ON DISCRETIZATION OF C*-ALGEBRAS

CHRIS HEUNEN AND MANUEL L. REYES

ABSTRACT. The C*-algebra of bounded operators on the separable Hilbert space cannot be mapped to a W*-algebra in such a way that each unital commutative C*-subalgebra $C(X)$ factors normally through $\ell^\infty(X)$. Consequently, there is no faithful functor discretizing C*-algebras to W*-algebras this way.

1. INTRODUCTION

In operator algebra it is common practice to think of a C*-algebra as a noncommutative analogue of a topological space, and to think of a W*-algebra as a noncommutative analogue of a measure space. In particular, just like any topological space embeds into a discrete one, atomic W*-algebras are often viewed as 'noncommutative sets' that can carry the 'noncommutative topology' of a C*-subalgebra, see *e.g.* [7, 1]. To make this precise, one needs a way to embed a C*-algebra into a W*-algebra. A standard way is the universal enveloping W*-algebra given by the adjunction

$$\mathbf{Cstar} \;\underset{\hookleftarrow}{\overset{(-)^{**}}{\underset{\perp}{\longrightarrow}}}\; \mathbf{Wstar}$$

between the category of unital C*-algebras with unital $*$-homomorphisms and the subcategory of W*-algebras with normal $*$-homomorphisms, see [6, 3.2]. This construction has the drawback that the resulting W*-algebra is very large. It does not restrict to the commutative case as the embedding $\eta\colon C(X) \to \ell^\infty(X)$. This leads to the following notion, in keeping with the recent programme of taking commutative subalgebras seriously [18, 4, 19, 3] that has recently been successful [11, 9, 12, 10].

Definition. A *discretization* of a unital C*-algebra A is a unital $*$-homomorphism $\phi\colon A \to M$ to a W*-algebra M whose restriction to each commutative unital C*-subalgebra $C \cong C(X)$ factors normally through $\ell^\infty(X)$, so that the following diagram commutes.

$$\begin{array}{ccc}
A & \xrightarrow{\quad\phi\quad} & M \\
\Big\uparrow\!\!\rfloor & & \Big\uparrow \text{normal } *\text{-homomorphism} \\
C(X) & \xrightarrow[\quad\eta\quad]{} & \ell^\infty(X)
\end{array}$$

This short note proves that this construction degenerates in prototypical cases.

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Theorem. *If $\phi\colon B(H) \to M$ is a discretization for a separable infinite-dimensional Hilbert space H, then $M = 0$.*

Stated more concretely, this obstruction means that $A = B(H)$ has no representation on a Hilbert space $K \neq 0$ such that every (maximal) commutative $*$-subalgebra of A has a basis of simultaneous eigenvectors in K.

Consequently, discretization cannot be made into a faithful functor.

Corollary. *Let $F\colon$ **Cstar** $\to$ **Wstar** be a functor, and $\eta_A\colon A \to F(A)$ natural unital $*$-homomorphisms. Suppose there are isomorphisms $F(C(X)) \cong \ell^\infty(X)$ for each compact Hausdorff space X that turn $\eta_{C(X)}$ into the inclusion $C(X) \to \ell^\infty(X)$. If a unital C*-algebra A has a unital $*$-homomorphism $f\colon B(H) \to A$ for an infinite-dimensional Hilbert space H, then $F(A) = 0$.*

The proof of the Theorem relies on the existence of normal states in W*-algebras. Intriguingly, this means that it does not rule out faithful functors F as above from **Cstar** to the category of AW*-algebras (see [12, Section 2] for the appropriate morphisms). A rather different approach to the problem of extending the embeddings $C(X) \to \ell^\infty(X)$ to noncommutative C*-algebras has recently appeared in [16]. We also remark that since the identity functor discretizes all finite-dimensional C*-algebras, this truly infinite-dimensional obstruction is independent of the Kochen-Specker theorem, a key ingredient in some previous spectral obstruction results [18, 4].

The rest of this note proves the Theorem and its Corollary.

2. Proof

We begin with a lemma that characterizes atomic measures. Let (X, Σ) be measurable space with a finite measure μ. Recall that an *atom* for μ is a measurable set $V \in \Sigma$ such that $\mu(V) > 0$ and for every measurable $U \subseteq V$, either $\mu(U) = 0$ or $\mu(U) = \mu(V)$. It follows that for every decomposition of V into a finite (or countably infinite) disjoint union of measurable sets $V = \bigsqcup V_i$, one of the V_i has measure $\mu(V)$ and the rest have measure zero.

The measure μ is said to be *diffuse* if it has no atoms, and *atomic* if every nonnegligible measurable set contains an atom. Define an *interval* for a finite measure μ on (X, Σ) to be a one-parameter family of measurable sets $U_t \in \Sigma$ with $t \in [0, M]$ for a positive real number M such that $s \leq t$ implies $U_s \subseteq U_t$ and $\mu(U_t) = t$ for all $s, t \in [0, M]$.

Lemma 1. *Let (X, Σ, μ) be a finite measure space. Then (X, Σ, μ) has an interval if and only if μ is not atomic.*

Proof. First suppose that μ is not atomic. Any finite measure μ decomposes uniquely as $\mu = \mu_a + \mu_d$ into an atomic measure μ_a and a diffuse measure μ_d [14, 2.6]. Moreover, μ_a and μ_d are *singular* [13, 3.3]. This means [8, p126] that (X, Σ, μ) is a disjoint union of an atomic measure space and a diffuse one. The latter is nonempty by assumption and we may restrict to it without loss of generality. But nonempty finite diffuse measure spaces always have an interval, see [2, Lemma 2.5] or [5, Lemma 4.1].

Now suppose that $\{U_t \mid t \in [0, M]\}$ is an interval in (X, Σ, μ). Scaling μ by $1/M$ and restricting to S_M, we may assume $M = 1$ and $U_1 = X$. For any positive integer n, the sets $K_1 = U_{1/n}$ and $K_i = (U_{i/n}) \setminus (U_{(i-1)/n})$ for $i = 2, \ldots, n$ partition

X into n disjoint subsets of measure $1/n$ each. If V were an atom of μ, because $V = \bigsqcup_n V \cap K_n$ it must be the case that $\mu(V) = \mu(V \cap K_i) \leq \mu(K_i) \leq 1/n$ for some i. As n was arbitrary, this means $\mu(V) = 0$. Thus μ is not atomic. □

Now let X be a compact Hausdorff space, and let ψ be a state on $C(X)$. We say that ψ is *atomic* if $\psi = \sum \lambda_\rho \rho$ for pure states ρ of $C(X)$ and nonnegative scalars λ_ρ with $\sum \lambda_\rho = 1$. The Riesz–Markov theorem shows that $\psi(f) = \int_X f \, d\mu$ for a unique regular Borel probability measure μ on X. Any atoms of such a measure μ must be singleton sets $\{x\}$ for $x \in X$ [15, 2.IV]. Note that the pure states ρ on $C(X)$ precisely correspond to Dirac measures δ_x for $x \in X$. Thus the state ψ is atomic if and only if the corresponding probability measure μ is atomic, in which case it has the form $\mu = \sum_{x \in X} \lambda_x \delta_x$ for scalars $\lambda_x \geq 0$ with $\sum \lambda_x = 1$.

For the separable Hilbert space $H = L^2[0,1]$, write $A = B(H)$ for the algebra of bounded operators on H, write $C = L^\infty[0,1]$ for the corresponding continuous maximal abelian subalgebra of A, and write D for the discrete maximal abelian subalgebra of A generated as a W*-algebra by the projections q_n onto the Fourier basis vectors $e_n = \exp(2\pi i n -)$ for $n \in \mathbb{Z}$.

Lemma 2. *Let $\psi \colon A \to \mathbb{C}$ be a state. If its restriction to D is pure, then its restriction to C cannot be atomic.*

Proof. By Kadison–Singer [17], a pure state on D extends *uniquely* to a state on A via the canonical conditional expectation $E \colon A \to D$ that sends an operator a to its diagonal part $\sum q_n a q_n$ with respect to the Fourier basis e_n. So $\psi = \psi \circ E$, as we assumed ψ to be pure on D. Letting p_t be the projection $\chi_{[0,t]}$ in C for $t \in [0,1]$:

$$\begin{aligned}
\langle p_t e_n, e_n \rangle &= \langle \chi_{[0,t]} \cdot \exp(2\pi i n -), \exp(2\pi i n -) \rangle \\
&= \int_0^1 \chi_{[0,t]}(x) \cdot e^{2\pi i n x} \cdot \overline{e^{2\pi i n x}} \, dx \\
&= \int_0^1 \chi_{[0,t]}(x) |e^{2\pi i n x}|^2 \, dx \\
&= \int_0^t 1 \, dx \\
&= t.
\end{aligned}$$

Thus $E(p_t) = \sum q_n p_t q_n = \sum \langle p_t e_n, e_n \rangle q_n = \sum t q_n = t \cdot 1_A$. It now follows that $\psi(p_t) = \psi(E(p_t)) = \psi(t \cdot 1_A) = t$.

Under an isomorphism $C \cong C(X)$ for a compact Hausdorff space X, the projections in the chain $\{p_t\}$ correspond to characteristic functions for clopen subsets $\{U_t\}$ of X and the state ψ corresponds to a state $f \mapsto \int_X f \, d\mu$ for some regular Borel measure μ on X. The condition $\psi(p_t) = t$ means $\mu(U_t) = \int \chi_{U_t} \, d\mu = t$, making $\{U_t \mid t \in [0,1]\}$ an interval of clopen sets in X. Lemma 1 implies that μ is not atomic, so ψ cannot be atomic. □

The first two lemmas suffice to establish the Theorem.

Proof of Theorem. Let M be a W*-subalgebra of $B(K)$ for a Hilbert space K. Write $C \cong C(X)$ and $D \cong C(Y)$ for compact Hausdorff spaces X and Y. The discretization $\phi \colon A \to M \subseteq B(K)$ is accompanied by the following commutative diagram.

$$\begin{array}{ccc}
C = L^\infty[0,1] \cong C(X) & \longrightarrow & \ell^\infty(X) \\
\downarrow & & \downarrow h \\
B(H) = A & \xrightarrow{\ \phi\ } & M \subseteq B(K) \\
\uparrow & & \uparrow g \\
D = \ell^\infty(\mathbb{Z}) \cong C(Y) & \longrightarrow & \ell^\infty(Y)
\end{array}$$

Given $y \in Y$, the atomic projection $\chi_{\{y\}} \in \ell^\infty(Y)$ has image $q_y = g(\chi_{\{y\}}) \in M$. Suppose for a contradiction that $q_y \neq 0$. Choose a unit vector $v_y \in K$ in its range. This induces a state $\psi_y(a) = \langle a v_y, v_y \rangle$ on A. For $d \in D$, considering $d \in C(Y) \subseteq \ell^\infty(Y)$ we have $d\chi_{\{y\}} = d(y)\chi_{\{y\}}$, and thus:

$$\begin{aligned}
\psi_y(d) &= \langle d v_y, v_y \rangle \\
&= \langle d q_y v_y, v_y \rangle \\
&= \langle d(y) q_y v_y, v_y \rangle \\
&= \langle d(y) v_y, v_y \rangle \\
&= d(y) \|v_y\|^2 \\
&= d(y).
\end{aligned}$$

That is, ψ_y restricts to the pure state $d \mapsto d(y)$ on D. It follows from Lemma 2 that ψ_y is *not* atomic on C.

On the other hand, for $x \in X$ consider the atomic projection $\chi_{\{x\}} \in \ell^\infty(X)$ and its image $p_x = h(\chi_{\{x\}}) \in M$. Since $\sum p_x = 1$, we can decompose $K = \bigoplus_x K_x$ along the ranges K_x of p_x. Write $v_y = \sum \lambda_x w_x$ for unit vectors $w_x \in K_x$ and $\lambda_x \in \mathbb{C}$ satisfying $\sum_x |\lambda_x|^2 = 1$. For $c \in C$, we have $c p_x = c(x) p_x$ (considering $c \in C(X) \subseteq \ell^\infty(X)$ as before) and $c w_x = c p_x w_x = c(x) w_x$, so that:

$$\begin{aligned}
\psi_y(c) &= \langle c v_y, v_y \rangle \\
&= \sum_{x,x'} \overline{\lambda_x} \lambda_{x'} \langle c w_x, w_{x'} \rangle \\
&= \sum_{x,x'} \overline{\lambda_x} \lambda_{x'} c(x) \langle w_x, w_{x'} \rangle \\
&= \sum_{x} |\lambda_x|^2 c(x).
\end{aligned}$$

Thus the restriction of ψ_y to C *is* an atomic state.

This is a contradiction, so every atomic projection $\chi_{\{y\}} \in \ell^\infty(Y)$ must have image $g(\chi_{\{y\}}) = q_y = 0$ in M. Hence the normal $*$-homomorphism $g \colon \ell^\infty(Y) \to M$ is the zero map. But then $1_M = \phi(1_A) = g(\eta(1_A)) = g(1_Y) = 0$, so $M = 0$. □

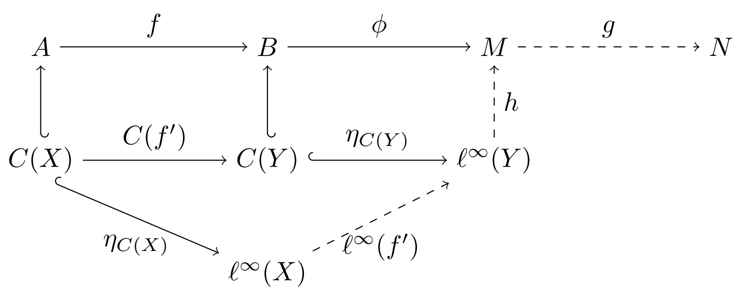
We thank an anonymous referee for informing us that the Theorem can be proved without the full force of Kadison–Singer, as follows. Identifying the algebra $C(\mathbb{T})$ of continuous functions on the unit circle $\mathbb{T}$ with the subalgebra of $C[0,1]$ satisfying $f(0) = f(1)$, it is known that $C(\mathbb{T})$ supports unique extensions of pure states of the discrete masa $D \subseteq B(H)$. (Indeed, the algebra of Fourier polynomials—or more generally, the Wiener algebra $A(\mathbb{T})$—is a dense subalgebra of $C(\mathbb{T})$ and lies in the algebra $M_0 \subseteq B(H)$ of operators that are l_1-bounded in the sense of Tanbay [20] with respect to the Fourier basis $\{e_n \mid n \in \mathbb{Z}\}$. Thus $C(\mathbb{T})$ lies in the norm closure

M of M_0, and the results of [20] imply that pure states on D extend uniquely to M.) A computation as in Lemma 2 shows that this extended state corresponds to the Lebesgue measure on $\mathbb{T}$, hence is not atomic on $C(\mathbb{T})$. The Theorem may now be proved in essentially the same manner, replacing the algebra C with $C(\mathbb{T})$.

The proof of the Corollary uses the following 'stability' of discretizations.

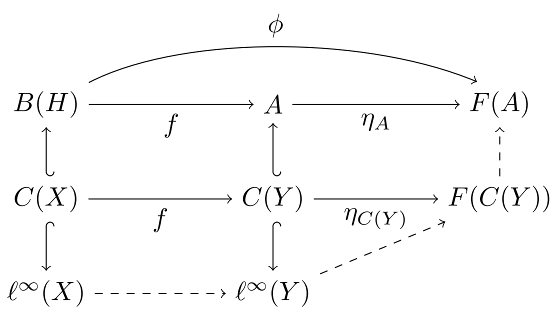
Lemma 3. *Discretizations are stable under precomposition with $*$-homomorphisms and postcomposition with normal $*$-homomorphisms: if $\phi\colon B \to M$ discretizes B, $f\colon A \to B$ is a morphism in* **Cstar***, and $g\colon M \to N$ is a morphism in* **Wstar***, then $g \circ \phi \circ f$ discretizes A.*

Proof. If $C(X) \cong C \subseteq A$ is a commutative C*-subalgebra, then so is $C(Y) \cong f[C] \subseteq B$, making the top squares of the following diagram commute (where $f'\colon Y \to X$ is a continuous function between compact Hausdorff spaces derived from $f\colon C \to f[C]$ via Gelfand duality).



The bottom triangle commutes by naturality of η. As all dashed arrows are normal, so is their composite. □

Proof of Corollary. We first prove that $\phi = \eta \circ f\colon B(H) \to F(A)$ is a discretization. If $C(X)$ is a commutative C*-subalgebra of $B(H)$, its image under f is a commutative C*-subalgebra of A and hence of the form $C(Y)$. Consider the following diagram.



The top-left square commutes by definition, and the top-right square commutes by naturality of η. The bottom-left square commutes by naturality of the inclusion $C(X) \hookrightarrow \ell^\infty(X)$, and the bottom-right triangle commutes by assumption. Finally, the dashed arrows are normal: the horizontal one because it is in the image of the functor ℓ^∞, the vertical one because it is in the image of the functor F, and the diagonal one because it is an isomorphism. Thus ϕ is a discretization.

Since H is infinite-dimensional, it is unitarily isomorphic to $L^2[0,1] \otimes H$. This gives rise to a unital $*$-homomorphism $i\colon B(L^2[0,1]) \to B(L^2[0,1]) \otimes B(H) \cong B(H)$

given by $i(a) = a \otimes 1$. Precomposing ϕ with this map induces a discretization $\phi \circ i\colon B(L^2[0,1]) \to F(A)$ according to Lemma 3, so the Theorem guarantees that $F(A) = 0$. □

We leave open whether there exists any state on $B(H)$ that restricts to an atomic state on each (maximal) abelian $*$-subalgebra.

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Department of Computer Science, University of Oxford, Wolfson Building, Parks Road, OX1 3QD, Oxford, UK

E-mail address: heunen@cs.ox.ac.uk

Department of Mathematics, Bowdoin College, Brunswick, ME 04011–8486, USA

E-mail address: reyes@bowdoin.edu