Mathematical Tripos
Part III Lecture Courses in 2012-2013

Department of Pure Mathematics & Mathematical Statistics

Department of Applied Mathematics & Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is ***no*** requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year 2012-13. Details for subsequent years are often broadly similar, but *not* necessarily identical. The courses evolve from year to year.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. The lectures define the syllabus. Each course lecturer has discretion to vary the material covered.
- This document was last updated in August 2012. Further changes to the list of courses will be avoided if at all possible, but may be essential.
- Some graduate courses have no writeup available. Hopefully, the title of the course is sufficiently explanatory.

Table of Contents

Algebra

Lie Algebras and their representations (M24)

C.J.B. Brookes

While groups arise from the study of symmetries, Lie algebras concern infinitesimal symmetries. They are non-associative algebraic structures arising in many branches of mathematics and theoretical physics.

The core material for this course is the theory of finite dimensional Lie algebras and their finite dimensional representations. This involves some basic structure theory: nilpotent, soluble and semisimple Lie algebras, the Killing form, derivations, Borel and Cartan subalgebras, Weyl groups. The classification of finite dimensional complex semisimple Lie algebras arises from considering finite root systems and their classification using Dynkin diagrams.

My intention this year is to spend less time on developing the structure theory and more on the representation theory.

An introduction to the representation theory of semisimple Lie algebras starts with the consideration of weights and highest weight modules, of which Verma modules are a special case. I shall discuss the universal enveloping algebra which is the associative algebra underlying the representation theory of Lie Algebras. In general for finite dimensional Lie algebras the enveloping algebra can be viewed as a non-commutative polynomial algebra.

Time permitting I may also say something about Lie algebras in characteristic p.

Pre-requisite Mathematics

To develop the theory of Lie algebras one just needs basic linear algebra. But for the representation theory it would certainly be helpful, but not absolutely essential, to have some experience of representation theory including some module theory.

Books

1. Dixmier J, Enveloping algebras North Holland (1977)
2. Erdmann K, Wildon MJ, Introduction to Lie algebras, Springer (2006)
3. Humphreys JE, Introduction to Lie algebras and representation theory, Springer (1972)
4. Jaconson N, Lie algebras, Dover (1979)
5. Serre J-P, Complex semisimple Lie algebras, Springer (1987)

Commutative Algebra (M24)

N. I. Shepherd-Barron

Commutative algebra (the study of commutative rings and modules over them) is an elegant subject in its own right, and is one that illuminates and is illuminated by, for example, algebraic geometry, number theory and representation theory. You might, therefore, like this course if you already like abstract algebra. If, on the other hand, you are indifferent to abstract algebra but like geometry and/or number theory then this course should give you tools with which to do these subjects more easily.

More precisely, I intend to flesh out the content of Atiyah and Macdonald and also discuss Kähler differentials and such homological things as the Koszul complex and the idea of a Cohen–Macaulay ring.

From a geometrical perspective the aim of this course is to teach you enough to make you happy with, for example, the idea of a smooth n-dimensional algebraic variety over a field, and also be in a position to approach the notion of duality on algebraic varieties. Books: (1): For this course:

M. Atiyah and I. Macdonald, *Introduction to Commutative Algebra*. This is brisk and efficient and perfect for what it covers. But it does not cover enough.

H. Matsumura, *Commutative Rings* (and another book, *Commutative Algebra*). For differentials and the Cohen–Macaulay material.

(2): Looking ahead and outward:

In algebraic geometry, Hartshorne (*Algebraic Geometry*) has been the standard introduction to the subject from an algebraic viewpoint for 35 years. As such, it provides a geometric counterpoint to this course.

In algebraic number theory there are many outstanding books at this level; you are spoilt for choice. Samuel, Fröhlich and Taylor, Marcus,...

Topics in Group Theory (L24)

Jan Saxl

This is a second course on Group theory, with emphasis on finite groups. After a general first half, the second half of the course will concentrate on finite simple group and their structure - in particular the classical simple groups. Here is an outline of the course:

Normal structure: composition series; Jordan-Hölder theorem; nilpotent and soluble groups; Hall's theorems for finite soluble groups.

Transfer and fusion. The Schur Zassenhaus theorem.

Permutation groups: primitivity; suborbits, orbitals and double cosets; permutation characters.

General linear groups: finite fields; Jordan decomposition; conjugacy classes; simplicity of PSL.

Classical groups: symplectic, orthogonal and unitary groups.

Subgroup structure theorems in finite almost simple groups, in particular the Aschbacher – O'Nan – Scott theorem.

Pre-requisite Mathematics

Some knowledge of Group theory will be assumed. Knowledge of some representation theory would be useful, but is not essential.

Literature

J.L. Alperin and R.B. Bell, Groups and Representations, Springer 1995.

P.J. Cameron, Permutation Groups, CUP 1999.

M. Suzuki, Group Theory, Springer 1982.

R.A. Wilson, The Finite Simple Groups, Springer 2009.

Topics in representation theory (L24)

Ian Grojnowski

This course will be an introduction to some classical topics in representation theory.

One possibility is we'll study Hecke algebras (finite, affine, double, and so on) from various points of view.

These algebras arise in many ways. They have simple definitions by generators and relations. They are deformations ('quantizations') of the group algebra of the symmetric group S_n, or more generally the Weyl group, and of its affine or double affine counterpart. They arise as quotients of the fundamental group of the regular semisimple elements of GL_n mod conjugacy, as endomorphism algebras of the principal series of $GL_n(F_q)$ and of its p-adic and other generalisations. They are central to representation theory and the Langlands program. The double affine variants (first defined by Cherednik, their degenerations are called symplectic reflection algebras) are the subject of much research in the last decade, and are still mysterious. They have incredibly pretty combinatorics (Macdonald 2-variable symmetric functions), and a lovely relation to the moduli space of zero dimensional subschemes of an algebraic surface (the 'Hilbert Scheme').

Possible alternate topics we may cover are Iwasawa-Lazard algebras, quantum groups, and finite groups.

No particular background in algebra, algebraic geometry, or representation theory will be assumed— we will start at the beginning, and explain any technology we need to use.

Representation theory (L24)

Stuart Martin

This is a lecture course on the representation theory of the symmetric group and the general linear group, originally due to Frobenius and Schur and later generalised by Weyl. I will also cover some related material in classical invariant theory. My aim is to treat as much of the basic theory as time permits without developing a lot of modular machinery, so, for example, I will usually restrict myself to working over the complex numbers. At least two examples classes will be offered to support the lectures.

For the symmetric group: a review of semisimple algebras; Young symmetrizers, partitions, Young tableaux and diagram calculus; the (irreducible) Specht modules, character computations; the Hook Formula for the dimension of the Specht module and other combinatorial algorithms (if time permits).

For the general linear group: a review of multilinear algebra including tensor products; Schur-Weyl duality; the decomposition of tensors; rational and polynomial representations of $\mathrm{GL}(V)$; Weyl's character formula.

For the invariant theory: basic examples, symmetric polynomials; the First Fundamental Theorem of invariant theory; (if time permits) Gordan's theorem and the computation of covariants.

Pre-requisite Mathematics

No previous knowledge is required beyond a smattering of rings and modules, undergraduate representation theory (meaning complex character theory of finite groups) and basic algebraic geometry (the Nullstellensatz and affine varieties).

Level

Part III

Literature

1. M.F. Atiyah and I.G. Macdonald. Introduction to commutative algebra. Addison-Wesley, 1969. Has background on ring theory and on algebraic geometry.

2. D.J. Benson, Polynomial invariants of finite groups. CUP 1993.

3. J.A. Dieudonné and J.B. Carrell. Invariant theory old and new. Academic Press 1971.

4. J.H. Grace and A. Young, The algebra of invariants. CUP 1903 (plus subsequent reprints). A classic.

5. G.D. James and A. Kerber, The representation theory of the symmetric group. CUP 1984.

6. I.G. Macdonald. Symmetric functions and Hall polynomials (2nd edn.) OUP 1995.

7. M.D. Neusel and L. Smith. Invariant theory of finite groups. AMS 2002.

8. H. Weyl, The classical groups: their invariants and representations. Princeton U.P. 1946 (plus subsequent reprints). The greatest book on algebra you will ever read.

Sporadic and Related Groups. (L16)

Non-Examinable (Graduate Level)

R. Parker

The Classification of Finite Simple groups essentially states that every finite simple group is either cyclic of prime order, an alternating group, some sort of linear group over a finite field or one of the 26 Sporadic Groups.

Most sporadic groups can be considered to be the symmetry group of an altogether exceptional object, with unique properties like no other.

This lecture series surveys these simple groups and the objects they act on, constructing many of them, and pointing out their interesting properties.

Starting with the Mathieu groups (acting on Steiner Systems and codes) we move on to rank 3 groups acting on graphs using A5 and M22 as an example. Then the Fischer (2,3 transposition) groups are examined.

The Leech Lattice - a packing of 196560 24-dimensional spheres - is then constructed, and its automorphism group (2.Co1) investigated and its main properties found.

The Monster is then described (it is a bit too big to construct), along with its main subgroups.

Finally the series closes with a look at the six "Pariahs" - the remining groups that seem to stand alone and not really act on anything natural that I know of.

Some knowledge of group theory is an advantage, but the topics will be developed (albeit at a fast pace) where needed and little specific material is assumed.

Analysis

Topics in Kinetic Theory (M24)

Amit Einav and Chanwoo Kim

Description

Kinetic equation are a particular type of, usually non linear, Partial Differential Equations (PDEs) that arise in Statistical Physics. Their goal is to describe the time evolution of systems consisting of large amount of objects, such as Plasmas, Galaxies and Dilute Gases. This course is an introductory course to the modern analysis of kinetic equations, aiming to present some results on the fundamentally important Boltzmann equation from the subject of gas dynamics.
The course is suitable for both Pure Mathematics and Applied Mathematics students. We hope to cover the following topics:

1. Introduction:
   - Microscopic, Macroscopic and Mesoscopic Viewpoints and Kinetic Theory.
   - From ODEs to PDEs.
2. Derivation of Kinetic Equations:
   - Newtonian and Statistical Viewpoints.
   - The Characteristic Method.
   - The Many Particle Limit and Mean Field Models.
3. Linear Transport Equations:
   - Lagrangian and Eulerian Viewpoints.
   - Dispersion Estimations.
   - Averaging Lemma and Phase Mixing.
4. The Linear Boltzmann Equation:
   - A Probabilistic Interpretation.
   - The Cauchy Theory.
   - The Maximum Principle.
   - Relaxation to Equilibrium.
5. Additional Topics.

Pre-requisite Mathematics

Knowledge of basic Measure Theory, Functional Analysis and simple methods in Ordinary Differential Equations (as in the 1A course 'Differential Equations') is required. Any advanced knowledge in the above topics, as well as knowledge in PDEs, Sobolev spaces and Fourier Analysis, can benefit the student, but is not mandatory. Students are welcome to discuss any pre-requisite requirements with the Lecturers prior to the beginning of the course.

Literature

The course is mainly self contained and requires no textbook. However, there are numerous textbooks that will compliment the material of the course, or help bring the student up to pace with the pre requisites of it. Interested students are welcome to discuss this with the Lectures.

Aspects of Analysis (L24)

Dr D.J.H. Garling

This will be a mixture of abstract and complex analysis. It will include SOME of the following topics.

The Hahn-Banach theorem, and Banach space duality.

Product topologies. Filters, ultrafilters and Tychonoff's theorem.

The weak*-topology and the Banach-Alaoglu theorem.

The Stone Čech compactification.

Measures on compact spaces, and the dual of $C(K)$.

Fixed point theorems.

The Krein-Mil'man theorem

The Ryll-Nardzewski fixed point theorem.

Haar measure on a compact Hausdorff space.

Maximal inequalities. Applications to harmonic analysis.

Khinchine's and other inequalities. Applications to Banach space theory.

Desirable Previous Knowledge

Basic knowledge of analysis and general topology. Results from the the Part II Linear Analysis course will be used, as will some of the standard results from measure theory which are taught in the Part II Probability and Measure course.

Introductory Reading

1. B.Bollobas. Linear Analysis. CUP 1990.
2. R.M.Dudley. Real analysis and probability. CUP 2002.

Reading to complement course material, and further reading

1. B.Bollobas. Linear Analysis. CUP 1990.
2. R.M.Dudley. Real analysis and probability. CUP 2002.
3. D.J.H Garling. Inequalities: a journey into linear analysis. CUP 2007.
4. W. Rudin. Functional Analysis. McGraw-Hill 1973-90.

Elliptic Partial Differential Equations (M24)

Non-Examinable (Graduate Level)

Brian Krummel

This course is intended as an introduction to the theory of linear elliptic partial differential equations. Elliptic equations play an important role in geometric analysis and a strong background in linear elliptic equations provides a foundation for understanding other topics including minimal submanifolds, harmonic maps, and general relativity. We will discuss both classical and weak solutions to elliptic equations, considering when solutions to the Dirichlet problem exist and are unique and considering the regularity of solutions. This involves establishing maximum principles, Schauder estimates, and other estimates on solutions. As time permits, we will discuss the De Giorgi-Nash theory, which can be used to prove the Harnack inequality and establish Hölder continuity for weak solutions. Note that the basic properties of Sobolev spaces that we will need for weak solutions will be covered as part of the course.

Pre-requisite Mathematics

Lebesgue integration, Lebesgue spaces, and basic functional analysis.

Literature

1. David Gilbarg and Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order. Springer-Verlag (1983).
2. Lawrence Evans, Partial Differential Equations. AMS (1998)
3. Qing Han and Fanghua Lin, Elliptic partial differential equations. Courant Lecture Notes, Vol. 1 (2011).

Analysis of Operators (L24)

Non-Examinable (Graduate Level)

Antony Wassermann

Starting from the spectral theorem for compact self–adjoint operators on Hilbert space, this course will study operators that occur in different parts of analysis, including partial differential equations, group representation theory and geometric complex function theory. Topics will include:

- Sobolev spaces on $\mathbb{T}^n$, elliptic regularity, Fredholm and Toeplitz operators
- Representations of SU(2) and U(2), index theorems
- The Heisenberg group, Fourier transform, Stone-von Neumann theorem, pseudodifferential operators
- SU(1,1) and SL(2,$\mathbb{R}$), oscillator representation and semigroup
- Singular integral operators, Hilbert transform on $\mathbb{T}$ and $\mathbb{C}$, Cauchy transform, applications to univalent functions

Previous versions of this course, which cover roughly half of these topics from a slightly different point of view, can be found at

http://www.dpmms.cam.ac.uk/∼ajw/

TeXed lecture notes will be available for the course after lectures have finished.

Pre-requisite Mathematics

Pre-requisite Description to be added

Literature

1. F.W. Warner, *Foundations of differentiable manifolds and Lie groups.* Graduate Texts in Mathematics, **94**, Springer–Verlag, 1983 (Sobolev spaces)
2. G.B. Folland, *Harmonic analysis in phase space,* Annals of Mathematics Studies, **122**, Princeton Univ. Press, 1989 (Heisenberg group, pseudodifferential operators, oscillator semigroup)
3. L. Hrmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis,* Grundlehren der Mathematischen Wissenschaften, **256**, Springer–Verlag, 1990 (Fourier transform)

4. M.E. Taylor, *Noncommutative harmonic analysis,* Mathematical Surveys and Monographs, **22**, American Mathematical Society, 1986 (group representations for analysts)

5. J.B. Garnett, *Bounded analytic functions,* Graduate Texts in Mathematics, **236**, Springer, 2007 (Hilbert transform on $\mathbb{T}$)

6. S.R. Bell, *The Cauchy transform, potential theory, and conformal mapping,* Studies in Advanced Mathematics, CRC Press, 1992 (Cauchy transform)

7. F.D. Gakhov, *Boundary value problems,* Dover Publications, 1990 (singular integral operators)

8. L.V. Ahlfors, *Lectures on quasiconformal mappings,* University Lecture Series, **38**, American Mathematical Society, 2006 (Hilbert transform on $\mathbb{C}$)

9. P.L. Duren, *Univalent functions,* Grundlehren der Mathematischen Wissenschaften, **259**, Springer–Verlag, 1983 (univalent functions)

10. O. Lehto, *Univalent functions and Teichmüller spaces,* Graduate Texts in Mathematics, **109**, Springer–Verlag, 1987 (univalent functions)

Introduction to Fourier Analysis (M24)

T. W. Körner

This will be first course in Fourier Analysis with a number theoretic tinge. A first draft of the course is available on my website website

http://www.dpmms.cam.ac.uk/∼twk/

Pre-requisite Mathematics

I will avoid the use of measure theory and functional analysis, but I will make substantial use of the contents of a first course in complex variable

Literature

There is a bibliography attached to my notes, but a browse through the early parts of H. Dym and H. P. McKean *Fourier Series and Integrals* or Y. Katznelson *An Introduction to Harmonic Analysis* might be helpful.

Products of random i.i.d. matrices (L16)

Non-Examinable (Graduate Level)

Péter Varjú

Let $X_1, X_2, \ldots \in SL_d(\mathbf{R})$ be a sequence of random independent identically distributed matrices. The purpose of the course is to understand the large scale behaviour of the product $Y_l = X_1 X_2 \cdots X_l$. This can be thought of as a non-commutative analogue of the classical theory of sums of independent random variables.

As an illustration, here is the sketch of a typical result: Under certain (very general) conditions, there is a positive number $\gamma > 0$ called the first *Lyapunov exponent* such that for any $0 \neq v \in \mathbf{R}^d$, the length of $|Y_l v|$ behaves like $e^{l\gamma}$. Moreover, the distribution of the direction of $Y_l v$ converges to a probability measure ν on $\mathbf{P}^{d-1}(\mathbf{R})$ called the *stationary measure* and it is independent of v. The speed of convergence is also well understood.

Using such results, one can give good estimates for the probability that the random product is degenerate in some sense, e.g. a fixed subspace is invariant for it. Such estimates has been used recently in the study of random walks on non-commutative groups in various settings. The theory has many other (older) applications, as well, for example to the spectral theory of Schrödinger operators.

Topics:

- Lyapunov exponents
- Convergence to the stationary measure
- Speed of convergence
- Regularity of the stationary measure
- The criterion of Goldsheid and Margulis for the simplicity of the Lyapunov exponents

Pre-requisite Mathematics

Basic courses in linear algebra and measure theory.

Literature

Most of the material is covered in the book of Bougerol and Lacroix and the survey of Goldsheid and Margulis.

- P. Bougerol and J. Lacroix, *Products of random matrices with application to Schrödinger operators*, Progress in Probability and Statistics **8**, Birkhäuser, 1985
- I. Ya. Gol'dsheid and G.A. Margulis *Lyapunov indices of a product of random matrices,* Russian Math. Surveys **44** No.5.(1989), 11–71

Ornstein Theory (L16, E8)

Non-Examinable (Graduate Level)

Yonatan Gutman

Consider the process of flipping an (unfair) coin repeatedly and independently. At each point of time $t = \dots, -2, -1, 0, 1, 2, \dots$ you mark **H**(eads) or **T**(ail) according to the outcome. Now replace the coin by a dice with n faces. Assume that the probability that $k = 1, 2 \dots, n$ shows up is p_k (so it holds $\sum_{k=1}^{n} p_k = 1$). Again record at each time $t = \dots, -2, -1, 0, 1, 2, \dots$ the outcome. These processes are examples of *Bernoulli Schemes* - the "most random possible" processes. A natural question is: When are two Bernoulli schemes isomorphic ("the same")? The celebrated Ornstein Theorem (1970) answers this question fully. But this is only the starting point of *Ornstein Theory.* It turns out that many deterministic processes are actually Bernoulli. By *deterministic* we mean that the time evaluation of the process is specified by some definite rule ("algorithm") which does not involve any randomness. For example consider ("ideal") gas molecules bouncing around in a box. Although the interactions are ruled by Newton's laws the dynamics resulting is Bernoullian (this can be made precise...).

Topics of the Lent course:

1. Basics of measure preserving systems.
2. The Rokhlin Lemma.
3. Independence and ϵ-independence
4. Entropy
5. Bernoulli Shifts.

6. The Ornstein Isomorphism Theorem.

Topics of the Easter course:

1. Finitely determined partitions.
2. Weak and very week Bernoulli partitions.
3. A mixing Markov shift is Bernoulli.
4. The Geodesic Flow on a Riemann surface of negative curvature is Bernoulli.

Pre-requisite Mathematics

A basic course in measure theory.

Literature

The Ornstein Theorem is a cornerstone of modern ergodic theory. Therefore its proof appears in quite a few books. Let me mention [1],[6],[3],[4] and [5]. While some of these proofs are very slick, they do not necessarily fit for an unacquainted learner. We will therefore closely follow the very accessible [2]. This is a true gem and it is even available free of charge from the webpage of the author!

References

[1] Ornstein, Donald S., *Ergodic theory, randomness, and dynamical systems*, Yale Mathematical Monographs, No. 5, 1974.

[2] Shields, Paul., *The theory of Bernoulli shifts*, Chicago Lectures in Mathematics, 1973.

[3] Glasner, Eli. *Ergodic theory via joinings*, Mathematical Surveys and Monographs 101, AMS, 2003.

[4] Petersen, K. *Ergodic theory*, Cambridge Studies in Advanced Mathematics 2, Cambridge University Press 1983.

[5] Downarowicz, Tomasz, *Entropy in dynamical systems*, New Mathematical Monographs 18, Cambridge University Press 2011.

[6] Rudolph, Daniel J., *Fundamentals of measurable dynamics*, OUP 1990.

The Kakeya universe and incidence problems (L24)

Michael Bateman

An old construction of Besicovitch shows that a subset of the plane containing a unit line segment in every direction can have arbitrarily small measure. Nevertheless it can be shown that such a set must have Hausdorff dimension two. The analogue for three and higher dimensions is unknown, and the statement "A subset of $\mathbf{R}^n$ ($n \geq 3$) containing a unit line segment in every direction must have Hausdorff dimension n" is typically referred to as the Kakeya Conjecture. The past $\sim$ 20 years have seen an explosion of work on this conjecture and its relatives, as well as connections to a number of other areas – additive combinatorics, Fourier series, PDE, number theory, etc.

The goal of this course is to explore the family of problems that have Kakeya-type flavor in two and higher dimensions. This could be loosely described a s "Any problem involving lots of long skinny rectangles/tubes that overlap a lot." The survey by Tao listed below is recommended for anyone interested in the course.

Topics will include most of the following.

- *The* 2 *dimensional Kakeya problem.* Basic notions of dimension. The Besicovitch construction. Fefferman's application of Besicovitch sets to convergence of Fourier series. Bourgain's sum-product theorem for unions of finitely many intervals, and its connection to two-dimensional Kakeya-type sets.
- *The* 3+ *dimensional Kakeya problem.* Specific estimates on the size of Besicovitch sets in dimensions ≥ 3, connections with these estimates and additive combinatorics. The multilinear Kakeya problem.
- *Algebraic techniques.* The Szemeredi-Trotter theorem on point-line incidences and its application to the sum-product theorem for finite sets of reals. The finite field Kakeya problem and its solution by Dvir. Connections to point line incidence theory in $\mathbf{R}^3$, the Erdos distance problem and its solution by Guth and Katz.
- *Siblings of the Kakeya problem.* Restriction of the Fourier transform to spheres. Local smoothing of the wave equation. Fourier multipliers (i.e., the Bochner-Riesz problem).

Pre-requisite Mathematics

This class should be very elementary in some sense, although we will discuss rather recent results. Students will hopefully have some knowlegde of measure theory, and have met the L^p spaces and Fourier series.

Optimal Transportation (L24)

Non-Examinable (Graduate Level)

Amit Einav

The subject of Optimal Transportation was first born in France, in a 1781 paper by the French mathematician Gaspard Monge. In his paper, Monge considered the problem of transporting a fixed quantity of soil, extracted from the ground, to places where it will be incorporated in construction. The location of the extraction and construction sites are given, as well as the cost function to send the soil from extraction site x to construction site y. Monge's goal was to find the most economically efficient transportation plan. Many years later, the problem was rediscovered by the Russian mathematician Leonid Vitaliyevich Kantorovich, and since then it has become a classical subject in Probability, Economics and Optimization. More recently, new connections were found between Optimal Transportation and Functional Analysis, Partial Differential Equations, Kinetic Theory and Fluid Mechanics. It gives a new perspective and new approaches to many unsolved problems, allowing us to build bridges between different subjects.
We will attempt to cover the following topics:

- The Monge and Kantorovich Optimal Transportation Problems.
- Kantorovich Duality.
- The Transport Plan in the Quadratic Case.
- Displacement Interpolation and Displacement Convexity.
- The Wasserstein Distance.
- Overview of Transportation Inequalities.

Pre-requisite Mathematics

Knowledge of basic Measure Theory is required, as well as basic Point Set Topology and Functional Analysis. Some knowledge in Probability may help, but is not necessary. The course is mostly self-contained. Students are welcome to discuss any pre-requisite requirement with the Lecturer prior to the beginning of the course.

Literature

- Villani, Cédric. Topics in optimal transportation. Graduate Studies in Mathematics, 58. American Mathematical Society, Providence, RI, 2003. xvi+370 pp. ISBN: 0-8218-3312-X.
- Villani, Cédric. Optimal Transport: Old and New. Volume 338 of Grundlehren der mathematischen Wissenschaften, Springer, 2009, ISBN 978-3-540-71049-3 (Can be found online at http://math.univ-lyon1.fr/ villani/surveys.html).

Combinatorics

Combinatorics (M16)

Prof I.B.Leader

The flavour of the course is similar to that of the Part II Graph Theory course, although we shall not rely on many of the results from that course.

We shall study collections of subsets of a finite set, with special emphasis on size, intersection and containment. There are many very natural and fundamental questions to ask about families of subsets; although many of these remain unsolved, several have been answered using a great variety of elegant techniques.

We shall cover a number of 'classical' extremal theorems, such as those of Erdős-Ko-Rado and Kruskal-Katona, together with more recent results concerning isoperimetric inequalities and intersecting families. The aim of the course is to give an introduction to a very active area of mathematics.

We hope to cover the following material.

Set Systems

Definitions. Antichains; Sperner's lemma and related results. Shadows. Compression operators and the Kruskal-Katona theorem. Intersecting families; the Erdős-Ko-Rado theorem.

Isoperimetric Inequalities

Harper's theorem and the edge-isoperimetric inequality in the cube. Inequalities in the grid. The classical isoperimetric inequality on the sphere. The 'concentration of measure' phenomenon. Applications.

Intersecting Families

Katona's t-intersecting theorem. The Ahlswede-Khachatrian theorem. Restricted intersections. The Kahn-Kalai counterexample to Borsuk's conjecture.

Desirable Previous Knowledge

The only prerequisites are the very basic concepts of graph theory.

Introductory Reading

1. Bollobás, B., Combinatorics, C.U.P. 1986.

Additive Combinatorics (M24)

Prof. B. J. Green

Additive Combinatorics may be viewed as the study of approximate algebraic structures and their applications.

It is well-known that a nonempty finite set A in some group G is a subgroup if and only if $xy^{-1} \in A$ whenever $x, y \in A$. One possible definition of an approximate group considers the possibility that this only happens some of the time: the proportion of pairs $(x, y) \in A$ for which $xy^{-1} \in A$ is at least 1/10, say. There is a surprisingly rich theory of such objects, and of analogous notions such as those of an approximate polynomial and approximate field, with many applications.

Topics will include most of the following.

- *Approximate groups.* Basic definitions. Ruzsa's estimates and the Balog-Szemerédi-Gowers theorem. Freiman's theorem on approximate subgroups of $\mathbf{Z}$. Helfgott's theorem on approximate subgroups of $\mathrm{SL}_2(\mathbf{F}_p)$ and application to construction of expanders.

- *Approximate polynomials.* The Gowers norms and the notion of an approximate polynomial. Proof of Roth's theorem that a subset of $\{1,\ldots,N\}$ of size at least $cN/\log\log N$ contains a 3-term arithmetic progression. The Behrend example. Discussion of inverse theorems for the higher Gowers norms and nilsequences. Outline proof of the inverse theorem for the U^3-norm and Szemerédi's theorem for 4-term arithmetic progressions.
- *Approximate fields.* Approximate fields and the sum-product phenomenon. The Bourgain-Glibichuk-Konyagin estimates for exponential sums over multiplicative subgroups of $\mathbf{F}_p$.
- *Further Topics.* The Croot-Sisask theorem. Recent results of Sanders establishing the best-known bounds for Roth's theorem.

Pre-requisite Mathematics

Very few formal prerequisites. The course will feature some Fourier analysis and a little group theory, but everything we need will be developed from scratch.

Literature

The book *Additive Combinatorics* by Tao and Vu, CUP 2006, covers most but not all of the material, albeit in a very different order. I plan to produce printed notes for much if not all of the course.

Extremal Graph Theory (Lent 24L, Part III)

A. Thomason

Extremal graph theory is an umbrella title for the study of graph properties and their dependence on the values of graph parameters. This course builds on the material introduced in the Part II Graph Theory course, in particular Turán's theorem and the Erdős-Stone theorem, as well as developing the use of randomness in combinatorial proofs. Further techniques and extensions to hypergraphs will be discussed. It is intended to cover some reasonably large subset of the following.

The Erdős-Stone theorem and stability. Supersaturation. Szemerédi's Regularity Lemma, with applications. The number of complete subgraphs.

Hypergraphs. Erdős's r-partite theorem. Instability. The Fano plane. Razborov's flag algebras. Hereditary properties and their sizes.

Probabilistic tools: the Local Lemma and concentration inequalities. The chromatic number of a random graph. The semi-random method, large independent sets and the Erdős-Hanani problem. Dependent random choice.

Pre-requisite Mathematics

A knowledge of the basic concepts, techniques and results of graph theory, such as that afforded by the Part II Graph Theory course.

Literature

No book covers the course but the following can be helpful.

B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics **184**, Springer-Verlag, New York (1998), xiv+394 pp.

N. Alon and J. Spencer, The Probabilistic Method, Wiley, 3rd ed. (2008)

Geometry and Topology

Algebraic Geometry (M24)

Caucher Birkar

This course is intended to serve as an introduction to modern algebraic geometry. Essentially, algebraic geometry is about studying the solutions of systems of polynomial equations. However, much of this study involves geometric intuition and advanced algebraic techniques. The methods of algebraic geometry are so fruitful that they are applied to subjects far beyond algebraic geometry such as number theory, analytic and differential geometry, topology, mathematical physics, mathematical logic, cryptography, etc.

Topics I hope to cover: sheaves, schemes, varieties, morphisms, divisors, differential forms, cohomology, duality, Riemann-Roch theorem, quotient by group actions, algebraic groups, etc.

Pre-requisite Mathematics

Previous familiarity with algebraic geometry is not necessary but it would be very helpful. If you have not encountered algebraic geometry before, it is recommended that prior to the start of the course you browse through chapter I of [H] or through [S]. On the other hand, commutative algebra is used systematically.

Related courses

The part III *commutative algebra* is strongly recommended.

Literature

[AM] M. Atiyah, I. Macdonald. *Introduction to commutative algebra.* Westview Press, 1994.

[H] R. Hartshorne. *Algebraic geometry.* Springer, 1977. (Much of the course is based on chapters II-III of this book.)

[S] I. Shafarevich. *Basic algebraic geometry I.* Springer, 1994.

Algebraic Topology (M24)

Jacob Rasmussen

Algebraic topology assigns algebraic invariants (groups and homomorphisms) to topological spaces and continuous maps between them. The most important example of such an invariant is ordinary homology theory, which is part of the basic language of geometry today. This course will cover homology and cohomology, together with applications to the topology of manifolds and vector bundles. The emphasis will be on learning to compute and use these invariants in a variety of examples. A tentative syllabus is as follows:

- *Homology.* Singular homology and cohomology. Eilenberg-Steenrod axioms and cellular homology. Universal coefficient theorem. Künneth theorem and cup products.
- *Vector Bundles.* Vector bundles and principal bundles. Long exact sequence on homotopy groups. Classifying spaces for bundles. Euler class and the Thom isomorphism. Leftshetz fixed point theorem.
- *Topology of Manifolds.* Handle decompositions and Morse theory. Poincaré duality. Cobordism groups.

Pre-requisite Mathematics

The only required background is basic point-set topology. The material in the Michaelmas term Differential Geometry course will be useful as well.

Literature

1. A. Hatcher, *Algebraic Topology*, CUP (2002).
2. J.W. Vick, *Homology Theory*, Springer (1994).
3. R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer (1982).
4. J.P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press (1999).

Differential Geometry (M24)

Mihalis Dafermos

This course is a basic introduction to differential geometry. Particular emphasis will be on Riemannian geometry, the natural generalization of the classical differential geometry of curves and surfaces. A major theme of the course will be the interaction of local and global geometry and topology. Related extensions of these ideas to general relativity will be explored. A tentative syllabus is as follows:

1. Local Analysis and Differential Manifolds. Definition and examples of manifolds. Tangent vectors, tangent and cotangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Stokes' Theorem, de Rham cohomology.

2. Local Riemannian Geometry. Riemannian metrics, Levi-Civita connection, parallel transport. Riemann curvature tensor, Ricci curvature. Laplace-Beltrami operator, Green's theorem.

3. Global Riemannian Geometry. Geodesics, exponential map, Gauss' Lemma. Jacobi fields, second variation, comparison theorems. The Bochner technique.

4. Singularity theorems in general relativity.

The lectures will be supplemented by four example classes.

Pre-requisite Mathematics

Some familiarity with the classical theory of curves and surfaces will be useful.

Literature

1. J. Lee, Introduction to Smooth Manifolds, GTM 218, Springer, 2003
2. I.Chavel, Riemannian geometry: a modern introduction. CUP, 1994.
3. V.Guillemin, A.Pollack, Differential topology. Prentice-Hall Inc., 1974.
4. B.O'Neill, Semi-Riemannian geometry. With applications to relativity. Academic Press, 1983.

Spectral geometry (L24)

D. Barden

The aim of this course is to give an overview of the work that has blossomed in response to Mark Kac' naive sounding question, first posed in 1966: 'Can one hear the shape of a drum?' In other, more general, words can one determine the geometry of a Riemannian manifold from the spectrum, the set of eigenvalues together with their multiplicities, of the Laplacian operator. The answer is

unsurprisingly, no: many pairs, and even continuous families, of manifolds have since been constructed that are isospectral (have the same spectrum) yet are not isometric. BUT

surprisingly, almost yes: these examples are very special, usually highly symmetric, so that it is still possible that generically (a term that may be defined to suit the context) manifolds are spectrally determined. In fact this has already been shown to be the case in certain contexts.

Contents

After definitions and basic results, most of the following will be discussed, the selection to some extent being determined by the audience.

• Computation of spectra. Explicit computations of spectra are very rare, but we can achieve it for 'round' spheres and flat tori. In particular we can show that flat 2-tori are spectrally determined.

• Examples of isospectral non-isometric (INI) planar domains and flat tori. The planar domains have boundaries that are only piecewise smooth so do not answer Kac' question since he required smooth boundaries: his precise question remains open. The INI flat tori will be shown to exist in all dimensions greater than 3 and are known not to exist in that dimension.

• The heat kernel and the spectral determination of dimension and volume.

• Sunada's work which resulted in a plethora of INI pairs with a common (Riemannian) covering manifold. In general this requires the residuality of 'bumpy' metrics which Sunada also proved.

• The use of Sunada's technique to produce INI Riemann surfaces.

• Wolpert's theorem that, despite the previous results, generic Riemann surfaces are spectrally determined.

• Gordon and Schuett's work generalising Sunada's ideas to torus bundles.

• Bérand's work on 'transplantation' and isospctrality, generalising the technique used to obtain the INI planar domains above.

Pre-requisite Mathematics

This subject is very much inter-disciplinary involving (Riemannian) geometry, analysis and topology as well as some algebra and minor forays into other subjects. However the results needed will mostly be stated without proof, so that the level of knowledge required will be that which is sufficient to understand and apply the statements of the thorems, rather than knowing or understanding their proofs.

Literature

Nothing is truly apposite for preliminary reading: full (indeed overfull) notes will be produced during the course, including a long reference list; the 'Survey of Isospectral Manifolds' by Carolyn Gordon published in Vol.I of the Handbook of Differential Geometry (published by Elsevier Science in 2000) gives an excellent overview of the subject; the LMS Student Text no. 31 by S.Rosenberg entitled 'The Laplacian on a Riemannian Manifold' overlaps with some of the course; 'Eigenvalues in Riemannian Geometry' by Isaac Chavel gives some idea of the state of play c.1980, but is more analytic than the course will be.

Complex Manifolds (L24)

P.M.H. Wilson

A preliminary outline of the course is as follows, but this will almost certainly be subject to change.

- Basic concepts of complex manifolds, holomorphic vector bundles, holomorphic tangent and cotangent bundles (for which corresponding concepts from the real smooth manifolds will be assumed). Canonical line bundles, normal bundle for a submanifold and the adjunction formula.

- Brief description of sheaf cohomology, with deduction of de Rham and Dolbeault cohomology for complex manifolds.
- Hermitian metrics, connections, curvature and Chern classes for complex vector bundles. Case of holomorphic vector bundles.
- Harmonic forms: the Hodge theorem and Serre duality (general results on elliptic operators will be assumed).
- Compact Kähler manifolds. Hodge and Lefschetz decompositions on cohomology, Kodaira–Nakano vanishing, Kodaira embedding theorem.

Pre-requisite Mathematics

A knowledge of basic Differential Geometry from the Michaelmas Term course will be highly desirable. The main books for this course will be as below.

Literature

1. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1978).
2. R.O. Wells, *Differential Analysis on Complex Manifolds*, Springer (1980).
3. F. Zheng, *Complex Differential Geometry*, AMS (2000).
4. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Volume II*, Wiley (1969).
5. D. Huybrechts, *Complex Geometry*, Springer 2005.

Derived Algebraic Geometry (L24)

Non-Examinable (Graduate Level)

J.P.Pridham

Whereas rings are the basic building blocks for classical algebraic geometry, in the guise of affine schemes, the building blocks for derived algebraic geometry are simplicial or differential graded rings. The main motivations for derived algebraic geometry come from intersection theory and moduli theory, especially obstructions. Associated to any classical moduli problem, Deligne, Drinfel'd and Kontsevich conjectured that there should be a derived moduli stack, smooth in an appropriate sense, permitting the construction of virtual fundamental classes on the classical moduli stack.

The course will begin with a description of higher and derived stacks in terms of Duskin-Glenn hypergroupoids, together with some of their basic properties. This will then be compared with various other formulations. I will then introduce Lurie's representability theorem (a derived analogue of Artin's representability theorem), and give some applications. Likely examples include derived moduli of vector bundles and of polarised projective varieties.

Pre-requisite Mathematics

The foundational parts of the course should involve a little algebraic topology, and no algebraic geometry more advanced than étale morphisms, although some familiarity with algebraic stacks is desirable. Later on, some knowledge of moduli theory will be helpful, as would some familiarity with simplicial techniques.

Literature

1. Charles A. Weibel. An introduction to homological algebra, Cambridge University Press, Cambridge, 1994. Chapter 8.

2. J. P. Pridham. Notes characterising higher and derived stacks concretely. arXiv:1105.4853v2 [math.AG], 2011.

3. Gabriele Vezzosi. What is...a derived stack? Notices Amer. Math. Soc. 58 (7), pp. 955–958, 2011.

4. Bertrand Toën. Higher and derived stacks: a global overview. arXiv math/0604504v3, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math (80), pp. 435–487, Amer. Math. Soc., Providence, RI, 2009.

5. Ionuţ Ciocan-Fontanine and Mikhail Kapranov. Derived Quot schemes. Ann. Sci. École Norm. Sup. (4), 34(3):403–440, 2001.

6. Paul G. Goerss and John F. Jardine. Simplicial homotopy theory, volume 174 of Progress in Mathematics. Birkhäuser Verlag, Basel, 1999. Chapters I–III, VII.

7. J. Lurie. Derived Algebraic Geometry. Ph.D. thesis, M.I.T., 2004. www-math.mit.edu/ ∼lurie/papers/DAG.pdf.

Hodge structures and Mumford-Tate groups (L16)

Non-Examinable (Graduate Level)

C. Vial

The singular cohomology groups $H^n(X(\mathbf{C}),\mathbf{Z})$ of a complex smooth projective variety X are endowed with a very rich structure coming from linear algebra : a Hodge structure. The famous Hodge conjecture stipulates that every Hodge class in $H^{2n}(X(\mathbf{C}),\mathbf{Q})$ should be representable by a sum with rational coefficients of codimension-n subvarieties of X. It is a theorem of Lefschetz that the Hodge conjecture holds in codimension 1 and very little is known about the conjecture in higher codimensions.

The Mumford-Tate group $MT(H)$ of a rational Hodge structure H can be defined as the largest algebraic subgroup of $GL(H)$ that fixes all Hodge classes on H and all of its tensor-powers. Thus, the sub-Hodge structure of $H^{\otimes n}$ consisting of the Hodge classes is nothing but the invariant sub-group of $H^{\otimes n}$ whose elements are fixed by $MT(H)$. When one is able to compute the Mumford-Tate group of the cohomology of a variety X, it is therefore possible, by using invariant theory, to study the validity of the Hodge conjecture on the self-products of X. For instance, given an elliptic curve E, it is possible by this mean to prove that the Hodge classes on the rational cohomology of E^n are spanned by degree-2 Hodge classes. In particular, the Hodge conjecture holds for the self-products of an elliptic curve.

Another consequence of the study of Mumford-Tate groups is the following theorem of Deligne: If $S \subset \mathbf{P}^3$ is a very general surface of degree ≥ 5, then $H^2(S,\mathbf{Q})$ cannot be expressed with the help of abelian varieties. In particular, such a variety is not dominated by a product of curves.

Depending on time and taste of the audience, we will cover but not restrict ourselves to the following topics : mixed Hodge structures; Variations of Hodge structures and their Mumford-Tate groups, Noether-Lefschetz type theorems; Mumford-Tate groups of abelian varieties and $K3$-surfaces; the Mumford-Tate conjecture for $K3$-surfaces; André's notion of motivated cycle and application to Hodge classes on abelian varieties.

Pre-requisite Mathematics

Complex manifolds, Algebraic geometry.

Symplectic Topology of Stein manifolds (L24)

Non-Examinable (Graduate Level)

Yankı Lekili

This course will be an introduction to symplectic topology of Stein manifolds and h-principle. We will closely follow Cieliebak and Eliashberg's recent book. The main focus of the course will be the dichotomy between flexible and rigid Stein structures. After going through the basics of Stein manifolds from a symplectic viewpoint, we will cover various selected topics exemplifying this dichotomy.

Pre-requisite Mathematics

Basic knowledge of differential geometry and algebraic topology. An introductory knowledge of symplectic geometry would also be helpful.

Literature

¿From Stein to Weinstein and Back: Symplectic Geometry of Affine Complex Manifolds (Colloquium Publications) by Kai Cieliebak and Yakov Eliashberg

Introduction to the h-Principle (Graduate Studies in Mathematics) by Eliashberg and Mishachev

Partial Differential Relations by Mikhael Gromov

Topics on Algebraic Surfaces (E12)

Non-Examinable (Graduate Level)

M. Shen

Algebraic surfaces are two dimensional algebraic varieties. They form a bridge between the well established theory of algebraic curves and the mystery of higher dimensional algebraic varieties. After reviewing the basics of algebraic surfaces and the Enriques-Kodaira classification, we will focus on more specific topics. These include Hodge structures, Chow groups, Brauer groups and derived categories of algebraic surfaces. Special emphasis will be put on $K3$ surfaces.

Crystalline Cohomology and Applications (E16)

Non-Examinable (Graduate Level)

Nicolás Ojeda Bär

Crystalline cohomology was envisioned by A. Grothendieck as a way to understand p-adic aspects of the cohomology groups of algebraic varieties over fields of characteristic p (for which p-adic étale cohomology behaves somewhat mysteriously). In particular it was supposed to give information on the p-adic valuation of the zeros and poles of the Zeta function of such a variety. Subsequently, the study of the relationship between crystalline cohomology and p-adic étale cohomology (the problem of the "mysterious functor") gave birth to p-adic Hodge theory.

In this course we will construct crystalline cohomology and prove its basic theorems. We will also try to discuss a number of results and constructions that are crucial to the applications in p-adic Hodge theory.

- Definition of crystalline cohomology and fundamental theorems.
- Frobenius action, Berthelot-Ogus isomorphism and Cartier isomorphism.

- Mazur's theorem and Katz' conjecture.
- Grothendieck-Messing crystal and classification of p-divisible groups.

Pre-requisite Mathematics

- Basic scheme theory.
- Homological algebra: hypercohomology, spectral sequences, derived categories.

Literature

- A. Grothendieck, *Dix exposés sur la theorie des schémas,* North-Holland Publishing Company, Amsterdam, 1968.
- P. Berthelot, *Cohomologie cristalline des schémas de caracteristique* $p > 0$*,* Springer-Verlag, 1974.
- P. Berthelot, A. Ogus, *Notes on crystalline cohomology,* Princeton University Press, Princeton, 1978.
- B. Mazur, *Frobenius and the Hodge filtration,* Bull. of the Amer. Math. Soc., **78** (1972) and *Frobenius and the Hodge filtration (estimates),* Ann. of Math. (2), **98** (1973).

Logic

Category Theory (M24)

Dr Julia Goedecke

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories.

Locally small categories. The Yoneda lemma. Representations of functors.

Limits as terminal cones. Construction of limits from products and equalizers. Preservation and creation of limits.

Monomorphisms and Epimorphisms. Regular, split and strong mono- and epimorphisms.

Adjunctions. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. The Adjoint Functor Theorems.

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions.

Abelian categories. Kernels and cokernels. Additive categories. Image factorisation in abelian categories. Exact sequences, introduction to homological algebra.

Pre-requisite Mathematics

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. Mac Lane, S. *Categories for the Working Mathematician*, Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.

2. Awodey, S. *Category Theory*, Oxford U.P. 2006. A new treatment very much in the spirit of Mac Lane's classic, but rather more gently paced.

3. Borceux, F. *Handbook of Categorical Algebra*, Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

4. McLarty, C. *Elementary Categories, Elementary Toposes* (chapters 1–12 only), Oxford U.P. 1992. A very gently-paced introduction to categorical ideas, written by a philosopher for those with little mathematical background.

Topics In Set Theory (M24)

Dr Oren Kolman

Axiomatics The formal axiomatic system of ordinary set theory (ZFC). Models of set theory. Absoluteness. Simple independence results. Transfinite recursion. Ranks. Reflection principles. Constructibility. [4]
Infinitary combinatorics Cofinality. Stationary sets. Fodor's lemma. Solovay's theorem. Cardinal exponentiation. Beth and Gimel functions. Generalized Continuum Hypothesis. Singular Cardinals Hypothesis. Prediction principles (diamonds, squares, black boxes). Partial orders. Aronszajn and Suslin trees. Martin's Axiom. Suslin's Hypothesis. [6]
Forcing Generic extensions. The forcing theorems. Examples. Adding reals; collapsing cardinals. Introduction to iterated forcing. Internal forcing axioms. Proper forcing. [4]
Large cardinals Introduction to large cardinals. Ultrapowers. Scott's theorem. [2]
Partition relations and possible cofinality theory Partition relations. Model–theoretic methods. Ramsey's theorem; Erdős–Rado theorem. Kunen's theorem. Walks on ordinals. Todorcevic's theorem. Introduction to pcf theory. [4]
Applications Selection from algebra, analysis, geometry, and topology. [4]

Pre-requisite Mathematics

Logic and Set Theory is essential.

Literature

Basic material

- Drake, F. R., Singh, D., *Intermediate Set Theory*, John Wiley, Chichester, 1996.
- Eklof, P. C., Mekler, A. H., *Almost Free Modules*, rev. ed., North-Holland, Amsterdam, 2002.
- Halbeisen, L., *Combinatorial Set Theory With a Gentle Introduction to Forcing*, Springer, Berlin, 2012.
- Kanamori, A., *The Higher Infinite*, 2^{nd} ed., Springer, Berlin, 2009.
- Kunen, K., *Set Theory*, reprint, Studies in Logic, 34, College Publications, London, 2011.

Advanced topics

- Burke, M. R., Magidor, M., Shelah's pcf theory and its applications, Ann. Pure Appl. Logic 50 (1990), 207–254.
- Kanamori, A., Foreman, M., *Handbook of Set Theory*, Springer, Berlin, 2012.
- Shelah, S., *Proper and Improper Forcing*, 2^{nd} ed., Springer, Berlin, 1998. Chapters 1 and 2.
- Shelah, S. *Cardinal Arithmetic*, Oxford University Press, New York, 1994.
- Todorcevic, S., Combinatorial dichotomies in set theory, Bull. Symbolic Logic 17 (2011), 1–72.

Realizability and Topos Theory (L24, Graduate)

Non-Examinable (Graduate Level)

Prof. P.T. Johnstone

The idea that the collection of recursive (or computable) functions on the natural numbers encodes a kind of non-classical logic, the logic of recursive realizability, first appeared in a 1945 paper of S.C. Kleene. It was in the late 1970s that Martin Hyland first showed that this logic is actually the internal logic of a certain topos, and that similar toposes can be constructed from arbitrary Schönfinkel algebras (also called partial combinatory algebras). These toposes have properties very different from those of the Grothendieck toposes more familiar to topologists and geometers, but they can be studied by the same (essentially geometrical) techniques. My aim in this course is to develop the theory of realizability toposes from scratch, beginning with the basic theory of Schönfinkel algebras; if time permits, I shall also cover the topos-theoretic version of modified realizability, and the recent development of 'Herbrand realizability', due to Benno van den Berg.

Pre-requisite Mathematics

Part III Category Theory, or its equivalent, is essential. Some familiarity with the basic definitions of elementary topos theory is useful, but will not be assumed. Some slight knowledge of the theory of computable functions is also helpful, but this may be picked up from the simultaneous Part III course by Thomas Forster.

Literature

The only book on the subject is

J. van Oosten, *Realizability: An Introduction to its Categorical Side* (Elsevier, 2008).

However, for preliminary reading, I'd recommend Parts III and IV of

C. McLarty, *Elementary Categories, Elementary Toposes* (O.U.P., 1992),

particularly Chapter 24.

Computability and Logic (L24)

Dr Thomas Forster

The course is expanded to 24 lectures from the 16 lectures of 2010/11 and 2011/12, and it is not yet 100% clear what will be in. However the following picture should be pretty accurate.

Recursive datatypes. Structural and wellfounded induction. Finite state machines. Primitive Recursive functions. General Recursive functions. Turing Machines. Lambda-representable functions. Semidecidable and decidable sets. Unsolvability of the halting problem. Rice's theorem. Recursive inseparability and Tennenbaum's theorem. Automatic structures (automatic groups) and automatic theories. Recursive structures. Recursive ordinals and hierarchies of fast-growing functions. Axiomatisable and nonaxiomatisable theories. Trakhtenbrod's theorem. Incompleteness of arithmetic. Undecidability of Predicate calculus. Introductory Degree theory: Friedberg-Muchnik, Baker-Gill-Solovay.

Pre-requisite Mathematics

The course is designed to be the sequel to Part II Logic and Set Theory.

Literature

There are numerous textbooks with titles like this course, and I can't think of any that the prospective reader needs to be warned against. Two suitable books easily available locally with your student discount are:

G. Boolos and R. Jeffrey "Computability and Logic" CUP paperback

N Cutland "Computability" CUP paperback

Earlier editions of Boolos-and-Jeffrey are to be preferred to the latest version prepared by Burgess. Mendelson's *Introduction to Mathematical Logic* is a good general background.

Number Theory

Algebraic Number Theory (M24)

A J Scholl

In recent years one of the most growing areas of research in number theory has been Arithmetic Algebraic Geometry, in which the techniques of algebraic number theory and abstract algebraic geometry are applied to solve a wide range of deep number-theoretic problems. These include the celebrated proof of Fermat's Last Theorem, the Birch–Swinnerton-Dyer conjectures, the Langlands Programme and the study of special values of L-functions. In this course we will study one half of the picture: Algebraic Number Theory. I will assume some familiarity with the basic ideas of number fields, although these will be reviewed briefly at the beginning of the course. (The relevant algebra will also be found in the Commutative Algebra course.)

Topics likely to be covered:

Decomposition of primes in extensions, decomposition and inertia groups. Discriminant and different.

Completion, adeles and ideles, the idele class group. Application to class group and units.

Dedekind zeta function, analytic class number formula.

Class field theory (statements and applications). L–functions.

Pre-requisite Mathematics

A first course in number fields (or equivalent reading). Basic algebra up to and including Galois theory is essential.

Literature

1. J.W.S. Cassels and A. Fröhlich, Algebraic Number Theory. London Mathematical Society 2010 (2nd ed.)

2. A, Fröhlich, M.J. Taylor, Algebraic Number Theory. Cambridge, 1993.

3. J. Neukirch, Algebraic number theory. Springer, 1999.

Elliptic Curves (L24)

T.A. Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points. The first few lectures will include a review of the necessary geometric background (at the level of Chapters I and II of [2]). The following topics will be covered, and possibly others if time is available.

Weierstrass equations and the group law. Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process. Associativity via the identification with the Jacobian. Elliptic curves as group varieties.

Isogenies. Definition and examples. The degree of an isogeny is a quadratic form. The invariant differential and separability. Description of the torsion subgroup over an algebraically closed field.

Elliptic curves over finite fields. Hasse's theorem.

Elliptic curves over local fields. Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p, and the formal group of an elliptic curve. Singular Weierstrass equations.

Elliptic curves over number fields. The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Galois cohomology and Selmer groups. Descent by 2-isogeny. Numerical examples.

Pre-requisite Mathematics

Students should be familiar with the main ideas in the Part II courses *Galois Theory* and *Number Fields*. It would also be useful to have some rudimentary knowledge of algebraic curves and of the field of p-adic numbers.

Literature

1. J.W.S. Cassels, Lectures on Elliptic Curves, CUP, 1991.
2. J.H. Silverman, The Arithmetic of Elliptic Curves, Springer, 1986.
3. J.H. Silverman, J. Tate, Rational Points on Elliptic Curves, Springer, 1992.

Topics in analytic number theory (L24)

Bob Hough

L-functions are at the crossroads of several fields of study in modern number theory, with connections to arithmetic, harmonic analysis and representation theory. This course introduces the analytic theory of L-functions with a special emphasis on applications in the theory of quadratic forms.

Topics will be drawn from the following.

- *Class numbers.* Dirichlet L-functions, Dirichlet's class number formula and the distribution of class numbers of varying negative discriminant. Siegel's ineffective lower bound for the class number.
- *L-functions in the critical strip.* The approximate functional equation and Weil's explicit formula. Selberg's theorem on the log-normality of the Riemann zeta function on the half line. Applications to upper bounds for L-functions.
- *The class number one problem.* András Biró's solution of the class number one problem for two types of real quadratic fields. A brief introduction to modular forms, and the L-function of a modular form. The Goldfeld-Gross-Zagier effective solution of Gauss's class number one problem for imaginary quadratic fields.
- *Linnik's ergodic method.* The uniform distribution of shapes of ideals in imaginary quadratic fields.

Pre-requisite Mathematics

The class will be largely self-contained, but a working understanding of complex analysis at the level of Cauchy's residue theorem is required. Familiarity with the factorization of ideals in quadratic number fields is a bonus.

Literature

Davenport's book *Multiplicative Number Theory*, Springer 2000, is the standard reference. *Analytic Number Theory*, AMS 2004, by Iwaniec and Kowalski contains most of the advanced topics.

Complex Multiplication (L24)

Non-Examinable (Graduate Level)

T. Yoshida

The theory of complex multiplication of elliptic functions in 19c was one of the most important origins of modern algebraic number theory and algebraic geometry. Generalised to abelian varieties of arbitary dimension by Weil, Taniyama and Shimura, now we can think of it as a theory of 0-dimensional Shimura varieties. We try to cover the classical theory in a modern language (especially of schemes) and situate it in a larger context of Langlands correspondences.

- Review of global/local class field theory (including adeles/ideles)
- The Langlands correpondences for GL_1: ℓ-adic characters of Galois groups and Hecke characters (Grossencharacters)
- Abelian varieties over number fields, local fields and their rings of integers
- Main theorems of complex multiplication (as Shimura varieties for GU_1)

Pre-requisite Mathematics

- Algebraic number theory (number fields, local fields)
- Basic algebraic geometry (some familiarity with schemes, elliptic curves)

Literature

- J.P. Serre, *Complex Multiplication*, in: Cassels-Fröhlich, ed., *Algebraic Number Theory*, Academic Press, 1967.
- G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions* (Princeton, 1971), *Abelian Varieties with Complex Multiplication and Modular Functions* (Princeton, 1998).
- J.S. Milne, *Complex Multiplication*, available at `www.jmilne.org`.

Probability

Advanced Probability (M24)

Alan Sola

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

The main topics are as follows:

Review of measure and integration: sigma-algebras, measures, and filtrations; integrals and expectation; Fatou's lemma, monotone and dominated convergence; product measures, independence, and Fubini's theorem.

Conditional expectation: Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.

Martingales: Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.

Stochastic processes in continuous time: Kolmogorov's criterion, regularization of paths; martingales in continuous time.

Weak convergence: Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.

Sums of independent random variables: Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.

Brownian motion: Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.

Poisson random measures: Definitions, compound Poisson processes; Infinite divisibility, the Lévy-Khinchin formula, Lévy-Itô decomposition.

Prerequisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

Literature

- D. Applebaum, Lévy processes (2nd ed.), Cambridge University Press 2009.
- R. Durrett, Probability: Theory and Examples (4th ed.), CUP 2010.
- O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, 1997.
- D. Williams, Probability with martingales, CUP 1991.

Stochastic Calculus and Applications (L24)

Michael Tehranchi

This course is an introduction to the theory of continuous-time stochastic processes, with an emphasis on the central role played by Brownian motion. It complements the material in Advanced Probability, Advanced Financial Models, and Schramm–Loewner Evolutions.

- *Review of Brownian motion.* Wiener's existence theorem. Strong Markov property. Sample path properties.
- *Continuous stochastic calculus.* Adapted and previsible processes. Martingales and local martingales. Quadratic variation and co-variation processes. Itô's isometry and definition of stochastic integral. Itô's formula. Martingale inequalities.
- *Applications of Brownian motion.* Transience and recurrence. Martingale representation theorems. Lévy's characterization of Brownian motion. Dubins–Schwartz theorem. Dirichlet problem. Donsker's invariance principle. Girsanov's theorem.
- *Stochastic differential equations.* Strong and weak solutions. Notions of existence and uniqueness. Yamada–Watanabe theorem. Markov property. Feynmann–Kac partial differential equation. Ergodicity. The one-dimensional case.

Pre-requisite Mathematics

Knowledge of measure theoretic probability at the level of Part III Advanced Probability will be assumed, especially familiarity with discrete-time martingales and basic properties of Brownian motion.

Literature

1. I. Karatzas and S. Shreve. (1998) Brownian Motion and Stochastic Calculus. Springer.
2. D. Revuz and M. Yor. (2001) Continuous martingales and Brownian motion. Springer.
3. L.C. Rogers and D. Williams. (2002) Diffusions, Markov Processes and Martingales. Vol.1 and 2. Cambridge University Press.

Percolation and Related Topics (L16)

Geoffrey Grimmett

The percolation process is the simplest probabilistic model for a random medium in finite-dimensional space. It has a central role in the general theory of disordered systems arising in the mathematical sciences, and it has strong connections with statistical mechanics. Percolation has a reputation as a source of beautiful mathematical probems that are simple to state but seem to require new techniques for solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm–Loewner evolutions (SLE), and to the theory of phase transitions in physics.

The basic theory of percolation will be described in this course, with some emphasis on areas for future development. The fundamental techniques, including correlation and/or concentration inequalities and their ramifications, will be covered. The related topics may include self-avoiding walks, and further models from interacting particle systems, and certain physical models for the ferromagnet such as the Ising and Potts models.

Pre-requisite Mathematics

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature

The following texts will cover the majority of the course, and are available online.

Grimmett, G. R., *Probability on Graphs*, Cambridge University Press, 2010;
see `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`

Grimmett, G. R., *Three theorems in discrete random geometry*, Probability Surveys 8 (2011) 403–411, `http://arxiv.org/abs/1110.2395`

Schramm-Loewner Evolutions (L16)

Nathanaël Berestycki

Schramm-Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter $\kappa \geq 0$. These non-crossing curves are the fundamental tool used to describe the scaling limits of a host of natural probabilistic processes in two dimensions, such as critical percolation interfaces and random spanning trees. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory.

The course will focus on the definition and basic properties of SLE. The key ideas are conformal invariance and a certain spatial Markov property, which make it possible to use Itô calculus for the analysis. In particular we will show that, almost surely, for $\kappa \leq 4$ the curves are simple, for $4 \leq \kappa < 8$ they have double points but are non-crossing, and for $\kappa \geq 8$ they are space-filling. We will then explore the properties of the curves for a number of special values of κ (locality, restriction properties) which will allow us to relate the curves to other conformally invariant structures.

The fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed but recalled.

Literature

1. Nathanaël Berestycki and James Norris. Lecture notes on SLE.
`http://www.statslab.cam.ac.uk/~beresty/teach/SLE`
2. Wendelin Werner. *Random planar curves and Schramm-Loewner evolutions*, arXiv:math.PR/0303354, 2003.
3. Gregory F. Lawler. *Conformally Invariant Processes in the Plane*, AMS, 2005.

Lattice Models in Probability and Statistical Mechanics (M8)

Non-Examinable (Graduate Level)

Zhongyang Li

The course is about the phase transition exhibited in lattice models in statistical mechanics, focusing on the dimer model (perfect matching) and the Ising model. The planar dimer model is, from one point of view, a statistical mechanical model for random 2-dimensional interfaces in 3-dimensional space. In a concrete sense it is a natural generalization of the 1-dimensional simple random walk. While the simple random walk and its scaling limit, Brownian motion, permeate all of probability theory and many other parts of mathematics, higher dimensional models like the dimer model are much less understood. Only recently have tools been developed for gaining a mathematical understanding of two dimensional random fields. The dimer model is at the moment one of the most successful of these two dimensional theories. Applications of the techniques developed in the study of the dimer model lead to the solvability of the celebrated Lenz-Ising model.

The basic theory of the dimer model will be described in this course, with some emphasis on areas of future development. The fundamental techniques, including the Laplacian, Green's function and Gaussian measure, will be covered. Other related models include the Ising model, the general vertex model and the holographic algorithm.

Pre-requisite Mathematics

There are no essential pre-requisites beyond probability, linear algebra, and complex analysis at undergraduate levels.

Literature

1. R. Kenyon, an introduction to the dimer model, arXiv: math/0310326
2. R. Kenyon, A. Okounkov, S. Sheffield, Dimers and Amoeba, Annals of Mathematics Second Series, Vol. 163, No. 3 (May, 2006), pp. 1019-1056
3. B. M. McCoy, T. T. Wu, The two-dimensional Ising model, Harvard University Press, 1 Jan 1973

Concentration of Measure (E8 - 2 hours each lecture)

Non-Examinable (Graduate Level)

Nathanaël Berestycki and Richard Nickl

The concentration of measure phenomenon was first put forward in geometric functional analysis by Milman and Gromov, and has been subject to fascinating recent developments, particularly in probability theory. Roughly speaking, this phenomenon says that random variables in high or infinite-dimensional spaces tend to be "nearly constant". It can be quantified explicitly by so-called *concentration inequalities.*

The aim of this course is to investigate the basic mathematical principles behind the concentration of measure phenomenon. The arguments often rely on a combination of ideas from probability, geometry, analysis and statistics. This remarkable synthesis makes the subject both very elegant and powerful. We will then illustrate how to apply these results to some concrete examples. Topics to be covered include:

- Poincaré and isoperimetric inequalities; basic spectral geometry
- Entropy and Log-Soboloev inequalities
- Concentration of Gaussian measures (Borell's inequality)
- Talagrand's inequality and sharp concentration inequalities for product measures, including applications to empirical processes
- Sharp thresholds and the Kahn-Kalai-Linial theorem, including applications to first-passage percolation

Desirable Previous Knowledge

We shall only assume some basic notions of probability and measure theory. This being a non-examinable course, we plan to make this as informal and accessible as possible.

Lecture notes

A draft set of lecture notes is available at `http://www.statslab.cam.ac.uk/~beresty/teach/cm.html`.

A standard reference for part of this material is:

M. Ledoux (2001). *The concentration of measure phenomenon.* AMS monographs, Providence.

Statistics

Actuarial Statistics (M16)

S.M. Pitts

This course provides an introduction to various topics in non-life insurance mathematics. These topics feature mainly in the Institute and Faculty of Actuaries examination CT6.

Topics covered in lectures include

1. Loss distributions
2. Reinsurance
3. Aggregate claims
4. Ruin theory
5. Credibility theory
6. No claims discount systems

Pre-requisite Mathematics

This course assumes

an introductory probability course (including moment generating functions, probability generating functions, conditional expectations and variances)

a statistics course (including maximum likelihood estimation, Bayesian methods)

that you know what a Poisson process is

that you have met discrete time finite statespace Markov chains

Literature

1. S. Asmussen and H. Albrecher *Ruin Probabilities.* 2nd edition. World Scientific, 2010.
2. C.D. Daykin, T. Pentikäinen and E. Pesonen, *Practical Risk Theory for Actuaries and Insurers.* Chapman and Hall, 1993.
3. D.M. Dickson, *Insurance Risk and Ruin.* CUP, 2005.
4. J. Grandell, *Aspects of Risk Theory.* Springer, 1991.
5. T. Rolski, H. Schmidli, V. Schmidt and J. Teugels, *Stochastic Processes for Insurance and Finance.* Wiley, 1999.

Time Series and Monte Carlo Inference (2 units)

Time Series (M8)

Time Series and Monte Carlo Inference (2 units)

A. P. Dawid

The course consists of two components: Time Series and Monte Carlo Inference, each having 8 lectures. Together these make up one 2 unit (16 lecture) course. You must take the two components together for the examination.

Time series analysis refers to problems in which observations are collected at regular time intervals and there are correlations among successive observations. Applications cover virtually all areas of Statistics but some of the most important include economic and financial time series, and many areas of environmental or ecological data.

This course will cover some of the most important methods for dealing with these problems, including basic definitions of autocorrelations *etc.*, time-domain model fitting including autoregressive and moving average processes, and spectral methods.

Pre-requisite Mathematics

You should have attended introductory Probability and Statistics courses.

Literature

1. P. J. Brockwell and R. A. Davis, *Time Series: Theory and Methods.* Springer Series in Statistics (2006).
2. C. Chatfield, *The Analysis of Time Series: Theory and Practice.* Chapman and Hall (2004).
3. P. J. Diggle, *Time Series: A Biostatistical Introduction.* Oxford University Press (1990).
4. M. Kendall, *Time Series.* Charles Griffin (1976).

Monte Carlo Inference (L8)

Time Series and Monte Carlo Inference (2 units)

A. P. Dawid

The course consists of two components: Time Series and Monte Carlo Inference, each having 8 lectures. Together these make up one 2 unit (16 lecture) course. You must take the two components together for the examination.

Monte Carlo methods are concerned with the use of stochastic simulation techniques for statistical inference. These have had an enormous impact on statistical practice, especially Bayesian computation, over the last 20 years, due to the advent of modern computing architectures and programming languages. This course covers the theory underlying some of these methods and illustrates how they can be implemented and applied in practice.

The following topics will be covered: Techniques of random variable generation. Markov chain Monte Carlo (MCMC) methods for Bayesian inference. Gibbs sampling, Metropolis–Hastings algorithm, reversible jump MCMC.

Pre-requisite Mathematics

You should have attended introductory Probability and Statistics courses. A basic knowledge of Markov chains would be helpful. Prior familiarity with a statistical programming package such as `R` or `MATLAB` would also be useful.

Literature

1. J. E. Gentle, *Random Number Generation and Monte Carlo Methods* (Second Edition). Springer (2003).
2. B. D. Ripley, *Stochastic Simulation.* Wiley (1987).
3. D. Gamerman and H. F. Lopes, *Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference* (Second Edition). Chapman and Hall (2006).
4. C. P. Robert and G. Casella, *Monte Carlo Statistical Methods.* Springer (1999).

Applied Statistics (Michaelmas and Lent)

Robin Evans and Brian Tom

This is a *practical* course (3 units: 12 lectures and 12 classes) aiming to develop skills in analysis and interpretation of data, and communicating this in writing. Students are strongly encouraged to attend the course Statistical Theory for the theoretical background to the results used in the practical analysis of data.

The statistical methods listed below will be put into practice using `R`. In the practical classes, emphasis is placed on the importance of the clear presentation of the analysis, so that students are required to submit written solutions to the lecturer.

Syllabus

Michaelmas Term

Introduction to `Linux` and `R` on the Statistical Laboratory computing network. Use of LaTeX for report writing. Exploratory data analysis, graphical summaries.

Linear regression and its assumptions: relevant diagnostics: residuals, leverages, Q-Q plots, Cook's distances and related methods. Hypothesis tests for linear models, ANOVA, F-tests. Factors for categorical data. [3]

Dependent data, use of linear mixed effects models, restricted maximum likelihood. [2]

The essentials of generalized linear modelling. Discrete data analysis: binomial and Poisson regression. Multi-way contingency tables. [3]

Lent Term

Some special topics. Previous examples include generalized additive models, and longitudinal data analysis. [4]

Pre-requisite Mathematics

It is assumed that you will have done an introductory statistics course, including: elementary probability theory; maximum likelihood; hypothesis tests (t-tests, χ^2-tests, possibly F-tests); confidence intervals.

Literature

1. Dobson, A.J. (2002) *An Introduction to Generalized Linear Models.* Chapman & Hall/CRC. 2nd edition.

2. Agresti, A. (1990) *Categorical Data Analysis.* Wiley. 2nd edition.

3. McCullagh, P. and Nelder, J.A. (1989) *Generalized Linear Models.* Chapman & Hall. 2nd edition.

4. Venables, W.N. and Ripley, B.D. (2002) *Modern Applied Statistics with S.* Springer-Verlag. 4th edition.

5. Pawitan, Y. (2001) *In All Likelihood : Statistical Modelling and Inference Using Likelihood.* Oxford Science Publications.

Statistical Theory (M16)

Richard Samworth

This is a course on parametric statistical theory that goes hand in hand with the Lent term course on nonparametric statistical theory. We begin by reviewing briefly the classical methods and theory of inference based on the likelihood function. Although these methods are usually perfectly adequate for relatively low-dimensional models, they can fail badly in high-dimensions – in particular, when the dimension of the parameter space (usually denoted p) is larger than the number of observations, n. These 'large p, small n' problems occur in a very wide range of applications, from microarray experiments in biology to portfolio selection in finance, and are at the forefront of modern Statistics. We will outline some of the most important recent developments in this very active research area.

Classical theory: Review of linear models. Review of likelihood function and related quantities. Distribution theory in no nuisance parameter case. Generalised linear models. Traditional model selection methods (e.g. AIC). Basic results from measure theory and probability, such as modes of convergence, convergence theorems, differentiation under an integral, stochastic order notation. [5]

High dimensional problems: Shrinkage. Ridge regression. Cross-validation. Penalised likelihood. LASSO and associated theory (based on Karuch–Kuhn–Tucker and compatibility conditions) and algorithms. Other penalty functions, e.g. SCAD. Related problems, e.g. Group LASSO, Graphical LASSO, additive models. [7]

Multiple testing and other topics: Bonferroni correction. False discovery rate, Benjamini–Hochberg procedure. Storey's procedure. Other topics, e.g. Covariance matrix estimation, Low rank + sparse estimation. [4]

Pre-requisite Mathematics

Basic familiarity with statistical inference, including point estimation and hypothesis testing, will be assumed. Part IID Principles of Statistics is recommended as background. A small amount of measure theory and convex analysis/optimisation will be used in the course, though we will cover what we need as we go along.

Literature

1. L. Pace and A. Salvan, *Principles of Statistical Inference*, World Scientific (1997).
2. T. Hastie, R. Tibshirani and J. Friedman, *The Elements of Statistical Learning*, Springer (2009)
3. P. Bühlmann and S. van de Geer, *Statistics for High-Dimensional Data*, Springer (2011)

Biostatistics (3 units)

This course consists of two components: Survival Data and Statistics in Medical Practice. Together these make up one 3 unit (24 lecture) course. You must take both components together for the examination. Survival Data has 14 lectures; Statistics in Medical Practice has 10 lectures.

Survival Data (L14)

P. Treasure

Fundamentals of Survival Analysis:

Characteristics of survival data; censoring. Definition and properties of the survival function, hazard and integrated hazard. Examples.

Review of inference using likelihood. Estimation of survival function and hazard both parametrically and non-parametrically.

Explanatory variables: accelerated life and proportional hazards models. Special case of two groups. Model checking using residuals.

Current Topics in Survival Analysis:

In recent years there have been lectures on: frailty, cure, relative survival, empirical likelihood, counting processes and multiple events.

Principal book

1. D. R. Cox & D. Oakes, *Analysis of Survival Data,* London: Chapman & Hall (1984).

Other books

2. P. Armitage, J. N. S. Matthews & G. Berry, *Statistical Methods in Medical Research* (4th ed.), Oxford: Blackwell (2001) [Chapter on Survival Analysis for preliminary reading].
3. M. K. B. Parmar & D. Machin, *Survival Analysis: A Practical Approach* (1995), Chichester: John Wiley.

Statistics in Medical Practice (L10)

S. Bird, I. Whilte, R. Turner, L. Sharples, J. Wason & J. Bowden

Each lecture will be a self-contained study of a topic in biostatistics, which may include study design (including randomization and evaluation of interventions), meta-analysis, clinical trials and multi-state models. The relationship between the medical issue and the appropriate statistical theory will be illustrated.

Appropriate books

There are no appropriate books, but relevant medical papers will be made available beforehand for prior reading. It would be very useful to have some familiarity with media coverage of medical stories involving statistical issues, e.g. from Behind the Headlines on the NHS Choices website:
`http://www.nhs.uk/News/Pages/NewsIndex.aspx`

Nonparametric Statistical Theory (L16)

Adam Bull

In parametric statistics, it is assumed the data comes from a known finite-dimensional family of distributions. While that assumption is often convenient, it may not always be true; in this course, we will ask whether it is possible to do statistics without it. We will see that, in many cases, the standard maximum-likelihood approach fails, and we must instead use procedures designed specifically for nonparametric settings.

We will focus on the fundamental problems of estimating a distribution, density, or regression function, and describe techniques including empirical distribution functions, kernels, and wavelets. We will see that, while there are inherent limits to the nonparametric approach, we can nevertheless obtain some impressive results, and thereby perform statistics in much greater generality.

Distribution functions Basic empirical process theory, uniform laws of large numbers, Donsker and Kolmogorov-Smirnov theorems.

Minimax lower bounds Reduction to testing problems.

Approximation theory Convolution with kernels, series approximations, wavelets.

Density estimation and regression Kernel, local polynomial and wavelet estimators.

Choice of smoothing parameters Cross-validation, variable bandwidths, wavelet thresholding.

Pre-requisite Mathematics

Basic knowledge of probability, statistics and analysis is required. Measure theory and linear analysis are also useful, but the relevant material can be learnt as needed. This course complements the Michaelmas term course on Statistical Theory.

Literature

Complete notes will be available online; other relevant works include the following.

1. Tsybakov, A.B. *Introduction to Nonparametric Estimation*, Springer, 2009.
2. Van der Vaart, A.W. *Asymptotic Statistics*, Cambridge University Press, 1998.

Applied Bayesian Statistics (L11+5)

David Spiegelhalter

This course will count as a 2-unit (16 lecture) course. There will be 11 lectures and five practical classes.

- Bayes theorem; principles of Bayesian reasoning; probability as a subjective construct
- Exact conjugate analysis; exponential family; mixture priors
- Likelihood principle; alternative theories of inference
- Assessment of prior distributions; imaginary observations
- Monte Carlo analysis;
- Conditional independence; graphical models
- Markov chain Monte Carlo methods; convergence
- Regression analysis (linear, GLM, nonlinear)
- Model criticism and comparison; Bayesian P-values; information criteria
- Hierarchical models (GLMMs)

The practical classes will use WinBUGS.

Pre-requisite Mathematics

This course assumes that students have a working knowledge of non-Bayesian applied statistics, such as the Applied Statistics course. It will be helpful but not essential to attend the Monte Carlo Inference course. Full familiarity with properties and manipulations of probability distributions will be assumed, including marginalisation, change of variable, Fisher information, iterated expectation, conditional independence, and so on.

Literature

1. Lunn, D., Jackson, C., Best, N.G., Thomas, A. and Spiegelhalter, D.J. (2012) *The BUGS Book: A Practical Introduction to Bayesian Analysis.* Chapman and Hall.
2. Gelman A., Carlin, J.B., Stern, H.S., and Rubin, D.B. (2003) *Bayesian Data Analysis.* 2nd Edition.Chapman and Hall.

Design of Experiments (L16)

R. A. Bailey

This course is about how to design real experiments, and includes issues about statistical consultancy as well as the necessary mathematics. It includes enough about the analysis of data from an experiment to show what we need to think about when designing the experiment. The following topics will be covered.

- The problem of deciding how to allocate treatments to experimental units.
- Bias, variance, blocking and randomization.
- Linear model and analysis of variance.
- Factorial treatments: main effects and interactions.
- Complete-block designs, row-column designs, split-plot designs, false replication.
- General theory of orthogonal designs, including Hasse diagrams for factors, null analysis of variance and skeleton analysis of variance.
- Incomplete-block designs.
- Fractional factorial designs.

Pre-requisite Mathematics

Introductory statistics, including estimation, bias and variance. Introductory probability, including the normal, χ^2, t- and F-distributions. Introductory linear algebra over the real numbers, including the eigespaces of real symmetric matrices and orthogonal projection onto subspaces. Arithmetic in the integers modulo n.

Literature

Main Text: *Design of Comparative Experiments* by R. A. Bailey, CUP, 2008.

Other texts: *Planning of Experiments* by D. R. Cox, Wiley, 1958; *Design and Analysis of Experiments* by George W. Cobb, Springer, 1998; *Experimental Designs* by W. G. Cochran and G. M. Cox, Wiley, 1957.

Contemporary sampling techniques and compressed sensing (L24)

Anders Hansen

This is a graduate course on sampling theory and compressed sensing for use in signal processing and medical imaging. Compressed sensing is a theory of randomisation, sparsity and non-linear optimisation techniques that breaks traditional barriers in sampling theory. Since its introduction in 2004 the field has exploded and is rapidly growing and changing. Thus, we will take the word contemporary quite literally and emphasise the latest developments, however, no previous knowledge of the field is assumed. Although the main focus will be on compressed sensing, it will be presented in the general framework of sampling theory. The course will also present related areas of sampling theory such as generalised sampling and sampling at a finite rate of innovation.

Although the course will be rather mathematical, it will be fairly self contained, and applications will be emphasised (in particular, signal processing and Magnetic Resonance Imaging (MRI)). Students from other disciplines than mathematics are encouraged to participate.

Desirable Previous Knowledge

Sampling theory and compressed sensing require a mix of mathematical tools from approximation theory, harmonic analysis, linear algebra, functional analysis, optimisation and probability theory. The course will contain discussions of both finite-dimensional and infinite-dimensional/analog signal models and thus linear algebra, Fourier analysis and functional analysis (at least basic Hilbert space theory) are important. The course will be self-contained, but students are encouraged to refresh their memories on properties of the Fourier transform as well as basic Hilbert space theory. Some basic knowledge of wavelets is useful as well as very basic probability (Bernstein's inequality, Hoeffding's inequality).

Introductory Reading

For a quick and dense review of basic Fourier analysis and functional analysis chapters 5 and 8 of "Real Analysis" (Folland) are good choices. For an introductory exposition to Hilbert space theory one may use "An Introduction to Hilbert Space" (Young). And for a review of wavelets see chapters 1 and 2 of "A First Course on Wavelets" (Hernandez, Weiss). The course will cover some of the chapters of "Compressed Sensing" (Eldar, Kutyniok), so to get a feeling about the topic one may consult chapter 1 as a start.

1. Eldar, Y and Kutyniok, G., Compressed Sensing, CUP
2. Folland, G. B., Real Analysis, Wiley.
3. Hernandez, E. and Weiss, G., A First Course on Wavelets, CRC
4. Young, N., An Introduction to Hilbert Space, CUP

Reading to complement course material

1. Adcock, B and Hansen, A., Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon, Appl. Comp. Harm. Anal., 32 (2012)
2. Candès, E., Romberg, J. and Tao, T., Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory 52 (2006)
3. Donoho, D., Compressed sensing, IEEE Trans. Inform. Theory 52 (2006)
4. Körner, T. W., Fourier Analysis, CUP
5. Reed, M. and Simon, B., Functional Analysis, Elsevier

Operational Research and Mathematical Finance

Stochastic Networks (M24)

Frank Kelly

This course uses stochastic models to shed light on important issues in the design and control of communication networks. Randomness arises in communication systems at many levels: for example, the initiation and termination times of calls in a telephone network, or the statistical structure of the arrival streams of packets at routers in the Internet. How can routing, flow control and connection acceptance algorithms be designed to work well in uncertain and random environments?

The first two parts of the course will describe a variety of classical models that can be used to help understand the performance of large-scale communication networks. Queueing and loss networks will be studied, as well as random access schemes. Parallels will be drawn with models from physics, and with models of traffic in road networks.

The third part of the course will more recently developed models of packet traffic and of congestion control algorithms in the Internet. This is an area of some practical importance, with network operators, hardware and software vendors, and regulators actively seeking ways of delivering new services reliably and effectively. The complex interplay between end-systems and the network has attracted the attention of economists as well as mathematicians and engineers.

Desirable previous knowledge

Mathematics that will be assumed to be known before the start of the course: Part IB Optimization and Markov Chains. Familiarity with Part II Applied Probability would be useful, but is not assumed.

Introductory reading

A feeling for some of the ideas of the course can be taken from

The mathematics of traffic in networks. In *Princeton Companion to Mathematics* (Edited by Timothy Gowers; June Barrow-Green and Imre Leader, associate editors) Princeton University Press, 2008. 862-870.

Literature

1. B. Hajek *Communication Network Analysis.*

2. P. Robert *Stochastic Networks and Queues.* Springer-Verlag, 2003. Chapter 4.

3. H. Chen and D.D. Yao *Fundamentals of Queueing Networks.* Springer-Verlag, 2001.

4. S. Asmussen *Applied Probability and Queues* - second edition. Springer-Verlag, 2003.

5. R. Srikant *The Mathematics of Internet Congestion Control.* Birkhauser, 2004.

6. S. Shakkottai and R. Srikant *Network Optimization and Control.* Foundations and Trends in Networking, NoW Publishers, 2007.

Mathematics of Operational Research (M24)

F. Fischer

This course is accessible to a candidate with mathematical maturity who has no previous experience of operational research; however it is expected that most candidates will already have had exposure to some of the topics listed below.

- Lagrangian sufficiency theorem. Lagrange duality. Supporting hyperplane theorem. Sufficient conditions for convexity of the optimal value function. Fundamentals of linear programming. Linear program duality. Shadow prices. Complementary slackness. [2]
- Simplex algorithm. Two-phase method. Dual simplex algorithm. Gomory's cutting plane method. [3]
- Complexity of algorithms. NP-completeness. Exponential complexity of the simplex algorithm. Polynomial time algorithms for linear programming. [2]
- Network simplex algorithm. Transportation and assignment problems, Ford-Fulkerson algorithm, max-flow/min-cut theorem. Shortest paths, Bellman-Ford algorithm, Dijkstra's algorithm. Minimum spanning trees, Prim's algorithm. MAX CUT, semidefinite programming, interior point methods. [5]
- Branch and bound. Dakin's method. Exact, approximate, and heuristic methods for the travelling salesman problem. [3]
- Cooperative and non-cooperative games. Two-player zero-sum games. Existence and computation of Nash equilibria, Lemke-Howson algorithm. Bargaining. Coalitional games, core, nucleolus, Shapley value. Mechanism design, Arrow's theorem, Gibbard-Satterthwaite theorem, VCG mechanisms. Auctions, revenue equivalence, optimal auctions. [9]

Books

1. M.S. Bazaraa, J.J. Jarvis and H.D. Sherali: Linear Programming and Network Flows, Wiley (1988).
2. D. Bertsimas, J.N. Tsitsiklis. Introduction to Linear Optimization. Athena Scientific (1997).
3. N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani. Algorithmic Game Theory. Cambridge University Press (2007).
4. M. Osborne, A. Rubinstein: A Course in Game Theory. MIT Press (1994).

Advanced Financial Models (M24)

Michael Tehranchi

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability, Stochastic Calculus and Applications, and Optimal Investment.

- **Discrete time models**. Filtrations and martingales. Arbitrage, state price densities and equivalent martingale measures. Attainable claims and market completeness. European and American claims. Optimal stopping.
- **Brownian motion and stochastic calculus.** Brief survey of stochastic integration. Girsanov's theorem. Itô's formula. Martingale representation theorem.
- **Continuous time models**. Admissible strategies. Pricing and hedging in Markovian models. The Black–Scholes model. Local and stochastic volatility models.
- **Interest rate models.** Short rates, forward rates, and bond prices. Markovian short rate models. The Heath–Jarrow–Morton drift condition.

Pre-requisite Mathematics

A knowledge of probability theory at the level of Part II Probability and Measure will be assumed. Familiarity with Part II Stochastic Financial Models is helpful.

Literature

Lecture notes will be distributed. Additionally, the following books may be helpful.

1. M. Baxter & A. Rennie. (1996) Financial calculus: an introduction to derivative pricing. Cambridge University Press
2. M. Musiela and M. Rutkowski. (2006) Martingale Methods in Financial Modelling. Springer.
3. D. Kennedy. (2010) Stochasic Financial models. Chapman & Hall
4. Lamberton, D. & B. Lapeyre. (1996) Introduction to stochastic calculus applied to finance. Chapman & Hall
5. S. Shreve. (2005) Stochastic Calculus for Finance: Vol. 1 and 2. Springer-Finance

Optimal Investment (L16)

L C G Rogers

The course will study a wide range of optimal investment/consumption problems that arise in theory and practice, and will discuss the usefulness of the conclusions. Most examples studied will be in a continuous-time setting. The following provisional list of topics indicates some of the intended content; not all the topics on this list will necessarily be covered, and topics may be covered that are not on this list.

- Self-financing portfolios and the wealth equation;
- the Merton problem and its solution in the CRRA case, using the Hamilton-Jacobi-Bellman approach;
- the Merton problem, general case, by martingale representation;
- the Merton problem, general case, using state-price density approach;
- (Davis-Varaiya) martingale principle of optimal control;
- dual methodology and the Pontryagin principle;
- equilibrium pricing;
- the equity premium puzzle;
- utility-indifference pricing;
- Lagrangian martingale characterisation of optimal solutions;
- imperfections: transaction costs, portfolio constraints, parameter uncertainty, infrequent rebalancing;
- varied objectives: ratcheting of consumption, habit formation, recursive utility;
- backward SDEs and optimal control;
- How good are any of these rules in practice?

Pre-requisites

A firm grasp of martingale theory, and a working knowledge of (Brownian) stochastic calculus will be required in the course.

Literature

1. I. Karatzas & S. E. Shreve: *Methods of Mathematical Finance*, Springer, 1998.

2. L. C. G. Rogers: *Optimal Investment*, Springer, 2012?

Particle Physics, Quantum Fields and Strings

The courses on Symmetry and Particles, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of Symmetry and Particles.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, 'Golden Rule' for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year	*Courses*	
Second	*Essential*:	Quantum Mechanics, Special Relativity, Methods, Complex Methods.
	Helpful:	Electromagnetism.
Third	*Essential*:	Principles of Quantum Mechanics, Classical Dynamics.
	Very helpful:	Applications of Quantum Mechanics, Statistical Physics, Electrodynamics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Quantum Field Theory (M24)

Professor A. C. Davis

Quantum Field Theory is the language in which all of modern physics is formulated. It represents the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using the Lagrangian language and Noether's theorem.

The quantisation of the basic non-interacting free fields is developed in terms of operators which create and annihilate particles and anti-particles and the associated Fock space of quantum physical states is explained.

Interactions are introduced using perturbative techniques and the role of Feynman diagrams is explained. This is first illustrated for theories with a purely scalar field interaction, and then for a Yukawa coupling between scalar fields and fermions. Finally Quantum Electrodynamics, the theory of interacting photons, electrons and positrons, is introduced and elementary scattering processed are computed.

Necessary Previous Knowledge

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Books

1. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996).
2. L.H. Ryder, *Quantum Field Theory*, Cambridge University Press (1996).
3. S. Weinberg, *The Quantum Theory of Fields Vol I*, Cambridge University Press (1995)
4. A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, (2003)

Symmetries, Fields and Particles (M24)

N. S. Manton

The course provides introductory material on properties of groups and their applications to particle physics. In particular Lie groups and Lie algebras, and their representations in terms of matrices, are discussed in some detail. The examples of SU(2) and SU(3) play a particularly important rôle. The observed elementary particles and their interactions are first reviewed. The particles are divided into hadrons and leptons. Hadrons are characterised by having strong interactions and are composite particles made up of quarks. Hadrons and leptons also undergo electromagnetic and weak interactions, which are responsible for their decay. Various quantum numbers which distinguish particles and their interactions are described.

Basic properties of Lie groups and Lie algebras are introduced. The geometric structures underlying the theory of Lie groups and algebras are presented, and properties of representations explained. Various aspects of angular momentum, which is associated with the group SU(2), are derived. Similarly, properties of SU(3) are examined in some detail. This is applied to hadrons, which are classified in terms of representations of global symmetry groups such as SU(2) and SU(3). This motivates the description of their states in terms of quarks. The role of Lie groups in understanding spacetime symmetry is also discussed and the Lorentz group and Poincaré group are described. This leads to consideration of Dirac matrices and spinors. Lie groups also play an essential role in non-abelian gauge theories. Local symmetry requires vector gauge fields, and the corresponding quantum field theories are the basis of modern particle physics. The weak and electromagnetic interactions are described by the "electro-weak" theory with gauge group $SU(2) \times U(1)$, while the strong interaction is described by "QCD", which has the gauge group SU(3). At the end of the course, the systematic classification of simple Lie algebras is reviewed. The notion of roots and weights and their properties are explained and their important role in the classification of representations is presented. The emphasis here is on the essential ideas rather than mathematical proofs.

Desirable Previous Knowledge

Familiarity with the treatment of angular momentum in quantum mechanics, the states $|jm\rangle$ and the action of the operators J_3, $J_\pm$ on them. Understanding how products of angular momentum states can be combined using Clebsch-Gordan coefficients is also important.

Introductory Reading

1. Perkins, D.H., Introduction to High energy Physics, 4th ed. CUP (2000).

Reading to complement course material

1. Georgi, H., Lie Algebras in Particle Physics, Westview Press (1999).
2. Fuchs, J, and Schweigert C., Symmetries, Lie Algebras and Representations, 2nd ed. CUP (2003).
3. Jones H.F., Groups, Representations and Physics, 2nd ed., IOP Publishing (1998).

Supersymmetry (L16)

B.C. Allanach

This course provides an introduction to the use of supersymmetry in quantum field theory. Supersymmetry combines commuting and anti-commuting dynamical variables and relates fermions and bosons.

Firstly, a physical motivation for supersymmetry is provided. The supersymmetry algebra and representations are then introduced, followed by superfields and superspace. 4-dimensional supersymmetric Lagrangians are then discussed, along with the basics of supersymmetry breaking. The minimal supersymmetric standard model will be introduced.

Three examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries in Particle Physics courses, or be familiar with the material covered in them.

Introductory Reading

1. The first chapters of `http://arxiv.org/abs/hep-ph/0505105`

Reading to complement course material

For more advanced topics later in the course, it will helpful to have a knowledge of renormalisation, as provided by the Advanced Quantum Field Theory course. It may also be helpful (but not essential) to be familiar with the structure of The Standard Model in order to understand the final lecture on the minimal supersymmetric standard model.

Beware: most of the supersymmetry references contain errors in minus signs, aside (as far as I know) Wess and Bagger.

1. Course lecture notes from last year: `http://www.damtp.cam.ac.uk/user/examples/3P7.pdf`
2. Videos of a very similar lecture course: follow the links from `http://users.hepforge.org/∼allanach/teaching.html`
3. Supersymmetric Gauge Field Theory and String Theory, Bailin and Love, IoP Publishing (1994) has nice explanations of the physics. An erratum can be found at `http://www.phys.susx.ac.uk/∼mpfg9/susyerta.htm`
4. Introduction to supersymmetry, J.D. Lykken, `hep-th/9612114`. This introduction is good for extended supersymmetry and more formal aspects.
5. Supersymmetry and Supergravity, Wess and Bagger, Princeton University Press (1992). Note that this terse and more mathematical book has the opposite sign of metric to the course.

6. A supersymmetry primer, S.P. Martin, `hep-ph/9709256` is good and detailed for phenomenological aspects, although with the opposite sign metric to the course.

Advanced Quantum Field Theory (L24)

H. Osborn

Quantum field theory (QFT) is the basic theoretical framework for describing elementary particles and their interactions (excluding gravity) and is essential in the understanding of string theory. It is also used in many other areas of physics including condensed matter physics, astrophysics, nuclear physics and cosmology. The Standard Model, which describes the basic interactions of particle physics, is a particular type of QFT known as a gauge theory. Gauge theories are invariant under symmetry transformations defined at each point in spacetime which form a Lie Group under composition. To quantise a gauge theory, it is necessary to eliminate non-physical degrees of freedom and this requires additional theoretical tools beyond those developed in the introductory quantum field theory course.

A variety of new concepts and methods are first introduced in the simpler context of scalar field theory. The functional integral approach provides a formal non-perturbative definition of any QFT which also reproduces the usual Feynman rules. The course discusses in a systematic fashion the treatment of the divergences which arise in perturbative calculations. The need for regularisation in QFT is explained, and the utility of dimensional regularisation in particular is emphasised. It is shown how renormalisation introduces an arbitrary mass scale and renormalisation group equations which reflect this arbitrariness are derived. Various physical implications are then discussed.

The rest of the course is concerned specifically with gauge theories. The peculiar difficulties of quantising gauge fields are considered, before showing how these can be overcome using the functional integral approach in conjunction with ghost fields and BRST symmetry. A renormalisation group analysis reveals that the coupling constant of a quantum gauge theory can become effectively small at high energies. This is the phenomenon of asymptotic freedom, which is crucial for the understanding of QCD: the gauge theory of the strong interactions. It is then possible to perform perturbative calculations which may be compared with experiment. Further properties of gauge theories are discussed, including the possibility that a classical symmetry may be broken by quantum effects, and how these can be analysed in perturbation theory. Such anomalies have important implications for the way in which gauge particles and fermions interact in the Standard Model.

Desirable Previous Knowledge

Knowledge of the Michaelmas term course "Quantum Field Theory" will be assumed. Familiarity with the content of "Symmetry and Particle Physics" (M24) would be very helpful.

Introductory Reading

1. Section 9.1 and Chapter 15 of An Introduction to Quantum Field Theory, Peskin M E and Schroeder D V (Addison-Wesley 1996)

Reading to complement course material

1. Quantum Field Theory, Ryder L H (2nd edn CUP 1996)
2. An Introduction to Quantum Field Theory, Peskin M E and Schroeder D V (Addison-Wesley 1996)
3. Quantum Theory of Fields, Vols. 1 &; 2, Weinberg S (CUP 1996)

Standard Model (L24)

M.B. Wingate

The Standard Model of particle physics is, by far, the most successful application of quantum field theory. As this booklet goes to press, this model accurately describes all elementary particle physics measurements involving strong, weak, and electromagnetic interactions.

The Standard Model is the quantum theory of the gauge group $SU(3) \times SU(2) \times U(1)$ with fermion fields for the leptons and quarks. The course aims to demonstrate how this model is realised in nature. It is intended to complement the more general Advanced QFT course.

This course begins by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content in terms of spin 1/2 leptons and quarks and also the spin 1 gauge bosons. The parity P, charge conjugation C and time-reversal T transformation properties of the theory are investigated. These need not be symmetries manifest in Nature; e.g. only left-handed particles feel the weak force in violation of parity symmetry. We show how CP violation becomes possible when there are three generations of particles.

Ideas of spontaneous symmetry breaking are applied to discuss the Higgs Mechanism; the weakness of the weak force is due to the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry. The recent measurements of what appear to be Higgs boson decays will be presented.

We show how to obtain cross-sections and decay rates from the matrix element squared of a process. Various scattering and decay processes can be calculated in the electroweak sector using perturbation theory because of the smallness of the couplings. We touch upon the topic of neutrino masses and oscillations, an important window into physics beyond the Standard Model.

The strong interactions are based upon the gauge theory with (unbroken) gauge group $SU(3)$, called quantum chromodynamics (QCD). At low energies quarks are confined, forming bound states called hadrons. In such a non-abelian theory, the coupling constant decreases in higher energy processes to the point where perturbation theory can be used. As an example we consider electron-positron annihilation to final state hadrons at high energies. Nonperturbatively, progress can be made in the limits of very small and very large quark masses, making use of chiral and heavy quark symmetries.

Throughout the course, we touch upon open questions. Very high energy experiments and very precise experiments are currently striving to observe effects not describable by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model.

Examples sheets and examples classes complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries in Particle Physics courses, or be familiar with the material covered in them. It is advantageous to attend the Advanced Quantum Field Theory course during the same term as attending this course, or to study renormalisation and non-abelian gauge fixing.

Reading to complement course material

1. M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley (1996).
2. T-P. Cheng and L-F. Li, Gauge Theory of Elementary Particle Physics, Oxford University Press (1984).
3. J.F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model, Cambridge University Press (1994).
4. I.J.R. Aitchison and A.J.G. Hey, Gauge Theories in Particle Physics, IoP Publishing (1989).

5. F. Halzen and A.D. Martin, Quarks and Leptons: An Introductory Course in Modern Particle Physics, John Wiley and Sons (1984).
6. A.V. Manohar and M.B. Wise, Heavy quark physics, Cambridge University Press (2000).

String Theory (L24)

P. K. Townsend

String theory is an ambitious project. It purports to be an all-encompassing theory of the universe, unifying the forces of nature, including gravity, in a single quantum mechanical framework.

While string theory is often paraded as the ultimate theory of everything, a less trumpeted facet is the way in which the theory reveals insights and connections between other, seemingly unrelated, aspects of physics. Much of modern research in theoretical physics uses string theory as a tool to understand more down-to-earth physical systems, most notably strongly coupled quantum field theories.

This course will provide an introduction to various topics in string theory. We aim to cover how to construct both the bosonic string and the superstring and examine the spectrum of states in these theories. We then aim to look at some simple calculations of scaterring amplitudes. If time permits, we will then move on and examine some more interesting topics: how it is that string theory encompasses general relativity, D-brane technology and how one starts to construct models of elementary particles starting from string theory.

Desirable Previous Knowledge

You will need knowledge of the material from the Quantum Field Theory course and, as it progresses, the Advanced Quantum Field Theory course. You should be comfortable with general relativity.

Reading to complement course material

1. M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory," Cambridge University Press (1987).
2. E. Kiritsis, "String Theory in a Nutshell," Princeton University Press, (2007).
3. K. Becker, M. Becker and J.H. Schwarz, "String Theory and M-Theory: A Modern Introduction," Cambridge University Press, (2007).

Classical and Quantum Solitons. (E 16)

N. Dorey

Solitons are solutions of the classical field equations with particle-like properties. In particular, they are localised in space, have finite energy and are stable against decay into radiation. After quantisation, they give rise to new particle states which are typically very massive at weak coupling but can become light at strong coupling. Solitons play a key role in many recent advances in field theory and string theory, especially in the phenomenon of duality which relates the strong-coupling behaviour of one theory to the weak-coupling behaviour of another. In this course we will study the properties of classical solitons and their quantum counterparts. We will focus mainly on the case of integrable theories in two dimensional spacetime where an exact analytic description is possible.

Desirable Previous Knowledge

Quantum Field Theory. Advanced Quantum Field Theory.

Introductory Reading

1. Topological Solitons, N. Manton and P. Sutcliffe (CUP 2004), Chapters 1, 4 and 5

Relativity and Gravitation

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Year	*Courses*	
Second	*Essential*:	Methods, Special Relativity, Principles of Dynamics, Quantum Mechanics.
	Helpful:	Electromagnetism, Geometry, Complex Methods.
Third	*Very helpful*:	General Relativity, Statistical Physics, Electrodynamics, Methods of Mathematical Physics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Cosmology (M24)

Daniel Baumann

Cosmology has become a precision science. The basic Big Bang picture provides quantitative explanations for the expansion of the universe, the origin of the cosmic microwave background radiation, the synthesis of light chemical elements and the formation of stars, galaxies and large-scale structures. Moreover, there is growing evidence that all of the large-scale structures we see around us originated from microscopic quantum fluctuations, stretched to cosmic sizes during a period of inflationary expansion. However, there are still important gaps in our understanding, including the nature of the dark matter, the cause of the observed late-time acceleration of the universe, the classic puzzle of the initial singularity and the physical origin of inflation.

This course will develop the standard Big Bang cosmology and review its major successes and some of the challenges now faced at the cutting-edge of the field. We will emphasize the point of view that cosmology provides some of the best tests of modern ideas in particle physics.

Course website: www.damtp.cam.ac.uk/user/db275/Cosmology/

A tentative syllabus is the following:

Part I: The Homogeneous Universe

- ▷ Geometry and Dynamics
- ▷ Thermal History
- ▷ Inflation

Part II: The Inhomogeneous Universe

▷ Cosmological Perturbation Theory

▷ Initial Conditions from Inflation

▷ Structure Formation

▷ Cosmic Microwave Background

Desirable Previous Knowledge

Basic knowledge of relativity, particle physics and statistical mechanics will be helpful. However, the course will be presented in a self-contained way, so students with less experience in any of these fields should have no problem to catch up.

Introductory Reading

1. Weinberg, *The First Three Minutes*.
2. Overbye, *Lonely Hearts of the Cosmos*.

Reading to complement course material

1. Dodelson, *Modern Cosmology*.
2. Kolb and Turner, *The Early Universe*.
3. Weinberg, *Cosmology*.
4. Mukhanov, *Physical Foundations of Cosmology*.
5. Peter and Uzan, *Primordial Cosmology*.

General Relativity (M24)

H.S. Reall

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.

Course website: `www.damtp.cam.ac.uk/user/hsr1000`

Desirable Previous Knowledge

This course will be self-contained, so previous knowledge of General Relativity is not essential. However, many students have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before then it is strongly recommended that you study an introductory book (e.g. Hartle or Rindler) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Familiarity with Newtonian gravity and special relativity is essential. Knowledge of the relativistic formulation of electrodynamics is desirable. Familiarity with finite-dimensional vector spaces, the calculus of functions $f : R^m \to R^n$, and the Euler-Lagrange equations will be assumed.

Introductory Reading

1. Gravity: An introduction to Einstein's General Relativity, J.B. Hartle, Addison-Wesley, 2003.
2. Relativity: Special, General, and Cosmological, 2nd ed., W. Rindler, OUP, 2006.

Reading to complement course material

There are many excellent books on General Relativity. The following is an incomplete list:

1. General Relativity, R.M. Wald, Chicago UP, 1984.
2. Spacetime and geometry: an introduction to General Relativity, S.M. Carroll, Addison-Wesley, 2004.
3. Advanced General Relativity, J.M. Stewart, CUP, 1993.
4. Gravitation and Cosmology, S. Weinberg, Wiley, 1972.

Our approach will be closest to that of Wald. Carroll's book is a very readable introduction. Stewart's book is based on a previous version of this course. Weinberg's book gives a good discussion of the Equivalence Principle.

Applications of Differential Geometry to Physics. (L16)

Maciej Dunajski

This is a course designed to develop the Differential Geometry required to follow modern developments in Theoretical Physics. The following topics will be discussed.

- Differential Forms and Vector Fields.
  1. One parameter groups of transformations.
  2. Vector fields and Lie brackets.
  3. Exterior algebra.
  4. Hodge Duality.
- Geometry of Lie Groups.
  1. Group actions on manifolds.
  2. Homogeneous spaces and Kaluza Klein theories.
  3. Metrics on Lie Groups.
- Fibre bundles and instantons.
  1. Principal bundles and vector bundles.
  2. Connection and Curvature.
  3. Twistor space.

Desirable Previous Knowledge

Basic General Relativity (Part II level) or some introductory Differential Geometry course (e.g. Part II differential geometry) is essential. Part III General Relativity is desirable.

Reading to complement course material

1. `http://www.damtp.cam.ac.uk/research/gr/members/gibbons/gwgPartIII_DGeometry2011-1.pdf`
2. Flanders, H. Differential Forms. Dover
3. Dubrovin, B., Novikov, S. and Fomenko, A. Modern Geometry. Springer
4. Eguchi, T., Gilkey, P. and Hanson. A. J. Physics Reports 66 (1980) 213-393
5. Arnold. V. Mathematical Methods of Classical Mechanics. Springer.
6. Dunajski. M. Solitons, Instantons and Twistors. OUP.

Black Holes (L24)

G. W. Gibbons

A black hole is a region of space-time that is causally disconnected from the rest of the Universe. The study of black holes reveals many surprising and beautiful properties, and has profound consequences for quantum theory. The following topics will be discussed:

1. Gravitational collapse. Why black holes necessarily form under certain circumstances.
2. Causal structure, asymptotic flatness, Penrose diagrams, the event horizon.
3. Exact black hole solutions: Schwarzschild, Reissner-Nordstrom and Kerr solutions.
4. Energy, angular momentum and charge. The positive energy theorem.
5. The laws of black hole mechanics. The analogy with laws of thermodynamics.
6. The Hawking effect. Black hole evaporation, the information paradox.

Examples sheets will be distributed during the course. Examples classes will be held to discuss these.

Desirable Previous Knowledge

Familiarity with the contents of the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Introductory Reading

1. R.M. Wald, *General Relativity* (University of Chicago Press, 1984), Chapter 6.

Reading to complement course material

1. P.K. Townsend, *Black holes: lecture notes*, arXiv:gr-qc/9707012.
2. R.M. Wald, *General relativity*, University of Chicago Press, 1984.
3. S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
4. *Spacetime and Geometry*, S.M. Carroll, Addison Wesley, 2004 (an earlier draft of this book is available online at pancake.uchicaco.edu/∼carroll/notes)
5. N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
6. V.P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.

Applications of General Relativity (L16)

Non-Examinable (Graduate Level)

Irena Borzym

This graduate course will introduce some useful mathematical tools and discuss some of the most important applications of these tools in general relativity. The lecture notes will be accompanied by exercises.

An outline of the course is as follows.

1. Two component spinors, their algebra and interpretation.
2. Translating between spinor and the more usual spacetime tensorial formulations.
3. Petrov Classification.
4. A brief introduction to NP formalism.
5. Shear and focusing.
6. Goldberg-Sachs.
7. Plane and Plane fronted waves.
8. Asymptopia for Minkowski space.
9. Asymptotic simplicity.
10. Conformal transformation formulae
11. Geometry of Scri.
12. Conformal compactification.
13. Scri and peeling.
14. A choice of conformal gauge.
15. A spin basis for scri.
16. Bondi mass and ADM Mass.
17. Bondi 4-momentum and positivity of Bondi mass.
18. Trapped surfaces.

Desirable Previous Knowledge

The Part 3 general relativity course is a prerequisite.

Introductory Reading

1. C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation. Freeman, 1973.
2. R.M. Wald, General Relativity. Chicago UP, 1984.

Reading to complement course material

1. J.M. Stewart, Advanced General Relativity. CUP, 1993.
2. Penrose and Rindler Spinors and Spacetime Volume 1
3. S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Spacetime. CUP, 1973.

Additional more specific references will be given in the lecture notes.

Astrophysics

Astrophysical Fluid Dynamics. (M24)

John Papaloizou

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. While many fluid dynamical effects can be seen in nature or the laboratory, there are other phenomena that are peculiar to astrophysics, for example self-gravitation, the dynamical influence of the magnetic field that is frozen in to a highly conducting plasma, and the dynamo effect driven by electromagnetic induction in a resistive fluid. The basic physical ideas introduced and applied in this course are those of Newtonian gas dynamics and magnetohydrodynamics (MHD) for a compressible fluid. The aim of the course is to provide familiarity with the basic phenomena and techniques that are of general relevance to astrophysics. Wherever possible the emphasis will be on simple examples, physical interpretation and application of the results in astrophysical contexts.

Examples of topics likely to be covered:

Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation. Microphysical basis and validity of a fluid description. Physical interpretation of MHD, with examples of basic phenomena, including dynamo theory. Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem. Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities. Spherically symmetric steady flows: stellar winds and accretion. Axisymmetric rotating magnetized flows: astrophysical jets. Waves and instabilities in stratified rotating astrophysical bodies.

Desirable Previous Knowledge

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of fluid dynamics, thermodynamics and electromagnetism will be assumed.

Introductory Reading

1. Choudhuri, A. R. (1998). The Physics of Fluids and Plasmas. Cambridge University Press.

Reading to complement course material

1. M.J.Thompson. An Introduction to Astrophysical Fluid Dynamics (2006). Imperial College Press.
2. Landau, L. D., and Lifshitz, E. M. (1987). Fluid Mechanics, 2nd ed. Pergamon Press.
3. Pringle, J. E., and King, A. R. (2007). Astrophysical Flows. Cambridge University Press.
4. Shu, F. H. (1992). The Physics of Astrophysics, vol. 2: Gas Dynamics. University Science Books.

Structure and Evolution of Stars (M16)

B. Davies and C. A. Tout

The structure of a star can be mathematically described by certain differential equations which can be derived from the principles of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics. Some familiarity with these theories will be assumed.

The basic equations of a spherical star will be derived in detail and the mode of energy transport, the equation of state, the physics of the opacity sources and the nuclear reactions will be discussed.

Approximate solutions of the equations for stellar structure will be given. Attention will be given to the virial theorem, polytropic gas spheres and homology principles. The procedure for numerical solution of the equations will be mentioned briefly.

The evolution of a star will be discussed with reference to its main-sequence evolution, the exhaustion of various nuclear fuels and the end points of evolution such as white dwarfs, neutron stars and black holes.

Throughout the course, reference will be made to the observational properties of stars and these will be discussed at appropriate times with particular reference to the Hertzsprung–Russell diagram, the mass-luminosity law and spectroscopic information.

There will be three examples sheets each of which will be discussed during an examples class.

Desirable Previous Knowledge

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics though detailed knowledge of all of these is not expected.

Introductory Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Reading to complement course material

1. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
2. Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.

Dynamics of Astrophysical Discs (L16)

S. Paardekooper

A disc of matter in orbital motion around a massive central body is found in numerous situations in astrophysics. For example, Saturn's rings consist of trillions of metre-sized iceballs that undergo gentle collisions as they orbit the planet and behave collectively like a (non-Newtonian) fluid. Protostellar or protoplanetary discs are the dusty gaseous nebulae that surround young stars for their first few million years; they accommodate the angular momentum of the collapsing cloud from which the star forms, and are the sites of planet formation. Plasma accretion discs are found around black holes in interacting binary star systems and in the centres of active galaxies, where they can reveal the properties of the compact central objects and produce some of the most luminous sources in the Universe. These diverse systems have much in common dynamically.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of extrasolar planets.

Provisional synopsis:

Occurrence of discs in various astronomical systems, basic physical and observational properties.

Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.

Evolution of an accretion disc.

Vertical disc structure, order-of-magnitude estimates and timescales, thin-disc approximations, thermal and viscous stability.

Shearing sheet, symmetries, shearing waves.

Incompressible dynamics: hydrodynamic stability, vortices.

Compressible dynamics (2D): density waves, gravitational instability.

Density waves in cylindrical geometry, Lindblad and corotation resonances.

Satellite-disc interaction, tidal potential, resonant torques, impulse approximation.

Magnetorotational instability.

Desirable previous knowledge

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is needed for the magnetorotational instability, but self-contained notes on this topic will be available.

Introductory reading

Much information on the astrophysical background is contained in

1. Frank, J., King, A. & Raine, D. (2002), Accretion Power in Astrophysics, 3rd edn, CUP.

Some of the basic theory of accretion discs is described in

1. Pringle, J. E. (1981), Annu. Rev. Astron. Astrophys. 19, 137.

Reading to complement course material

(There are no suitable textbooks.)

The Origin and Evolution of Galaxies (L24)

Martin Haehnelt

Galaxies are a fundamental building block of our Universe. The course will give an account of the physics of the formation of galaxies and their central supermassive black holes in the context of the standard paradigm for the formation of structure in the Universe.

Specific topics to be covered include the following:

- Observed properties of galaxies
- Cosmological framework and basic physical processes
- The interplay of galaxies and the intergalactic medium from which they form
- Numerical Methods for modeling galaxy formation
- Collapse of dark matter haloes and the inflow/outflow of baryons
- The hierarchical build-up of galaxies
- The origin and evolution of the central supermassive black holes in galaxies
- Towards understanding the origin of the Hubble sequence of galaxies

Desirable Previous Knowledge

The course is aimed at astronomers/astrophysicists (including beginning graduate students). It should be also suitable for interested physicists and applied mathematicians. The course is self-contained, but students who have previously attended introductory courses in General Relativity and/or Cosmology will have an easier start.

Introductory Reading

1. Ryden, B., *Introduction to Cosmology*, 2003, Addison-Wesley.
2. Sparke, L., Gallagher, J.S., *Galaxies in the Universe*, 2nd ed., 2007, Cambridge University Press.

Reading to complement course material

1. Mo, H., van den Bosch, F., White, S., *Galaxy Formation and Evolution*, 2010, Cambridge University Press.
2. Schneider, P., *Extragalactic Astronomy and Cosmology: An Introduction*, 2006, Springer.
3. Coles P., Lucchin F., *Cosmology - The Origin and Evolution of Cosmic Structure* (second edition), 2002, Wiley.

Binary Stars (L16)

Christopher Tout

A binary star is a gravitationally bound system of two component stars. Such systems are common in our Galaxy and a substantial fraction interact in ways that can significantly alter the evolution of the individual stellar components. Many of the interaction processes lend themselves to useful mathematical modelling when coupled with an understanding of the evolution of single stars.

In this course we begin by exploring the observable properties of binary stars and recall the basic dynamical properties of orbits by way of introduction. This is followed by an analysis of tides, which represent the simplest way in which the two stars can interact. From there we consider the extreme case in which tides become strong enough that mass can flow from one star to the other. We investigate the stability of such mass transfer and its effects on the orbital elements and the evolution of the individual stars. As a prototypical example we examine Algol-like systems in some detail. Mass transfer leads to the concept of stellar rejuvenation and blue stragglers. As a second example we look at the Cataclysmic Variables in which the accreting component is a white dwarf. These introduce us to novae and dwarf novae as well as a need for angular momentum loss by gravitational radiation or magnetic braking. Their formation requires an understanding of significant orbital shrinkage in what is known as common envelope evolution. Finally we apply what we have learnt to a number of exotic binary stars, such as progenitors of type Ia supernovae, X-ray binaries and millisecond pulsars.

Desirable Previous Knowledge

The Michaelmas term course on Structure and Evolution of Stars is essential. Knowledge of elementary Dynamics and Fluids will be assumed.

Introductory Reading

1. Pringle, J. E. and Wade, R. A. Interacting Binary Stars. CUP.

Reading to complement course material

1. Eggleton, P. P. Evolutionary Processes in Binary and Multiple Stars. CUP.

Quantum Information and Quantum Foundations

Quantum Foundations (L16)

Berry Groisman

In recent decades, there has been a renaissance of interest in foundational issues in quantum theory, particularly in relation to quantum information science, cosmology and quantum gravity. This course provides an introduction to modern research on quantum foundations.

The first part of the course covers central cornerstones of quantum mechanics: quantum entanglement, the measurement problem, the Einstein-Podolsky-Rosen argument, Bell's theorem and quantum non-locality, experimental tests of quantum non-locality and the failure of local hidden variable theories, and the delicate "peaceful co-existence" between quantum theory and the no-signalling principle in special relativity. It is followed by detailed account of the theory of quantum measurement, including the von Neumann's measurement paradigm, state-verification measurements and generalized measurements. We then further develop the connection between quantum theory and relativity: we discuss the restrictions posed by special relativity on instantaneous measurements of non-local variables - properties of composite systems with space-like separated parts - and explore some recent developments in this subject.

In the second part of the course we review major foundational perspectives on quantum theory, starting with the "orthodox" standard Copenhagen theory, followed up by an early attempt at an alternative to standard quantum theory, de Broglie-Bohm theory, and some of its problems. Then we go on to consider the physics of decoherence, some simple models of decoherence, and estimates of decoherence rates. This brings us to a more recent class of attempts at alternatives to quantum theory, the so-called 'dynamical collapse models' proposed by Ghirardi-Rimini-Weber, Pearle and others; we describe these models and review some of their problems. Finally, we discuss many-worlds quantum theory and the problem of making sense of probability in many-worlds theory. We will develop some of the above topics in more depth, while briefly touching the others.

Examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

A good understanding of undergraduate level quantum theory is required. (Cambridge 1B Quantum Mechanics course is a good starting point.)

Optional Introductory Reading

1. Benjamin Schumacher and Michael Westmoreland, Quantum Processes Systems, and Information, Cambridge University Press, Chapters 1-8. This is a good starter for those students who wish to review the core aspects of quantum theory in the context of quantum information. One might also find useful Benjamin Schumacher's lectures on Quantum Theory, video archived at http://pirsa.org/C10028/.

Optional reading to complement course material

1. John Bell, "Speakable and Unspeakable in Quantum Mechanics", Cambridge University Press, 2nd edition, Chapters 1,2 and 22.
2. Robert Spekkens, lectures on Foundations of Quantum Mechanics given to Perimeter Scholars International (2009). Video archived at pirsa.org, beginning with the first lecture at http://pirsa.org/09110168/
3. Yakir Aharonov and Daniel Rohrlich, "Quantum Paradoxes: Quantum Theory for the Perplexed," WILEY-VCH Verlag, Chapters 3, 7 and 14.
4. "Many Worlds? Everett, Quantum Theory, and Reality", Simon Saunders, Jonathan Barrett, Adrian Kent and David Wallace (eds.) (Oxford University Press, 2010), Chapter 8 (available at arXiv:0906.2718) and Chapter 10 (available at arXiv:0905.0624).

Quantum Computation (L16)

Richard Jozsa

Quantum mechanical processes can be exploited to provide new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. Notable examples include integer factorisation (and consequent efficient breaking of commonly used public key crypto systems) and database searching. In addition to such potential practical benefits, the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

The course will cover the following topics:

Notion of qubits, quantum logic gates, circuit model of quantum computation. Basic notions of quantum computational complexity, oracles, query complexity.

The quantum Fourier transform. Exposition of fundamental quantum algorithms including the Deutsch-Jozsa algorithm, Shor's factoring algorithm and Grover's searching algorithm.

A selection from the following further topics:
(i) Quantum teleportation and the measurement-based model of quantum computation;
(ii) Lower bounds on quantum query complexity;
(iii) Applications of phase estimation in quantum algorithms;
(iv) Quantum simulation;
(v) Introduction to quantum walks.

Desirable Previous Knowledge

It is desirable to have familiarity with the basic formalism of quantum mechanics especially in the simple context of finite dimensional state spaces (state vectors, composite systems, unitary matrices, Born rule for quantum measurements). Revision notes will be provided giving a summary of the necessary material including an exercise sheet covering notations and relevant calculational techniques of linear algebra. It would be desirable for you to look through this material at (or slightly before) the start of the course.
Any encounter with basic ideas of classical theoretical computer science (complexity theory) would be helpful but is not essential.

Reading to complement course material

1. Nielsen, M. and Chuang, I., Quantum Computation and Quantum Information. CUP.
2. John Preskill's lecture notes on quantum information theory, available at
   `http://www.theory.caltech.edu/people/preskill/ph219/`
3. Andrew Childs lecture notes on quantum algorithms available at
   `http://www.math.uwaterloo.ca/ amchilds/teaching/w11/qic823.html`

Philosophy of Classical and Quantum Mechanics (M8 and L8)

Non-Examinable (Graduate Level)

Jeremy Butterfield and Adam Caulton

This graduate course analyses some philosophical aspects of classical and quantum physics. Since philosophy of physics is an inter-disciplinary subject (and the course is not examinable!), we will let the content be influenced by the interests of those attending. But we will begin with elements of the quantum measurement problem (including density matrices, mixtures and decoherence). Then we will continue with such topics as: (i) quantization; (ii) uncertainty relations, including an analogue in classical mechanics;

(iii) symmetry principles, including spontaneous symmetry breaking, the CPT theorem and permutation symmetries (especially anyons).

Desirable Previous Knowledge

There are no formal prerequisites. Previous familiarity with the frameworks of classical and quantum mechanics will be essential; but the technicalities of each topic will be developed as needed in the lectures.

Introductory Reading

This list of introductory reading is approximately in order of increasing difficulty.

1. Weyl, H. Philosophy of Mathematics and Natural Science. Princeton University Press.
2. Bell, J. Speakable and Unspeakable in Quantum Mechanics. CUP.

Reading to complement course material

1. Brading, K. and Castellani, E. (eds.) Symmetries in Physics. CUP.
2. Butterfield, J. On Symplectic Reduction in Classical Mechanics, in J. Earman and J. Butterfield (eds.) The Handbook of Philosophy of Physics, 2 volumes, Elsevier; pp. 1 - 131. Available at: physics/0507194 and at http://philsci-archive.pitt.edu/archive/00002373/
3. Isham, C. Modern Differential Geometry for Physicists. World Scientific.
4. Landsman, N. Between classical and quantum. In Butterfield, J. and Earman, J. (eds.) Handbook of the Philosophy of Physics, 2 volumes, Elsevier. Available at: http://arxiv.org/abs/quant-ph/0506082, and at: http://philsci-archive.pitt.edu/archive/00002328/
5. Sternberg, S. Group Theory and Physics. CUP.

Philosophical Foundations of Quantum Field Theory (M8)

Non-Examinable (Graduate Level)

Nazim Bouatta and Nicholas Teh

Quantum field theory (QFT) is a wonderful mountain range, combining strikingly deep and unifying ideas with a panoply of powerful calculational tools. In recent decades, QFT has become the framework for several basic and outstandingly successful physical theories. But it has been largely unexplored by philosophy of physics, which has concentrated on conceptual questions raised by non-relativistic quantum mechanics and general relativity (and the focus of another graduate course). Here, we will introduce the philosophical aspects of quantum field theory. More specifically, we will conceptually address topics that have been central to quantum field theory's development in the last forty years, such as: the renormalization group, gauge symmetries and solitons.

Desirable Previous Knowledge

There are no formal prerequisites. Previous familiarity with the tools of quantum field theory, such as provided by the Part III courses, will be helpful.

Introductory Reading

This list of introductory reading is approximately in order of increasing difficulty.

1. Wallace, D. (2006), 'In defense of naiveté: The conceptual status of Lagrangian quantum field theory', *Synthese*, **151**(1):33-80, 2006. Preprint available online at: http://arxiv.org/pdf/quant-ph/0112148v1

2. Weinberg, S. (1997), 'What is Quantum Field Theory, and What Did We Think It Is?'. Available online at: http://arxiv.org/abs/hep-th/9702027

3. Fisher, M. (1998), 'Renormalization group theory: Its basis and formulation in statistical physics', *Rev. Mod. Phys* **70**, pp 653-681.

Reading to complement course material

1. Cao, T. ed. The Conceptual Foundations of Quantum Field Theory. Cambridge University Press, 1999.

2. Weinberg. S. The Quantum Theory of Fields, Vols I and II. Cambridge University Press, 1995 and 1996.

3. Ruetsche, L. Interpreting Quantum Mechanics. Oxford University Press, 2011.

4. Healey, R. Gauging What's Real. The Conceptual Foundations of Contemporary Gauge Theories. Oxford University Press, 2007.

Computational Complexity (M16)

Ashley Montanaro

Computational complexity theory is the study of the intrinsic difficulty of problem-solving: the ultimate goal of the field is to determine which problems can be solved efficiently by computer and which cannot. The subject is perhaps best exemplified by its most famous open problem, the P vs. NP question. Informally speaking, this asks whether there exist problems which are significantly more difficult to solve than to verify a claimed solution. Perhaps surprisingly, this sort of question can be made mathematically precise, and has led to an increasingly intricate classification of problems by difficulty.

This course, which is likely to appeal to students with an interest in theoretical computer science or discrete and combinatorial mathematics, aims to provide a broad grounding in the fundamentals of computational complexity theory, together with some very recent and more advanced results. Time permitting, the intention is to cover the following topics, among others.

The Turing machine model, decidability and the halting problem. Time-bounded complexity and the time hierarchy theorem. Nondeterminism and the complexity class NP. NP-completeness and the Cook-Levin theorem. The P vs. NP problem and obstacles to resolving it. Approximation algorithms and hardness of approximation. Randomised algorithms. Space-bounded complexity. Concrete models of complexity: decision trees, circuits and communication complexity.

Desirable previous knowledge

There are no prerequisites for this course. While the course will be helpful preparation for those students intending to take Quantum Computation, it deals only with classical complexity theory and no knowledge of quantum mechanics is required.

Reading to complement course material

The course will not follow any textbook closely and notes will be provided. However, the following books may be useful.

1. Computational Complexity: A Modern Approach, S. Arora and B. Barak. Cambridge University Press.
2. Computational Complexity, C. Papadimitriou. Addison-Wesley.
3. Computers and Intractability: A Guide to the Theory of NP-Completeness, M. Garey and D. Johnson. W. H. Freeman.

Applied and Computational Analysis

Distribution Theory and Applications (M16)

Dr A. Ashton

This course will provide an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the *use* of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will briefly look at Sobolev spaces in $\mathbf{R}^n$ and their description in terms of the Fourier transform of tempered distributions. Time permitting, the material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. We will also look at Hörmander's oscillatory integrals and use them to describe the singular support of a large class of distributions. The course will be supplemented with hand-outs and example sheets. There will be three examples classes.

Desirable Previous Knowledge

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Methods/Analysis). No knowledge of measure theory or functional analysis is required.

Introductory Reading

1. F. G. Friedlander and M. S. Joshi, *Introduction to the Theory of Distributions*, Cambridge Univ Pr, 1998.
2. M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge Univ Pr, 1958.
3. G. B. Folland, *Introduction to Partial Differential Equations*, Princeton Univ Pr, 1995.

Reading to complement course material

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vols I-II*, Springer Verlag, 1985.
2. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vols I-II*, Academic Press, 1979.
3. F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, Routledge, 1966.

Approximation Theory (M24)

A Shadrin

The course will give an overview of basic concepts of Approximation Theory, i.e., best and good approximation of a large family of functions by a smaller set (usually finitely generated, linear or nonlinear) in certain normed spaces (such as C and L_p), construction of good approximants (if possible), finding approximation order.

We start with the classical polynomial tools and then move the emphasis to univariate splines. Time allowing, we will pay some attention to multivariate splines and wavelets.

Desirable Previous Knowledge

Some elements of Functional Analysis (normed spaces, linear operators).

Reading to complement course material

1. E. W. Cheney, Approximation theory, McGraw-Hill, New-York, 1966.
2. R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, 1993.
3. C. de Boor, Lecture Notes on Approximation Theory, `www.cs.wisc.edu/~deboor`

Convex Optimisation with Applications in Image Processing. (M16)

Jan Lellmann

Convex optimisation problems have the intriguing property that they can be numerically solved to a *global* minimum in a reasonable amount of time, even for many large-scale, non-linear, non-differentiable problems. With some effort many interesting real-world applications can be modeled by, or at least approximated by, the task of solving a single convex problem.

This approach has two strong points: firstly, if the method fails, we know that it is clearly not a numerical issue, but that in fact the model was wrong. Secondly, more often than not we find that the resulting algorithms are very efficient and scale almost linearly in the number of variables. This is especially important in image processing, where the number of variables can easily go into the millions for a single image.

Convex optimization as a field is now relatively mature, which makes for a very polished theory, but it still keeps evolving with the increasing computational power and new architectures such as GPUs. The number of applications in image processing is enormous – removing noise from digital camera images, increasing the resolution of an image, cutting out objects from the background, tracking people in video sequences, reconstructing 3D objects from several views, and many more.

The course is laid out as an introduction into the theory and solution strategies together with a collection of interesting applications in image processing and their specific challenges. We will begin with the theory in a conic optimization setting, including the fundamental results about subdifferentials, optimality conditions, and duality. We will then cover the most important solvers including Interior Point- and min-cut/max-flow methods, and recent first-order developments. Depending on time and interest we might also look into some complexity results.

Desirable Previous Knowledge

A background in variational methods is helpful but not required, since we will mainly work in the finite-dimensional setting.

Introductory Reading

1. S. Boyd, L. Vandenberghe: Convex Optimization. Cambridge University Press, 2004 (available online).
2. R. T. Rockafellar, J.-B. Wets: Variational Analysis. Springer, 3rd ed., 2009.
3. A. Ben-Tal, A. Nemirovski: Lectures on Modern Convex Optimization. MPS-SIAM, 2001.
4. N. Paragios, Y. Chen, O. Faugeras: Handbook of Mathematical Models in Computer Vision. Springer, 2006.

Reading to complement course material

1. Y. Nesterov: Introductory Lectures on Convex Optimization. Kluwer, 2004.
2. D. P. Bertsekas: Network Optimization: Continuous and Discrete Models. Athena Scientific, 1998.
3. J. Nocedal, S. J. Wright: Numerical Optimization. Springer, 2006.
4. C. M. Bishop: Pattern Recognition and Machine Learning. Springer, 2006.

Numerical solution of differential equations (M24)

A. Iserles

The goal of this lecture course is to present and analyse efficient numerical methods for ordinary and partial differential equations. The exposition is based on few basic ideas from approximation theory, complex analysis, theory of differential equations and linear algebra, leading in a natural way to a wide range of numerical methods and computational strategies. The emphasis is on algorithms and their mathematical analysis, rather than on applications.

The course consists of three parts: methods for *ordinary differential equations* (with an emphasis on initial-value problems and a thorough treatment of stability issues and stiff equations), numerical schemes for *partial differential equations* (both boundary and initial-boundary value problems, featuring finite differences and finite elements) and, time allowing, *numerical algebra of sparse systems* (inclusive of fast Poisson solvers, sparse Gaussian elimination and iterative methods). We start from the very basics, analysing approximation of differential operators in a finite-dimensional framework, and proceed to the design of state-of-the-art numerical algorithms.

Desirable Previous Knowledge

Good preparation for this course assumes relatively little in numerical mathematics *per se,* except for basic understanding of elementary computational techniques in linear algebra and approximation theory. Prior knowledge of numerical methods for differential equations will neither be assumed nor is necessarily an advantage. Experience with programming and application of computational techniques will obviously aid comprehension but is neither assumed nor expected.

Fluency in linear algebra and decent understanding of mathematical analysis are a must. Thus, linear spaces (inner products, norms, basic theory of function spaces and differential operators), complex analysis (analytic functions, complex integrals, the Cauchy formula), Fourier series, basic facts about dynamical systems and, needless to say, elements from the theory of differential equations.

There are several undergraduate textbooks on numerical analysis. The following present material at a reasonable level of sophistication. Often they present material well in excess of the requirements for the course in computational differential equations, yet their contents (even the bits that have nothing to do with the course) will help you to acquire valuable background in numerical techniques:

Introductory Reading

1. S. Conte & C. de Boor, *Elementary Numerical Analysis,* McGraw–Hill, New York, 1980.
2. G.H. Golub & C.F. van Loan, *Matrix Computations,* 3rd edition. Johns Hopkins Press 1996.
3. M.J.D. Powell, *Approximation Theory and Methods,* Cambridge University Press, Cambridge, 1981.
4. G. Strang, *Introduction to Linear Algebra,* Wellesley-Cambridge Press, Cambridge (Mass.), 3rd ed. 2003..

Reading to complement course material

1. U. M. Ascher, *Numerical Methods for Evolutionary Differential Equations,* SIAM, Philadelphia, 2008.
2. O. Axelsson, *Iterative Solution Methods,* Cambridge University Press, Cambridge, 1996.
3. E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer-Verlag, Berlin, 2nd ed. 1993.
4. E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems,* Springer-Verlag, Berlin, 2nd ed. 1996.
5. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge University Press, Cambridge, 2nd ed. 2008.
6. K.W. Morton & D.F. Mayers, *Numerical Solution of Partial Differential Equations: An Introduction,* Cambridge University Press, Cambridge, 2005.
7. G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Wellesley–Cambridge Press, Cambridge (Mass.), 2nd ed. 2008.

Image Processing - Variational and PDE Methods (L16)

C.-B. Schönlieb

In our modern society the processing of digital images becomes more and more important, reflected in a myriad of applications: medical imaging (MRI, CT, PET), forensics, security, design, arts and many more. In this course we encounter some of the most powerful image processing methods and its underlying mathematical principles. In particular, we are interested in deterministic imaging approaches using variational calculus and partial differential equations (PDEs) [1-3].

The course is organised as follows: We start with mathematical representations of images (e.g., distributions, Sobolev functions, functions of bounded variation, level sets) and formulate inverse problems, i.e., optimization models, for image denoising, –decomposition, –inpainting and –segmentation (e.g., total variation minimization [4], Mumford-Shah, curve models, active contours). Then, we move on to PDEs for image processing (e.g., the heat equation, degenerate elliptic PDEs, Perona-Malik, diffusion filters, anisotropic diffusion, higher-order models involving curvature). Eventually, we discuss their numerical solution (steepest descent, iterative methods, dual solvers).

Relevant Courses

Useful: Functional Analysis, variational calculus, partial differential equations, numerical analysis.

References

[1] G. Aubert, and P. Kornprobst, *Mathematical Problems in Image Processing. Partial Differential Equations and the Calculus of Variations*, Springer, Applied Mathematical Sciences, Vol 147, (2006).

[2] T. F. Chan, and J. J. Shen, *Image Processing and Analysis - Variational, PDE, wavelet, and stochastic methods.* SIAM, (2005).

[3] Y. Meyer, *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations*, AMS 2001.

[4] L. Rudin, S. Osher, E. Fatemi, *Nonlinear Total Variation Based Noise Removal Algorithms*, Physica D: Nonlinear Phenomena, Vol. 60, Issues 1-4, 1992.

Novel Techniques for Boundary Value Problems (L16)

Non-Examinable (Graduate Level)

A. Fokas

This mini-course will discuss a new method for analyzing linear boundary value problems. This method, which has been acclaimed as "the most important development in the exact analysis of linear PDEs since the classical works of the 18nth century" is based on the "synthesis" as opposed to separation of variables.

It has led to the analytical solution of several non-separable, as well as non-self-adjoint boundary value problems. Furthermore, it has led to new numerical techniques for solving linear elliptic PDEs in the interior as well as in the exterior of polygons. The analytical and numerical implementation of the new method to both evolution and elliptic PDEs will be discussed.

A related topic is the emergence of new analytical methods for solving inverse problems arising in medicine, including techniques for PET (Positron Emission Tomography) and SPECT (Single Photon Emission Computerized Tomography). A summary of these developments will also be presented.

Applications of Functional Integration (L16)

Non-Examinable (Graduate Level)

D. M. A. Stuart

This course will be a mathematical introduction to the use of functional integration methods in theoretical physics and applied mathematics. After reviewing some necessary background material on integration and probability we will study (some of) the following topics (depending on time/preference):

(i) Feynman-Kac formula, eigenvalues and boundary value problems. (ii) Euclidean approach to quantum mechanics. (iii) Eigenvalue splitting and instantons for double well potential. (iv) Applications to statistical mechanics. (v) Turbulent advection.

Books/reviews: (i) B. Simon: Functional Integration And Quantum Physics (AMS Chelsea Publishing) (ii) J. Ginibre: Some applications of functional integration in statistical mechanics, C.M. DeWitt (ed.) R. Stora (ed.) , Statistical mechanics and quantum field theory , Gordon and Breach pp. 327-427 (iii)K. Gawedzki: Soluble models of turbulent advection, arXiv:nlin/0207058

Continuum Mechanics

Desirable previous knowledge

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice familiarity with the continuum assumption and the material derivative will be assumed, as will basic ideas concerning incompressible, inviscid fluids mechanics (e.g. Bernoulli's Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable.

For solid mechanics courses no previous knowledge of solid mechanics is required, but prior knowledge of some continuum mechanics (e.g. an introductory course in fluid dynamics) will be assumed.

For both fluid dynamics and solid mechanics courses previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is highly desirable.

In summary, knowledge of Chapters 1-8 of 'Elementary Fluid Dynamics' (D.J. Acheson, Oxford), plus Chapter 3 of 'Waves in Fluids' (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace's equation, Poisson's equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses.

Year	*Courses*
First	Differential Equations, Dynamics, Calculus & Methods.
Second	Methods, Complex Methods, Fluid Dynamics.
Third	Fluid Dynamics, Waves, Asymptotic Methods.

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses, which may be found on WWW with URL:

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Fluid dynamics of the environment (M24)

C. P. Caulfield & J. A. Neufeld

Understanding, predicting and minimizing the impact of human activity on the environment is a central challenge for sustainability. Many of the key issues are associated with fluid motions in the ocean, atmosphere and the earth itself, and this course provides an introduction to the basic fluid dynamics necessary to build mathematical models of the environment in which we live, focussing on flows which occur over sufficiently small time and length scales to be largely unaffected by the earth's rotation. The course begins by considering the governing equations of fluid flow in the presence of (typically relatively small) density variations. When there are density variations in a fluid, it is possible for 'internal gravity waves' to occur, since the density variations within the fluid provide the restoring force, and the course will highlight some of the rich and surprising dynamics of these waves. In particular, internal gravity waves radiate energy vertically as well as horizontally, and their interaction with boundaries can focus this energy and cause mixing far from where the energy was input.

The subtle dynamics of stratified mixing by turbulence is then introduced through an exploration of some of its basic characteristics including the complex interplay between kinetic and potential energy in a sheared, density stratified flow. Of course, density variations can also drive a flow, and the course will consider a particularly important class of such flows, where a relatively localised source drives the rise of a turbulent 'plume' of buoyant fluid. Volcanic eruption clouds and accidental releases of pollution are just two examples of such plumes, and their interaction with a stratified environment, such as the atmosphere and the ocean, will also be discussed. When turbulent plumes are confined along horizontal boundaries they form turbulent gravity currents as found in the avalanches, pyroclastic eruptions, and in turbidity currents driven by suspended sediment. These are dynamically distinct from their viscous counterparts, as exemplified by the flow of glacial ice and mud or magmatic currents. Finally, density differences, in the form of thermal or compositional gradients, can drive convective motions with consequences for the rates of mixing, cooling and phase change (solidification or evaporation, for example) in many environmental settings.

Desirable Previous Knowledge

Undergraduate fluid dynamics.

Reading to complement course material

1. B. R. Sutherland, Internal gravity waves, Cambridge University Press (2010).
2. J. S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).

Biological Physics (M24)

R.E. Goldstein & U. Keyser

This course will provide an overview of the physics and mathematical description of living systems. The range of subjects and approaches, from phenomenology to detailed calculations, will be of interest to students from applied mathematics, physics, and computational biology. The topics to be covered will span the range of length scales from molecular to ecological, with emphasis on key paradigms. Introductory material on statistical mechanics will provide background for much of the course. The subsequent topics will include *Microscopic Physics* – van der Waals forces, screened electrostatics, Borwnian motion, fluctuation-dissipation theorem; *Fluctuation-Induced Forces* – polymer physics, random walks, entropic forces, stiff chains, self-avoidance, dynamics, protein folding; *Elasticity* – differential geometry of curves and suraces, linear elasticity theory, thin plates and rods, Helfrich model for membranes, elastohydrodynamics; *Chemical Kinetics and Pattern Formation*– Michaelis-Menten kinetics, oscillations, excitable media, ion channels, action potentials, reaction-diffusion dynamics, Fitzhugh-Nagumo model, spiral waves; *Dynamics*– life at low Reynolds numbers, chemoreception, advection-diffusion problems.

Desirable Previous Knowledge

Some familiarity with statistical physics will be helpful.

Introductory Reading

1. P. Nelson. *Biological Physics.* W.H. Freeman (2007).
2. J.D. Murray. *Mathematical Biology I. & II.* Springer (2007, 2008).
3. K. Dill & S. Bromberg. *Molecular Driving Forces.* Garland (2009).

Reading to complement course material

1. B. Alberts, A. Johnson, J. Lewis, M. Raff, K. Roberts and P. Walter. *Molecular Biology of the Cell.* 5th edition. Garland Science (2007).
2. J.N. Israelachvili. *Intermolecular and Surface Forces.* 2nd edition. Academic Press (1992).

3. E.J.W. Verwey and J.Th.G. Overbeek. *Theory of the Stability of Lyophobic Colloids.* Elsevier (1948).

4. M. Doi and S.F. Edwards. *The Theory of Polymer Dynamics.* OUP (1986).

5. A. Parsegian. *Van der Waals Forces.* CUP (2005).

6. D. Andelman & W. Poon. *Condensed Matter Physics in Molecular and Cell Biology.* Taylor & Francis (2006).

Perturbation and Stability Methods (M24)

J.M. Rallison and N. Peake

This first part of this course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

A number of the most useful mathematical tools for research will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

The second part of the course covers applications of perturbation methods to the study of fluid flows. So-called 'hydrodynamic stability' is a very broad discipline, and in this course we will concentrate on the stability of nearly parallel-flows (as for example arise in boundary-layer flows).

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide and conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. [6]

- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKBJLG' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium, including investigation of the rescaling required near 'hot spots', or'caustics'). [5]

- *The Summation of Series.* Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, Domb-Sykes plots. [1]

- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. At the end of this section further examples will be given of asymptotics beyond all orders. [6]

- *Stability Theory.* This section will review both eigenvalue and 'non-eigenvalue' aspects of stability theory as applied to fluid flows, concentrating on nearly-parallel flows. Aspects that will be covered include the concepts of 'causality' and the Briggs-Bers technique, the continuous spectrum, and the transitory algebraic growth that can follow from the fact that the operators in hydrodynamic stability theory are often not self-adjoint. [6]

In addition to the lectures, a series of examples sheets will be provided. The lecturers will run examples classes in parallel to the course.

Desirable Previous Knowledge

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve simple differential equations and partial differential equations and evaluate simple integrals.

Introductory Reading

1. E.J. Hinch. *Perturbation Methods*, Cambridge University Press (1991).
2. M.D. Van Dyke. *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975).

Reading to complement course material

1. C.M. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *Beware: Bender and Orszag call Stokes lines anti-Stokes lines, and vice versa.*
2. John P. Boyd. *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series* Acta Applicandae, **56**, 1-98 (1999), and also available at

   `http://www-personal.engin.umich.edu/~jpboyd/boydactaapplicreview.pdf`
3. M.V. Berry. *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).
4. P.G. Drazin and W. H. Reid. *Hydrodynamic Stability*, Cambridge University Press (1981 and 2004).
5. J. Kevorkian and J.D. Cole. *Perturbation Methods in Applied Mathematics*, Springer (1981).
6. Peter Schmid and Dan S. Henningson. *Stability and Transition in Shear Flows*, Springer-Verlag (2001).

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media will be analysed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Desirable Previous Knowledge

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Introductory Reading

1. D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7
2. G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.
3. L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 and 5.

Reading to complement course material

1. J. Happel and H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).
2. S. Kim and J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)
3. C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP (1992).
4. O.M. Phillips. *Flow and Reactions in Permeable Rocks.* CUP (1991).

Fluid Dynamics of Energy Systems. (L16)

Prof Andrew W Woods

This course will be divided into two main sections. Fist, it will explore some of the fluid dynamics involved in the energy supply sector, including oil, gas and geothermal energy, as well as a brief discussion of wind and tidal energy systems. Then it will examine some of the fluid mechanical challenges for efficient use of energy in buildings, especially through use of natural ventilation.

This will include a description of the fluid dynamics of oil and gas reservoir formation, including the formation of large sedimentary deposits from particle laden flows on the sea floor and their subsequent burial and compaction, followed by the natural migration of oil from source rock into reservoir rocks. The course will also examine the subsequent displacement of oil and gas such permeable rocks, either through primary pressure driven flow or as it is displaced by waterm , and reactive chemical solutions, injected into system. The emerging area of carbon sequestration will also be discussed, illustrating the dynamics controlling the dispersal of large volumes of CO2 in subsurface aquifers, and the longer term migration controlled by buoyancy forces. [8 lectures]

The fluid dynamics involved in the production of geothermal energy will also be discussed, illustrating how thermal energy can be transported through permeable rocks by the controlled flow of water and vapour. This will include a discussion of the phase changes involved in superheated systems, and of the deposition and dissolution of minerals as the temperature of the fluids migrating through such systems change. [2 lectures]

The course will then dscribe some of the fluid mechanical challenges for wind turbines and tidal turbines, especially related to efficiency of power generation and dynamics of the wakes [2 lectures].

Some of the challenges of energy efficiency will also be presented, including the use of natural flows in buildings to reduce the enormous energy demand of air-conditioning systems. Descriptions of the interaction between wind and buoyancy driven air flows, with heat exchange to the mass of a building will be discussed, as well as the detailed flow patterns within a building arising from localised and distributed sources of heating or cooling, which lead to turbulent plumes mixing the interior of confined spaces.[6 lectures]

Desirable Previous Knowledge

Part 1B and Part II Fluid Mechanics, and knowledge of partial differential equations.

Introductory Reading

1. OM Phillips, Flow and reactions in permeable rock. CUP 1991

1. JS Turner, Buoyancy effects in fluids, CUP, 1979.

Sound Generation and Propagation (L16)

E.J. Brambley

The application of wave theory to problems involving the generation, propagation and scattering of acoustic and other waves is of considerable relevance to many applications of practical significance. These include, for example, underwater sound propagation, aircraft noise, remote sensing, the effect of noise in built-up areas, and a variety of medical diagnostic applications. This course aims to provide the basic theory of wave generation, propagation and scattering, and an overview of the mathematical methods and approximations used to tackle these problems, with emphasis on applications to aeroacoustics. The course will cover some general aeroacoustic theory [3], sound generation by turbulence and moving bodies (including the Lighthill and Ffowcs Williams–Hawking acoustic analogies) [3], scattering (including the scalar Wiener-Hopf technique applied to the Sommerfeld problem of scattering by a sharp edge) [4], long-distance sound propagation including nonlinear and viscous effects [3], and wave-guides [3]. The lectures will be supplemented by three examples sheets and examples classes.

Desirable Previous Knowledge

This course assumes that students have attended some introductory courses in continuum mechanics and complex variable theory (especially Fourier transforms and the use of complex residues). Attendance at the Part III course Perturbation Methods would also be helpful, but is by no means essential.

Introductory Reading

1. Dowling, AP and Ffowcs Williams, JE. Sound and Sources of Sound, Ellis Horwood.
2. Landau, LD and Lifschitz, EM. Fluid Mechanics, Butterworth-Heinemann. [Chapter 8]

Reading to complement course material

1. Crighton, DG et al. Modern Methods in Analytical Acoustics, ASA.
2. Pierce, AD. Acoustics, McGraw–Hill.

Solidification of Fluids. (L16)

M.G. Worster

Many solid materials were once liquid: the frame of a bicycle; the polar ice caps; igneous rocks. The structure and composition of such solids are significantly affected by the way in which the liquid flows as it solidifies. This course aims to introduce the fluid-mechanical and thermodynamical interactions that occur during solidification and its converse, melting. The emphasis will be on mathematical modelling, starting from physical descriptions of the processes involved.

The course will begin by introducing the fundamental thermodynamic principles governing changes of phase between liquid and solid. From a mathematical perspective, the course will also introduce so-called free-boundary problems, in which the locations of the boundaries of the domains in which the equations are to be solved must themselves be determined as part of the solution. An important aspect of free- boundary problems is the possibility of morphological instability: an initially planar boundary can spontaneously become highly convoluted. This is the origin of the branched structure of snowflakes, for example.

The rates and patterns of solidification are dictated by heat transfer through both the solid and liquid phases. The latter is affected by flow of the liquid, which is commonly generated by buoyancy forces arising from the intrinsic thermal gradients established during solidification. Therefore, a significant part of the course will focus on the establishment and effects of convection in the melt.

Morphological instability is prevalent in multi-component systems, such as salt water. In consequence, solidification typically produces a matrix of solid crystals bathed in melt, called a mushy region. These two-phase, partially solidified, reactive porous media are predominant during the casting of alloys and in most geophysical systems, such as during the freezing of the oceans and solidification of lava. Fascinating and important interactions occur as the residual melt flows through the pores of a mushy layer.

The final part of the course will introduce the concept of interfacial premelting, whereby materials below their melting point are nevertheless liquid in microscopically thin layers at their surface or near their contact with a neighbouring material. The analysis of the dynamics of premelted liquid films involves the fluid mechanics of lubrication theory modified by long-range intermolecular forces that determine the film thicknesses.

Desirable Previous Knowledge

A basic understanding of viscous fluid dynamics. Mathematical methods, particularly the solution of ordinary and partial differential equations.

Reading to complement course material

1. Worster, M.G. Solidification of Fluids. In Perspectives in Fluid Dynamics a Collective Introduction to Current Research. Edited by GK Batchelor, HK Moffatt and MG Worster. pp. 393-446. CUP.

2. Davis, S.H. Thoery of Solidification. Cambridge Books Online.

The Physics Of The Polar Oceans, Sea Ice and Climate Change (L16)

Peter Wadhams

Course description

The course is designed to give a complete background on the physics of sea ice and its role in the climate system, also including ice mechanics, icebergs and the physics of oil-ice interaction. The course comprises 16 sessions, as shown below. Each session requires an hour of teaching.

1. Regional setting The geography, water structure, currents and ice cover in the Arctic and Antarctic oceans.

2. The physics of sea ice and ice formation What happens when sea water cools Growth of ice crystals Brine cells and brine rejection Salinity structure Summer melt processes First- and multi-year ice

3. Ice growth and decay Thermodynamic model Equilibrium thickness Sensitivity of thickness to changes in forcing Sensitivity to albedo.

4. Ice dynamics Ice motion - driving forces Free drift solution Ice interaction The dynamics of polynyas

5.The ice thickness distribution Ridge and lead formation Geometry of pressure ridges The probability density of ice thickness and its evolution Mathematical form of ridges and leads distributions

6. Ice mechanics The ridging and rafting process Ridge evolution and decay Ice interaction with structures Ice interaction with the seabed

7. The marginal ice zone Ice floes Waves in ice Modelling development of floe size distribution Eddies

8. Icebergs and ice islands Sources Distribution in Arctic and Antarctic Physical properties Dynamics Decay and breakup Role in the oceans and in sediment transport

9. Glacial ice threat to offshore structures Iceberg scouring depths, incidence, seabed interaction Mechanics of iceberg and ice island interaction with structures Upstream detection of ice islands

10. Oil spills under ice Scope of the under ice blowout problem Other sources of spills under and in ice Physical behaviour of crude oil in very cold water Dynamics of a rising oil-infested bubble plume Incorporation of oil in rough sea ice containment factors Ice growth under an oil layer Oil penetration into brine drainage channels Oil transport by ice The melt process and mode of final oil release Oil behaviour in pancake ice and the marginal ice zone

11. Regional ice studies Greenland and the Beaufort Sea East Greenland waters Greenland Sea convection zone South Greenland and the Storis Baffin Bay ice conditions Nares Strait The Lincoln Sea and waters north of Greenland The Beaufort Gyre and its variability Changes in ice conditions in central Beaufort Sea The Beaufort Sea coastal zone The summer Beaufort Sea as a new MIZ Methane release from seabed

12. Thinning and retreat of sea ice Satellite data on retreat Parkinson - retreat in sectors, Arctic and Antarctic What is found in Antarctic Thinning - the submarine and other evidence Model predictions of a future seasonal Arctic ice cover

13. Conclusions - how did it start and where will it end? Ice ages and their causes Earlier ice-free periods The physics of the greenhouse effect Is Man the only cause of current changes? What will happen in the longer term? Global sea level rise Potential feedbacks on feedbacks.

14-16. Special topics and examples, drawn from rest of course and inserted at appropriate intervals (probably after 5, 9 and 13).

Desirable previous knowledge

Basic fluid dynamics and some knowledge of, and interest in, climatology, oceanography and glaciology would be useful.

Introductory reading

Background reading of general introductory physical oceanographic books will be useful. The book of the course is
"Ice in the Ocean" by P Wadhams (Taylor and Francis, 2000).
This might be read through before the lectures begin.

Reading to complement course

Other very useful books which will be used in the course are
"Global Warming - the Complete Briefing" by Sir John Houghton, 3rd Edn (CUP)
and
"On Sea Ice" by Willy Weeks (Univ. Alaska Press).

During the course there will be specific references to material that should be pursued further in sources such as
The Geophysics of Sea Ice (ed. N Untersteiner)
The Physics of Ice-Covered Seas (Univ Helsinki)
The Drift of Sea Ice (M Lepparanta)
Field Techniques for Sea Ice Research (ed. H. Eicken)

Fluid Dynamics of Climate (L24)

P.F. Linden and J.R. Taylor

Understanding and predicting the Earth's climate is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the Earth's climate, helping to make the planet hospitable for life. However, the dynamical complexity of this motion and the wide range of space and time scales involved, makes predicting the climate system a very difficult endeavour.

This course provides an introduction to the basic fluid dynamics necessary to build mathematical models of the environment in which we live, focusing on the large-scale behaviour of stratified and rotating flows. The course begins by considering flows where the timescale for the motion is long compared with a day and the Earth's rotation plays an important role. The additional timescale introduced by the Earth's rotation modifies the dynamics in a profound way for both homogeneous and density stratified flows. The Coriolis force (a fictitious force arising from our use of a frame of reference rotating with the planet) causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics. We will then apply the theory for rotating, stratified fluids to describe the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the dynamics that give rise to eddies and storms in the ocean and atmosphere, ocean gyres and boundary currents like the Gulf Stream, the meridional (north/south) circulation in the ocean and atmosphere, and the transport of heat and other tracers across the globe.

Desirable Previous Knowledge

Undergraduate fluid dynamics

Reading to complement course material

1. A.E. Gill, Atmosphere-Ocean Dynamics. Academic Press (1982).
2. Marshall, J. and R.A. Plumb. Atmosphere, Ocean, and Climate Dynamics. Academic Press. 2008.
3. Pedlosky, J. Geophysical Fluid Dynamics. Springer. (1987).
4. J.S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).

Demonstrations in Fluid Mechanics. (L8)

Non-Examinable (Graduate Level)

Dr. S.B. Dalziel

While the equations governing most fluid flows are well known, they are often very difficult to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The purpose of this demonstration course is to help develop an intuitive 'feeling' for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- sedimentation and resuspension;
- avalanches and granular flows;
- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low Reynolds' number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account.

Desirable Previous Knowledge

Undergraduate Fluid Dynamics.

Reading to complement course material

1. M. Van Dyke. An Album of Fluid Motion. Parabolic Press.
2. G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. Multimedia Fluid Mechanics (Multilingual Version CD-ROM). CUP.
3. M. Samimy, K. Breuer, P. Steen, and L. G. Leal. A Gallery of Fluid Motion. CUP.

Granular Flows (L8)

Non-Examinable (Graduate Level)

N. Vriend

Course description

Granular flows are found everywhere in nature and industry. They are complex to describe in a comprehensive model, as their physical behavior changes strongly depending on the flow regime and its environment.

This is a non-examinable graduate lecture course which will give you an introduction to granular flows. Topics we will discuss include: (1) contact and interaction forces between grains, (2) behavior of a granular solid, including both quasistatic behavior and static and elastic behavior, (3) collisional granular gases, (4) dense inertial granular liquids, (5) suspensions of fluids and particles, (6) erosion, sedimentation and geomorphology. Each lecture will start with a small table-top experiment or a video of a large-scale granular flow in nature or industry and by end the of the lecture course you should have learned how Newton's craddle, an hourglass and a sand castle work!

Desirable previous knowledge

Physics and mathematics at the undergraduate level.