of the form $c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ for some $c_2, \ldots, c_n \in K$. Combining Claim 1 with Claims 5-6, we see that $F(V) = \langle f(\vec{x}_1), \ldots, f(\vec{x}_p) \rangle$, as desired. □

This completes the proof of the fundamental theorem of projective geometry.

3 Proof of fundamental theorem of symplectic projective geometry

Our proof of Theorem 2 is based on the following lemma. For a field K and $n \geq 3$, define $\mathcal{T}'_n(K)$ to be the subposet of $\mathcal{T}_n(K)$ consisting of subspaces $V \subset K^n$ such that $\dim(V) \in \{1, n-1\}$.

Lemma 3. *Let K be a field and $n \geq 3$. Then $\mathrm{Aut}(\mathcal{T}'_n(K)) = P\Gamma L_n(K)$.*

Proof. By the Fundamental Theorem of Projective Geometry (Theorem 1), it is enough to prove that every $F \in \mathrm{Aut}(\mathcal{T}'_n(K))$ can be extended to an automorphism of $\mathcal{T}_n(K)$. Consider $F \in \mathrm{Aut}(\mathcal{T}'_n(K))$.

Claim. *Consider lines $L_1, \ldots, L_p, L'_1, \ldots, L'_q \subset K^n$. Then $\langle L_1, \ldots, L_p \rangle \subset \langle L'_1, \ldots, L'_q \rangle$ if and only if $\langle F(L_1), \ldots, F(L_p) \rangle \subset \langle F(L'_1), \ldots, F(L'_q) \rangle$.*

Proof of claim. Since F is an automorphism of the poset $\mathcal{T}'_n(K)$, it is enough to express the condition $\langle L_1, \ldots, L_p \rangle \subset \langle L'_1, \ldots, L'_q \rangle$ entirely in terms of the poset structure on $\mathcal{T}'_n(K)$. This is easy:

- For all subspaces $V \subset K^n$ with $\dim(V) = n-1$, if $L'_1, \ldots, L'_q \subset V$, then $L_1, \ldots, L_P \subset V$. □

We now construct the desired extension of F to $\mathcal{T}_n(K)$. Consider a nonzero proper subspace $V \subset K^n$. Write $V = \langle L_1, \ldots, L_p \rangle$, where the L_i are lines in K^n. Set $F(V) = \langle F(L_1), \ldots, F(L_p) \rangle$. To see that this is well-defined, if we have a different expression $V = \langle L'_1, \ldots, L'_q \rangle$ with the L'_j lines in K^n, then applying the claim twice we see that

$$\langle F(L_1), \ldots, F(L_p) \rangle \subset \langle F(L'_1), \ldots, F(L'_q) \rangle$$

and

$$\langle F(L'_1), \ldots, F(L'_q) \rangle \subset \langle F(L_1), \ldots, F(L_p) \rangle,$$

so $\langle F(L_1), \ldots, F(L_p) \rangle = \langle F(L'_1), \ldots, F(L'_q) \rangle$ and F is well-defined. Another application of the claim shows that F is an automorphism of $\mathcal{T}_n(K)$, and we are done. □

Proof of Theorem 2. Consider $F \in \mathrm{Aut}(\mathcal{TP}_{2g}(K))$. It is enough to show that F is induced by some element of $\Gamma\mathrm{P}_{2g}(K)$. Define $F' \in \mathrm{Aut}(\mathcal{T}'_{2g}(K))$ as follows.

- For a subspace $L \subset K^{2g}$ with $\dim(L) = 1$, define $F'(L) = F(L)$.
- For a subspace $V \subset K^{2g}$ with $\dim(V) = 2g-1$, write $V = L^{\perp}$ for some line $L \subset K^{2g}$ (the orthogonal complement is with respect to the symplectic form), and define $F'(V) = (F(L))^{\perp}$.

To see that $F' \in \mathrm{Aut}(\mathcal{T}'_{2g}(K))$, we must check two things.

- That F' is a bijection of $\mathcal{T}'_{2g}(K)$. It is a bijection on lines since F is a bijection on lines, and it is a bijection on codimension-1 subspaces since the $\perp$-relation is a bijection between lines and codimension-1 subspaces.
- That F preserves the poset structure. Observe that if $L, L' \subset K^{2g}$ satisfy $\dim(L) = \dim(L') = 1$, then $L' \subset L^{\perp}$ if and only there exists an isotropic subspace $W \subset K^{2g}$ such that $L, L' \subset W$. This condition is preserved by F, so $F(L') \subset F(L^{\perp})$ if and only if $L' \subset L^{\perp}$.

Lemma 3 implies that there exists some $f \in \Gamma\mathrm{L}_{2g}(K)$ such that f induces F'.

For nonzero $\vec{v}, \vec{w} \in K^{2g}$, we have $\omega(\vec{v}, \vec{w}) = 0$ if and only if $\langle \vec{v} \rangle \subset \langle \vec{w} \rangle^{\perp}$. By assumption, this holds if and only if $\langle f(\vec{v}) \rangle \subset \langle f(\vec{w}) \rangle^{\perp}$, so we conclude that $\omega(\vec{v}, \vec{w}) = 0$ if and only if $\omega(f(\vec{v}), f(\vec{w})) = 0$, and thus that $f \in \Gamma\mathrm{P}_{2g}(K)$.

It remains to check that f induces F. We know that F and f agree on lines. Consider an arbitrary isotropic subspace $V \subset K^{2g}$. Set $U = \{L \in \mathcal{TP}_{2g}(V) \mid L \subset V \text{ and } \dim(L) = 1\}$. In terms of the poset structure on $\mathcal{TP}_{2g}(V)$, the element V is characterized as the unique element containing all $L \in U$. Since F is an automorphism of $\mathcal{TP}_{2g}(V)$, we see that $F(V)$ is the unique element containing all $L \in F(U)$. Clearly $f(V) \in \mathcal{TP}_{2g}(V)$ also is the unique element containing all $L \in F(U) = f(U)$, so we conclude that $f(V) = F(V)$, as desired. $\square$

References

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