

SCHRÖDINGER OPERATORS DEFINED BY INTERVAL EXCHANGE TRANSFORMATIONS

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ABSTRACT. We discuss discrete one-dimensional Schrödinger operators whose potentials are generated by an invertible ergodic transformation of a compact metric space and a continuous real-valued sampling function. We pay particular attention to the case where the transformation is a minimal interval exchange transformation. Results about the spectral type of these operators are established. In particular, we provide the first examples of transformations for which the associated Schrödinger operators have purely singular spectrum for every non-constant continuous sampling function.

1. INTRODUCTION

Consider a probability space (Ω, μ) and an invertible ergodic transformation $T : \Omega \rightarrow \Omega$. Given a bounded measurable sampling function $f : \Omega \rightarrow \mathbb{R}$, one can consider discrete one-dimensional Schrödinger operators acting on $\psi \in \ell^2(\mathbb{Z})$ as

$$(1.1) \quad [H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n),$$

where $\omega \in \Omega$, $n \in \mathbb{Z}$, and

$$(1.2) \quad V_\omega(n) = f(T^n \omega).$$

Clearly, each H_ω is a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$.

It is often convenient to further assume that Ω is a compact metric space and μ is a Borel measure. In fact, one can essentially force this setting by mapping $\Omega \ni \omega \mapsto V_\omega \in \tilde{\Omega}$, where $\tilde{\Omega}$ is an infinite product of compact intervals. Instead of T , μ , and f one then considers the left shift, the push-forward of μ , and the function that evaluates at the origin. In the topological setting, it is natural to consider continuous sampling functions.

A fundamental result of Pastur [26] and Kunz-Souillard [22] shows that there are $\Omega_0 \subseteq \Omega$ with $\mu(\Omega_0) = 1$ and $\Sigma, \Sigma_{\text{pp}}, \Sigma_{\text{sc}}, \Sigma_{\text{ac}} \subset \mathbb{R}$ such that for $\omega \in \Omega_0$, we have $\sigma(H_\omega) = \Sigma$, $\sigma_{\text{pp}}(H_\omega) = \Sigma_{\text{pp}}$, $\sigma_{\text{sc}}(H_\omega) = \Sigma_{\text{sc}}$, and $\sigma_{\text{ac}}(H_\omega) = \Sigma_{\text{ac}}$. Here, $\sigma(H)$, $\sigma_{\text{pp}}(H)$, $\sigma_{\text{sc}}(H)$, and $\sigma_{\text{ac}}(H)$ denote the spectrum, the pure-point spectrum, the singular continuous spectrum, and the absolutely continuous spectrum of H , respectively.

A standard direct spectral problem is therefore the following: given the family $\{H_\omega\}_{\omega \in \Omega}$, identify the sets $\Sigma, \Sigma_{\text{pp}}, \Sigma_{\text{sc}}, \Sigma_{\text{ac}}$, or at least determine which of them are non-empty. There is a large literature on questions of this kind and the area has been especially active recently; we refer the reader to the survey articles [9, 14] and references therein.

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Naturally, inverse spectral problems are of interest as well. Generally speaking, given information about one or several of the sets Σ , Σ_{pp} , Σ_{sc} , Σ_{ac} , one wants to derive information about the operator family $\{H_\omega\}_{\omega \in \Omega}$. There has not been as much activity on questions of this kind. However, a certain inverse spectral problem looms large over the general area of one-dimensional ergodic Schrödinger operators:

Kotani-Last Conjecture. *If $\Sigma_{\text{ac}} \neq \emptyset$, then the potentials V_ω are almost periodic.*

A function $V : \mathbb{Z} \rightarrow \mathbb{R}$ is called almost periodic if the set of its translates $V_m(\cdot) = V(\cdot - m)$ is relatively compact in $\ell^\infty(\mathbb{Z})$. The Kotani-Last conjecture has been around for at least two decades and it has been popularized by several people, most recently by Damanik [9, Problem 2], Jitomirskaya [14, Problem 1], and Simon [28, Conjecture 8.9]. A weaker version of the conjecture is obtained by replacing the assumption $\Sigma_{\text{ac}} \neq \emptyset$ with $\Sigma_{\text{pp}} \cup \Sigma_{\text{sc}} = \emptyset$. A proof of either version of the conjecture would be an important result.

There is an equivalent formulation of the Kotani-Last conjecture in purely dynamical terms. Suppose Ω , μ , T , f are as above. For $E \in \mathbb{R}$, consider the map

$$A_E : \Omega \rightarrow \text{SL}(2, \mathbb{R}), \quad \omega \mapsto \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}.$$

This gives rise to the one-parameter family of $\text{SL}(2, \mathbb{R})$ -cocycles, $(T, A_E) : \Omega \times \mathbb{R}^2 \rightarrow \Omega \times \mathbb{R}^2$, $(\omega, v) \mapsto (T\omega, A_E(\omega)v)$. For $n \in \mathbb{Z}$, define the matrices A_E^n by $(T, A_E)^n = (T^n, A_E^n)$. Kingman's subadditive ergodic theorem ensures the existence of $L(E) \geq 0$, called the Lyapunov exponent, such that $L(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|A_E^n(\omega)\| d\mu(\omega)$. Then, the following conjecture is equivalent to the one above; see, for example, [9, Theorem 4].

Kotani-Last Conjecture – Dynamical Formulation. *Suppose the potentials V_ω are not almost periodic. Then, $L(E) > 0$ for Lebesgue almost every $E \in \mathbb{R}$.*

The existing evidence in favor of the Kotani-Last conjecture is that its claim is true in the large number of explicit examples that have been analyzed. On the one hand, there are several classes of almost periodic potentials which lead to (purely) absolutely continuous spectrum. This includes some limit-periodic cases as well as some quasi-periodic cases. It is often helpful to introduce a coupling constant and then work in the small coupling regime. For example, if α is Diophantine, T is the rotation of the circle by α , g is real-analytic, and $f = \lambda g$, then for λ small enough, $\Sigma_{\text{pp}} \cup \Sigma_{\text{sc}} = \emptyset$; see Bourgain-Jitomirskaya [5] and references therein for related results. On the other hand, all non-almost periodic operator families that have been analyzed are such that $\Sigma_{\text{ac}} = \emptyset$. For example, if α is irrational and T is the rotation of the circle by α , then the potentials $V_\omega(n) = f(\omega + n\alpha)$ are almost periodic if and only if f is continuous — and, indeed, Damanik and Killip have shown that $\Sigma_{\text{ac}} = \emptyset$ for discontinuous f (satisfying a weak additional assumption) [10].

In particular, we note how the understanding of an inverse spectral problem can be advanced by supplying supporting evidence on the direct spectral problem side.

This point of view will be taken and developed further in this paper. We wish to study operators whose potentials are not almost periodic, but are quite close to being almost periodic in a sense to be specified, and prove $\Sigma_{\text{ac}} = \emptyset$. In some sense, we will study a question that is dual to the Damanik-Killip paper. Namely, what can be said when f is nice but T is not? More precisely, can discontinuity of T be exploited in a similar way as discontinuity of f was exploited in [10]?

The primary example we have in mind is when the transformation T is an interval exchange transformation. Interval exchange transformations are important and extensively studied dynamical systems. An interval exchange transformation is obtained by partitioning the unit interval into finitely many half-open subintervals and permuting them. Rotations of the circle correspond to the exchange of two intervals. Thus, interval exchange transformations are natural generalizations of circle rotations.

Note, however, that they are in general discontinuous. Nevertheless, interval exchange transformations share certain basic ergodic properties with rotations. First, every irrational rotation is minimal, that is, all its orbits are dense. Interval exchange transformations are minimal if they obey the Keane condition, which requires that the orbits of the points of discontinuity are infinite and mutually disjoint, and which holds under a certain kind of irrationality assumption and in particular in Lebesgue almost all cases. Secondly, every irrational rotation is uniquely ergodic. This corresponds to the Masur-Veech result that (for fixed irreducible permutation) Lebesgue almost every interval exchange transformation is uniquely ergodic; see [24, 29]. A major difference between irrational rotations and a typical interval exchange transformation that is not a rotation is the weak mixing property. While irrational rotations are never weakly mixing, Avila and Forni have shown that (for every permutation that does not correspond to a rotation) almost every interval exchange transformation is weakly mixing [2].

This leads to an interesting general question: If the transformation T is weakly mixing, can one prove $\Sigma_{\text{ac}} = \emptyset$ for most/all non-constant f 's? The typical interval exchange transformations form a prominent class of examples. For these we are able to use weak mixing to prove the absence of absolutely continuous spectrum for Lipschitz functions:

Theorem 1.1. *Suppose T is an interval exchange transformation that satisfies the Keane condition and is weakly mixing. Then, for every non-constant Lipschitz continuous f and every ω , we have $\sigma_{\text{ac}}(H_\omega) = \emptyset$.*

For more general continuous functions, it is rather the discontinuity of an interval exchange transformation that we exploit and not the weak mixing property. We regard it as an interesting open question whether the Lipschitz assumption in Theorem 1.1 can be removed and the conclusion holds for every non-constant continuous f .

Our results will include the following theorem, which we state here because its formulation is simple and it identifies situations where the absence of absolutely continuous spectrum indeed holds for every non-constant continuous f .

Theorem 1.2. *Suppose $r \geq 3$ is odd and T is an interval exchange transformation that satisfies the Keane condition and reverses the order of the r partition intervals. Then, for every non-constant continuous f and every ω , we have $\sigma_{\text{ac}}(H_\omega) = \emptyset$.*

Remarks. (i) Reversal of order means that if we enumerate the partition intervals from left to right by $1, \dots, r$, then $i < j$ implies that the image of the i -th interval under T lies to the right of the image of the j -th interval. Note that our assumptions on T leave a lot of freedom for the choice of interval lengths. Indeed, the Keane condition holds for Lebesgue almost all choices of interval lengths.

(ii) We would like to emphasize here that *all* non-constant continuous sampling functions are covered by this result. Of course, if f is constant, the potentials V_ω are constant and the spectrum is purely absolutely continuous, so the result is best possible as far as generality in f is concerned.

(iii) Even more to the point, to the best of our knowledge, Theorem 1.2 is the first result of this kind, that is, one that identifies an invertible transformation T for which the absolutely continuous spectrum is empty for all non-constant continuous sampling functions.¹ While one should expect that it should be easier to find such a transformation among those that are strongly mixing, no such example is known. Our examples are all not strongly mixing (this holds for interval exchange transformations in general) [15].²

¹There are non-invertible examples, such as the doubling map, defining operators on $\ell^2(\mathbb{Z}_+)$. In these cases, the absence of absolutely continuous spectrum follows quickly from non-invertibility; see [11].

²After a preliminary version of this paper was posted, Svetlana Jitomirskaya explained to us how to prove the absence of absolutely continuous spectrum for the two-sided full shift and every non-constant continuous sampling function. Her proof is also based on Kotani theory.

(iv) In particular, this shows that interval exchange transformations behave differently from rotations in that regularity (or any other property) of f and sufficiently small coupling do not yield the existence of some absolutely continuous spectrum.

The organization of the paper is as follows. We collect some general results about the absolutely continuous spectrum of ergodic Schrödinger operators in Section 2. In Section 3, we recall the formal definition of an interval exchange transformation and the relevance of the Keane condition for minimality of such dynamical systems. The main result established in Section 4 is that for minimal interval exchange transformations and continuous sampling functions, the spectrum and the absolutely continuous spectrum of H_ω are globally independent of ω . Section 5 is the heart of the paper. Here we establish several sufficient conditions for the absence of absolutely continuous spectrum. Theorems 1.1 and 1.2 will be particular consequences of these results. Additionally, we establish a few results in topological dynamics. In Section 6 we use Gordon’s lemma to prove almost sure absence of eigenvalues for (what we call) Liouville interval exchange transformations. This is an extension of a result of Avron and Simon.

2. ABSOLUTELY CONTINUOUS SPECTRUM: KOTANI, LAST-SIMON, REMLING

In this section we discuss several important results concerning the absolutely continuous spectrum. The big three are Kotani theory [17, 18, 19, 20, 21], the semicontinuity result of Last and Simon [23], and Remling’s earthquake³ [27].

Kotani theory and Remling’s work provide ways to use the presence of absolutely continuous spectrum to “predict the future.” Kotani describes this phenomenon by saying that *all potentials leading to absolutely continuous spectrum are deterministic*, while Remling uses absolutely continuous spectrum to establish an *oracle theorem*. Last and Simon discuss the effect of taking limits of translates of a potentials on the absolutely continuous spectrum. Their result may also be derived within the context Remling is working in; see [27] for the derivation.

Let us return to our general setting where (Ω, μ, T) ergodic is given and, for a bounded measurable function $f : \Omega \rightarrow \mathbb{R}$, the potentials V_ω and operators H_ω are defined by (1.2) and (1.1), respectively. We first recall a central result from Kotani theory. It is convenient to pass to the topological setting described briefly in the introduction. Choose a compact interval I that contains the range of f and set $\tilde{\Omega} = I^{\mathbb{Z}}$, equipped with the product topology. The shift transformation $\tilde{T} : \tilde{\Omega} \rightarrow \tilde{\Omega}$ is given by $[\tilde{T}\tilde{\omega}](n) = \tilde{\omega}(n + 1)$. It is clearly a homeomorphism. Consider the measurable map $\mathcal{K} : \Omega \rightarrow \tilde{\Omega}$, $\omega \mapsto V_\omega$. The push-forward of μ by the map \mathcal{K} will be denoted by $\tilde{\mu}$. Finally, the function $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{R}$ is given by $\tilde{f}(\tilde{\omega}) = \tilde{\omega}(0)$.

We obtain a system $(\tilde{\Omega}, \tilde{\mu}, \tilde{T}, \tilde{f})$ that generates Schrödinger operators $\{\tilde{H}_{\tilde{\omega}}\}_{\tilde{\omega} \in \tilde{\Omega}}$ with potentials $\tilde{V}_{\tilde{\omega}}(n) = \tilde{f}(\tilde{T}^n \tilde{\omega})$ in such a way that the associated sets $\tilde{\Sigma}, \tilde{\Sigma}_{\text{pp}}, \tilde{\Sigma}_{\text{sc}}, \tilde{\Sigma}_{\text{ac}}$ coincide with the original sets $\Sigma, \Sigma_{\text{pp}}, \Sigma_{\text{sc}}, \Sigma_{\text{ac}}$, but they come from a homeomorphism \tilde{T} and a continuous function \tilde{f} .

The following theorem shows that the presence of absolutely continuous spectrum implies (continuous) determinism. For proofs, see [21] and [9], but its statement and history can be traced back to the earlier Kotani papers [17, 18, 19, 20].

Theorem 2.1 (Kotani’s Continuous Extension Theorem). *Suppose $\Sigma_{\text{ac}} \neq \emptyset$. Then, the restriction of any $\tilde{\omega} \in \text{supp } \tilde{\mu}$ to either \mathbb{Z}_+ or \mathbb{Z}_- determines $\tilde{\omega}$ uniquely among elements of $\text{supp } \tilde{\mu}$ and the extension map $\mathcal{E} : \text{supp } \tilde{\mu}|_{\mathbb{Z}_-} \rightarrow \text{supp } \tilde{\mu}$, $\tilde{\omega}|_{\mathbb{Z}_-} \rightarrow \tilde{\omega}$ is continuous.*

Another useful result, due to Last and Simon [23], is that the absolutely continuous spectrum cannot shrink under pointwise approximation by translates:

³For lack of a better term, we quote Barry Simon here (OPSF9, Luminy, France, July 2007).

Theorem 2.2 (Last-Simon Semicontinuity). *Suppose that $\omega_1, \omega_2 \in \Omega$ and $n_k \rightarrow \infty$ are such that the potentials $V_{T^{n_k}\omega_1}$ converge pointwise to V_{ω_2} as $k \rightarrow \infty$. Then, $\sigma_{\text{ac}}(H_{\omega_1}) \subseteq \sigma_{\text{ac}}(H_{\omega_2})$.*

We will furthermore need the following proposition, which follows from strong convergence.

Proposition 2.3. *Suppose that $\omega_1, \omega_2 \in \Omega$ and $n_k \rightarrow \infty$ are such that the potentials $V_{T^{n_k}\omega_1}$ converge pointwise to V_{ω_2} as $k \rightarrow \infty$. Then, $\sigma(H_{\omega_1}) \supseteq \sigma(H_{\omega_2})$.*

Theorems 2.1, 2.2 and Proposition 2.3 will be sufficient for our purpose. We do want to point out, however, that all the results that concern the absolutely continuous spectrum, and much more, are proved by completely different methods in Remling's paper [27] — hence the term *earthquake*.

3. INTERVAL EXCHANGE TRANSFORMATIONS

In this section we give a precise definition of an interval exchange transformation and gather a few known facts about these maps. As a general reference, we recommend Viana's survey [31].

Definition 3.1. *Suppose we are given an integer $r \geq 2$, a permutation $\pi : \{1, \dots, r\} \rightarrow \{1, \dots, r\}$, and an element $\Lambda \in \Delta^r = \{(\lambda_1, \dots, \lambda_r), \lambda_i > 0, \sum_{j=1}^r \lambda_j = 1\}$. Let $\Omega = [0, 1)$ be the half-open unit interval with the topology inherited from \mathbb{R} . Then the interval exchange transformation $T : \Omega \rightarrow \Omega$ associated with (π, Λ) is given by $T : I_j \ni \omega \mapsto \omega - \sum_{i=1}^{j-1} \lambda_i + \sum_{i, \pi(i) < \pi(j)} \lambda_i$, where $I_j = \sum_{i=1}^{j-1} \lambda_i + [0, \lambda_j)$.*

Remarks. (i) If $r = 2$ and $\pi(1) = 2, \pi(2) = 1$, then T is conjugate to $\tilde{T} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}, \tilde{\omega} \mapsto \tilde{\omega} + \lambda_2$, that is, a rotation of the circle. More generally, the same is true whenever $\pi(j) - 1 = j + k \pmod{r}$ for some k . In this sense, interval exchange transformations are natural generalizations of rotations of the circle. We call π with this property of rotation class.

(ii) The permutation π is called reducible if there is $k < r$ such that $\pi(\{1, \dots, k\}) = \{1, \dots, k\}$ and irreducible otherwise. If π is reducible, T splits into two interval exchange transformations. For this reason, it is natural to only consider irreducible permutations.

Definition 3.2. *The interval exchange transformation associated with (π, Λ) satisfies the Keane condition if the orbits of the left endpoints of the intervals I_1, \dots, I_r are infinite and mutually disjoint.*

The following result is [31, Proposition 3.2,4.1]):

Proposition 3.3 (Keane). *If π is irreducible and Λ is rationally independent, then the interval exchange transformation associated with (π, Λ) satisfies the Keane condition.*

If an interval exchange transformation satisfies the Keane condition, then it is minimal. More precisely, for every $\omega \in \Omega$, the set $\{T^n\omega : n \geq 1\}$ is dense in Ω .

On the other hand, the Keane condition is not necessary for minimality; see [31, Remark 4.5] for an example. For a more detailed discussion, see [4, Section 2].

Lemma 3.4. *Assume that T is minimal. Given any $N \geq 1, \omega, \omega' \in \Omega, \epsilon > 0$, there exists $l \in \mathbb{Z}$ such that*

$$(3.1) \quad |T^m(\omega) - T^{m+l}(\omega')| < \epsilon$$

for any $-N \leq m \leq N$.

Proof. Since the set of discontinuities of $T^j, -N \leq j \leq N$ is discrete in $[0, 1)$ and we have chosen T to be right continuous, we can find an interval $[a, b) \subseteq [0, 1)$ containing ω such that T^j are isometries on $[a, b)$ for $-N \leq j \leq N$. Hence, for any $\tilde{\omega} \in I := [a, b) \cap (\omega - \epsilon, \omega + \epsilon)$, (3.1) holds. Since T is minimal and I has non-empty interior, we can find l such that $T^l\omega' \in I$, finishing the proof. \square

4. SOME FACTS ABOUT SCHRÖDINGER OPERATORS GENERATED BY INTERVAL EXCHANGE TRANSFORMATIONS

In this section, we prove some general results for potentials and operators generated by minimal interval exchange transformations. First, we show that all potentials belong to the support of the induced measure and then we use this observation to show that the absolutely continuous spectrum is globally constant, not merely almost surely with respect to a given ergodic measure. Note that there are minimal interval exchange transformations that admit several ergodic measures.

Lemma 4.1. *Suppose that T is a minimal interval exchange transformation and μ is T -ergodic. Then, for every continuous sampling function f , we have $\{V_\omega : \omega \in \Omega\} \subseteq \text{supp } \tilde{\mu}$, where $\tilde{\mu}$ is as defined in Section 2.*

Proof. Note first that since $\text{supp } \mu$ is closed, non-empty, and T -invariant, it must equal all of Ω because T is minimal. Now given any $\omega \in \Omega$, we wish to show that $V_\omega \in \text{supp } \tilde{\mu}$. Consider the sets $[\omega, \omega + \varepsilon)$ for $\varepsilon > 0$ small. Since $\text{supp } \mu = \Omega$, $\mu([\omega, \omega + \varepsilon)) > 0$ and hence there exists $\omega_\varepsilon \in [\omega, \omega + \varepsilon)$ such that $V_{\omega_\varepsilon} \in \text{supp } \tilde{\mu}$. It follows that there is a sequence ω_n which converges to ω from the right and for which $V_{\omega_n} \in \text{supp } \tilde{\mu}$. Since T is right-continuous and f is continuous, it follows that $V_{\omega_n} \rightarrow V_\omega$ pointwise as $n \rightarrow \infty$. Since $\text{supp } \tilde{\mu}$ is closed, it follows that $V_\omega \in \text{supp } \tilde{\mu}$. \square

Theorem 4.2. *Suppose that T is a minimal interval exchange transformation. Then, for every continuous sampling function f , we have that $\sigma(H_\omega)$ and $\sigma_{\text{ac}}(H_\omega)$ are independent of ω .*

Proof. We first show that for $\omega, \omega' \in \Omega$, we have $\sigma(H_\omega) = \sigma(H_{\omega'})$. By Lemma 3.4 and f being continuous, we can find a sequence n_k such that $\sup_{|l| \leq k} |f(T^{n_k+l}\omega) - f(T^l\omega')| \leq \frac{1}{k}$. Hence $V_{T^{n_k}\omega}$ converges pointwise to $V_{\omega'}$. Now Proposition 2.3 implies that $\sigma(H_\omega) = \sigma(H_{T^{n_k}\omega}) \supseteq \sigma(H_{\omega'})$. Interchanging the roles of ω and ω' now finishes the proof. To prove the claim about the absolutely continuous spectrum, repeat the above argument replacing Proposition 2.3 by Theorem 2.2 to obtain $\sigma_{\text{ac}}(H_\omega) = \sigma_{\text{ac}}(H_{T^{n_k}\omega}) \subseteq \sigma_{\text{ac}}(H_{\omega'})$. This implies ω independence of $\sigma_{\text{ac}}(H_\omega)$. \square

5. ABSENCE OF ABSOLUTELY CONTINUOUS SPECTRUM FOR SCHRÖDINGER OPERATORS GENERATED BY INTERVAL EXCHANGE TRANSFORMATIONS

In this section we identify situations in which the absolutely continuous spectrum can be shown to be empty. Particular consequences of the results presented below are the sample theorems stated in the introduction, Theorems 1.1 and 1.2. We will make explicit later in this section how these theorems follow from the results obtained here.

Let us first establish the tool we will use to exclude absolutely continuous spectrum.

Lemma 5.1. *Suppose T is a minimal interval exchange transformation and f is a continuous sampling function. Assume further that there are points $\omega_k, \hat{\omega}_k \in \Omega$ such that for some $N \geq 1$, we have*

$$(5.1) \quad \limsup_{k \rightarrow \infty} |f(T^N \omega_k) - f(T^N \hat{\omega}_k)| > 0,$$

and for every $n \leq 0$, we have

$$(5.2) \quad \lim_{k \rightarrow \infty} |f(T^n \omega_k) - f(T^n \hat{\omega}_k)| = 0.$$

Then, $\sigma_{\text{ac}}(H_\omega) = \emptyset$ for every $\omega \in \Omega$.

Proof. By passing to a subsequence of ω_k , we can assume that the limit in (5.1) exists. Fix any T -ergodic measure μ and assume that the corresponding almost sure absolutely continuous spectrum, Σ_{ac} , is non-empty. By Lemma 4.1, the potentials corresponding to the ω_k 's and $\hat{\omega}_k$'s belong to $\text{supp } \tilde{\mu}$. Then, by Theorem 2.1, the map from the restriction of elements of $\text{supp } \tilde{\mu}$ to their extensions is continuous. Moreover, since the domain of this map is a compact metric space, the map is uniformly

continuous. Combining this with (5.1) and (5.2), we arrive at a contradiction. Thus, the μ -almost sure absolutely continuous spectrum is empty. This implies the assertion because Theorem 4.2 shows that the absolutely continuous spectrum is independent of the parameter. \square

Our first result on the absence of absolutely continuous spectrum works for all permutations, but we have to impose a weak assumption on f .

Theorem 5.2. *Suppose T is an interval exchange transformation that satisfies the Keane condition. If f is a continuous sampling function such that there are $N \geq 1$ and $\omega_d \in (0, 1)$ with*

$$(5.3) \quad \lim_{\omega \uparrow \omega_d} f(T^N \omega) \neq \lim_{\omega \downarrow \omega_d} f(T^N \omega),$$

then $\sigma_{\text{ac}}(H_\omega) = \emptyset$ for every $\omega \in \Omega$. In other words, if f is continuous and $f \circ T^n$ is discontinuous for some $n \geq 1$, then $\sigma_{\text{ac}}(H_\omega) = \emptyset$ for every $\omega \in \Omega$.

Proof. Since f is continuous, it follows from (5.3) that ω_d is a discontinuity point of T^N . Since T satisfies the Keane condition, this implies that T^n , $n \leq 0$, are all continuous at ω_d . Thus, choosing a sequence $\omega_k \uparrow \omega_d$ and a sequence $\hat{\omega}_k \downarrow \omega_d$, we obtain the conditions (5.1) and (5.2). Thus, the theorem follows from Proposition 3.3 and Lemma 5.1. \square

Remarks. (i) If π is irreducible and not of rotation class, then the interval exchange transformation T has discontinuity points. Fix any such point ω_d and consider the continuous functions f that obey

$$(5.4) \quad \lim_{\omega \uparrow \omega_d} f(T\omega) \neq \lim_{\omega \downarrow \omega_d} f(T\omega).$$

This is clearly an open and dense set, even in more restrictive categories such as finitely differentiable or infinitely differentiable functions. Thus, the condition on f in Theorem 5.2 is rather weak.

(ii) If the permutation π is such that T is topologically conjugate to a rotation of the circle, then there is only one discontinuity point and the condition (5.4) forces f to be discontinuous if regarded as a function on the circle! Thus, in this special case, Theorem 5.2 only recovers a result of Damanik and Killip from [10].

Even though the condition on f in Theorem 5.2 is weak, it is of course of interest to identify cases where no condition (other than non-constancy) has to be imposed at all. The following definition will prove to be useful in this regard.

Definition 5.3. *We define a finite directed graph $G = (V, E)$ associated with an interval exchange transformation T coming from data (π, Λ) with π irreducible. Denote by ω_i the right endpoint of I_j , $1 \leq j \leq r - 1$. Set $V = \{\omega_1, \dots, \omega_{r-1}\} \cup \{0, 1\}$. The edge set E is defined as follows. Write $T_- \hat{\omega} = \lim_{\omega \uparrow \hat{\omega}} T\omega$ and $T_+ \hat{\omega} = \lim_{\omega \downarrow \hat{\omega}} T\omega$. Then there is an edge from vertex v_1 to vertex v_2 if $T_- v_1 = T_+ v_2$. In addition, there are two special edges, \tilde{e}_1 and \tilde{e}_2 . The edge \tilde{e}_1 starts at 0 and ends at ω_j for which $T_+ \omega_j = 0$, and the edge \tilde{e}_2 ends at 1 and starts at ω_k where $T_- \omega_k = 1$.*

Remarks. (i) Irreducibility of π ensures that the ω_j and ω_k that enter the definition of the special edges actually exist.

(ii) Note that the graph G depends only on π , that is, it is independent of Λ . Indeed,

$$T_+(\omega_j) = \begin{cases} 0 & \pi(j+1) = 1 \\ T_-(1) & \pi(j+1) - 1 = \pi(r) \\ T_-(\omega_k) & \pi(j+1) - 1 = \pi(k) \end{cases} \quad \text{and} \quad T_-(\omega_j) = \begin{cases} 1 & \pi(j) = r \\ T_+(0) & \pi(j) + 1 = \pi(1) \\ T_+(\omega_k) & \pi(j) + 1 = \pi(k+1) \end{cases}$$

for $j = 1, \dots, r - 1$.

(iii) T is defined to be right continuous, so we could simply replace $T_+ \hat{\omega}$ by $T\hat{\omega}$. In order to emphasize that the discontinuity of T is important, we did not do so in the above definition.

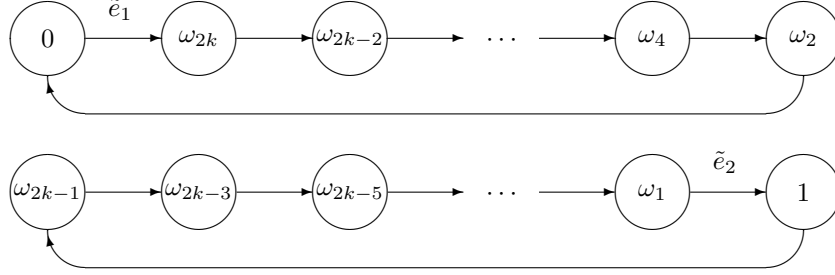
Examples. In these examples, we will denote permutations π as follows:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & r \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(r) \end{pmatrix}.$$

(i) Here are the permutations relevant to Theorem 1.2:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & 2k & 2k+1 \\ 2k+1 & 2k & 2k-1 & \cdots & 2 & 1 \end{pmatrix}.$$

The corresponding graph

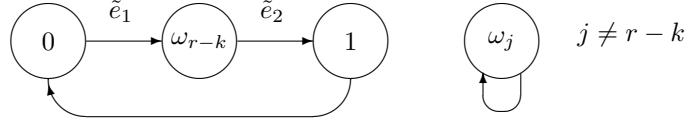


consists of two cycles, each of which contains exactly one special edge.

(ii) For the permutations

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & r-1 & r \\ r-k+1 & r-k+2 & \cdots & r & 1 & \cdots & r-k-1 & r-k \end{pmatrix},$$

which correspond to rotations of the circle, the graph looks as follows:



It consists of several cycles, each of which contains either no special edge or two special edges.

Theorem 5.4. *Suppose the permutation π is irreducible and the associated graph G has a cycle that contains exactly one special edge. Assume furthermore that Λ is such that the interval exchange transformation T associated with (π, Λ) satisfies the Keane condition. Then, for every continuous non-constant sampling function f , we have $\sigma_{ac}(H_\omega) = \emptyset$ for every $\omega \in \Omega$.*

Proof. Our goal is to show that under the assumptions of the theorem, every non-constant continuous sampling function satisfies (5.3) for $N \geq 1$ and $\omega_d \in (0, 1)$ suitable and hence the assertion follows from Theorem 5.2. So let us assume that f is continuous and (5.3) fails for all $N \geq 1$ and $\omega_d \in (0, 1)$. We will show that f is constant. If there is a non-special edge from v_1 to v_2 , where $v_1 \neq v_2$, the Keane condition implies that for every $n \geq 1$, T^n will be continuous at the point $T_-v_1 = T_+v_2$ and hence $T_-^n v_1 = T_+^n v_2$ for every $n \geq 1$, where we generalize the previous definition as follows, $T_-^n \hat{\omega} = \lim_{\omega \uparrow \hat{\omega}} T^n \omega$ and $T_+^n \hat{\omega} = \lim_{\omega \downarrow \hat{\omega}} T^n \omega$. Next let us consider the special edges. For the edge \tilde{e}_1 from 0 to ω_j (for j suitable), we have by construction and the Keane condition again that $T_+^{n+1} \omega_j = T_+^n 0$ for $n \geq 1$. Likewise, for the special edge \tilde{e}_2 from ω_k to 1 (for k suitable), we have $T_-^{n+1} \omega_k = T_-^n 1$ for $n \geq 1$. By our assumption on f , the function $\omega \mapsto f(T^n \omega)$ is continuous for every $n \geq 0$. In particular, we have $f(T_-^n \omega_m) = f(T_+^n \omega_m)$ for every $n \geq 0$ and $1 \leq m \leq r-1$. Consequently, if there is a path containing no special edges connecting vertex v_1 to vertex v_2 , then $f(T_\pm^n(v_1)) = f(T_\pm^n(v_2))$. On the other hand, if there is a path containing just \tilde{e}_1 (but not \tilde{e}_2) from

v_1 to v_2 , then $T_{\pm}v_1 = T_{\pm}^2v_2$. Likewise, if there is a path containing just \tilde{e}_2 from v_1 to v_2 , then $T_{\pm}^2(v_1) = T_{\pm}(v_2)$. Putting all these observations together, we may infer that if there is a cycle containing exactly one special edge and a vertex ω_j , then $f(T^n\omega_j) = f(T^{n+1}\omega_j)$ for every $n \geq 1$. The Keane condition implies that the set $\{T^n\omega_j : n \geq 1\}$ is dense; see Proposition 3.3. Thus, the continuous function f is constant on a dense set and therefore constant. \square

Proof of Theorem 1.2. Example (i) above shows that Theorem 1.2 is a special case of Theorem 5.4. \square

Remarks. (i) Of course, it is necessary to impose some condition on the permutation π aside from irreducibility in Theorem 5.4. It is known, and discussed in the introduction, that the assertion of Theorem 5.4 fails for permutations corresponding to rotations. Example (ii) above shows the graph associated with such permutations and demonstrates in which way the assumption of Theorem 5.4 fails in these cases.

(ii) There is another, more direct, way of seeing that Theorem 5.2 is not always applicable. Suppose that the interval exchange transformation T is not topologically weakly mixing. This means that there is a non-constant continuous function $g : \Omega \rightarrow \mathbb{C}$ and a real α such that $g(T\omega) = e^{i\alpha}g(\omega)$ for every $\omega \in \Omega$. At least one of $\Re g$ and $\Im g$ is non-constant. Consider the case where $\Re g$ is non-constant, the other case is analogous. Thus, $f = \Re g$ is a non-constant continuous function from Ω to \mathbb{R} such that $f(T^N\omega) = \Re g(T^N\omega) = \Re (e^{i\alpha}g(T^{N-1}\omega)) = \dots = \Re (e^{iN\alpha}g(\omega))$. Since the right-hand side is continuous in ω , so is the left-hand side, and hence (5.3) fails for all $N \geq 1$ and $\omega_d \in (0, 1)$. Nogueira and Rudolph proved that for every permutation π that does not generate a rotation and Lebesgue almost every Λ , the transformation T generated by (π, Λ) is topologically weakly mixing [25]. This result was strengthened, as was pointed out earlier, by Avila and Forni [2]. However, there are cases where topological weak mixing fails (see, e.g., [13]) and hence the argument above applies in these cases.

This example is even more striking, since if we write $g(\omega) = \lambda e^{i\theta}$, we see that $f(T^N\omega) = \lambda \Re(e^{iN\alpha+i\theta}) = \lambda \cos(\theta + N\alpha)$. Hence the operator defined by (1.1) is the almost Mathieu operator and it has purely absolutely continuous spectrum for $|\lambda| < 2$ for α irrational, and for all λ if α is rational; see [1] and references therein for earlier partial results.

Let us briefly discuss the results above from a dynamical perspective. To do so, we recall the definition of a *Type W* permutation; compare [7, Definition 3.2].

Definition 5.5. *Suppose π is an irreducible permutation of r symbols. Define inductively a sequence $\{a_k\}_{k=0,\dots,s}$ as follows. Set $a_0 = 1$. If $a_k \in \{\pi^{-1}(1), r+1\}$, then set $s = k$ and stop. Otherwise let $a_{k+1} = \pi^{-1}(\pi(a_k) - 1) + 1$. The permutation π is of *Type W* if $a_s = \pi^{-1}(1)$.*

It is shown in [7, Lemma 3.1] that the process indeed terminates and hence a_s is well defined. Observing [7, Theorem 3.5], [30, Theorem 1.11] then reads as follows:

Theorem 5.6 (Veech 1984). *If π is of Type W, then for almost every Λ , the interval exchange transformation associated with (π, Λ) is weakly mixing.*

We have the following related result, which has a weaker conclusion but a weaker, and in particular explicit, assumption:

Corollary 5.7. *Every interval exchange transformation satisfying the Keane condition and associated to a Type W permutation is topologically weakly mixing.*

Proof. We claim that π is Type W if and only if the associated graph G has a cycle that contains exactly one special edge. Assuming this claim for a moment, we can complete the proof as follows. Suppose Λ is such that the Keane condition holds for the interval exchange transformation T associated with (π, Λ) . We may infer from the proof of Theorem 5.4 that for every non-constant continuous function

f , there is an $n \geq 1$ such that $f \circ T^n$ is discontinuous. This shows that T does not have any continuous eigenfunctions and hence it is topologically weakly mixing. Thus, it suffices to prove the claim. Note that the procedure generating the sequence $\{a_k\}_{k=0, \dots, s}$ is the same as determining the arrows in the cycle containing the vertex 0 in the graph G associated with π . Thus, if π is not Type W, then before we close the cycle containing the vertex 0, it is identified to the vertex 1. This necessitates that this cycle contains both special edges. On the other hand, if π is Type W, then the cycle containing the vertex 0 closes up with just one special edge. \square

Recall now that a continuous map $S : Y \rightarrow Y$ is said to be a topological factor of $T : [0, 1) \rightarrow [0, 1)$ if there is a surjective continuous map $\pi : [0, 1) \rightarrow Y$ such that $\pi \circ T = S \circ \pi$. T is called topologically prime if it has no topological factors. If T has a factor, we can find a continuous function $f : [0, 1) \rightarrow \mathbb{R}$ such that also $f \circ T^n$ is continuous, by setting $f = g \circ \pi$ for some continuous $g : Y \rightarrow \mathbb{R}$. Hence, Theorem 5.4 also has the following consequence:

Corollary 5.8. *Every interval exchange transformation satisfying the Keane condition and associated to a Type W permutation is topologically prime.*

Let us now return to our discussion of sufficient conditions for the absence of absolutely continuous spectrum, presenting results that hold under additional assumptions on the sampling function f .

Theorem 5.9. *Assume that the graph G contains a path with ℓ vertices corresponding to distinct discontinuities. Then if the continuous function f is such that for every $x \in \mathbb{R}$, the set $f^{-1}(\{x\})$ has at most $\ell - 1$ elements, we have $\Sigma_{ac} = \emptyset$ for any T satisfying the Keane condition.*

Proof. Denote by $\tilde{\Omega}$ the set of ℓ distinct vertices, from the assumption of the theorem. If (5.3) fails, we see as in the proof of the last theorem that $f(\omega) \in \{f(\omega'), f(T^{-1}\omega'), f(T\omega')\}$ for $\omega, \omega' \in \tilde{\Omega}$. By the Keane condition this contradicts our assumption of the preimage of a single point under f containing a maximum of $\ell - 1$ points. \square

Theorem 5.10. *Suppose that f is non-constant and Lipschitz continuous and T is an interval exchange transformation for which $f \circ T^n$ is continuous for every n . Then, T is not measure theoretically weakly mixing.*

Proof. By assumption f satisfies $|f(x) - f(y)| \leq k|x - y|$ for all x, y . Assume that $f \circ T^n$ is continuous for every n . Consider $x, y \in \Omega$ with $x < y$. Denote by $\delta_1 < \dots < \delta_t$ the discontinuities of T^n between x and y . Then,

$$\begin{aligned} |f(T^n(x)) - f(T^n(y))| &\leq |f(T^n(x)) - f(T_-^n(\delta_1))| + |f(T_-^n(\delta_1)) - f(T_+^n(\delta_1))| \\ &\quad + |f(T_+^n(\delta_1)) - f(T_-^n(\delta_2))| + \dots + |f(T_+(\delta_t)) - f(T^n(y))| \\ &= |f(T^n(x)) - f(T_-^n(\delta_1))| + |f(T_+^n(\delta_1)) - f(T_-^n(\delta_2))| \\ &\quad + \dots + |f(T_+(\delta_t)) - f(T^n(y))| \\ &\leq k(|T^n(x) - T_-^n(\delta_1)| + |T_+^n(\delta_1) - T_-^n(\delta_2)| \\ &\quad + \dots + |T_+(\delta_t) - T^n(y)|) \\ &= k(|x - \delta_1| + |\delta_1 - \delta_2| + \dots + |\delta_t - y|) = k(y - x), \end{aligned}$$

where we used in the first equality that $f(T^n(\delta_i)) = f(T_+^n(\delta_i))$ since $f \circ T^n$ is continuous, in the second inequality that f is k -Lipschitz, and in the second equality that T^n is an isometry on (δ_i, δ_{i+1}) . So we have seen that, we also have that $|f(T^n x) - f(T^n y)| \leq k|x - y|$. We now proceed to show that T cannot be weak mixing. Since f is continuous and non-constant, there exist $c > 1$, E , d , such that $\frac{1}{10} < \text{Leb}(\{x : |f(x) - E| < d\}) < \text{Leb}(\{x : |f(x) - E| < cd\}) < \frac{1}{2}$. For any ω with

$T^n(\omega) \in \{x : |f(x) - z| < d\}$ and any $\tilde{\omega} \in B(\omega, \frac{(c-1)d}{k})$, we compute

$$|f(T^n \tilde{\omega}) - E| \leq |f(T^n \omega) - E| + |f(T^n \omega) - f(T^n \tilde{\omega})| < d + k \cdot \frac{(c-1)d}{k} = cd,$$

where we used that $f \circ T^n$ is k -Lipschitz continuous by the last lemma. So, we have that $T^n \tilde{\omega} \in \{x : |f(x) - E| < cd\}$. This implies that for the set $\left\{(\omega, \tilde{\omega}) : \tilde{\omega} \in B\left(\omega, \frac{(c-1)d}{k}\right)\right\} \subseteq \Omega \times \Omega$, which has positive measure, we have that its image under $T^n \times T^n$ never intersects the set $\{\omega : |f(\omega) - E| < d\} \times \{\tilde{\omega} : |f(\tilde{\omega}) - E| \geq cd\}$, which also has positive measure. Hence, $T \times T$ cannot be ergodic, and so T is not weakly mixing. \square

Remark. We only use that T is a piecewise isometry and $f \circ T^n$ is continuous for all n . In particular, the argument extends to rectangle exchange transformations and interval translation maps.

Proof of Theorem 1.1. The result follows from Theorems 5.2 and 5.10. \square

Let us say that a function f has a non-degenerate maximum at ω^{\max} if for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$(5.5) \quad f(\omega) \geq f(\omega^{\max}) - \delta \Rightarrow |\omega - \omega^{\max}| \leq \varepsilon.$$

Theorem 5.11. *Let π be a non-trivial permutation. Assume that $T = (\pi, \Lambda)$ satisfies the Keane condition and that f has a non-degenerate maximum at ω^{\max} and a continuous bounded derivative. Then, (5.3) holds. In particular, we have $\sigma_{\text{ac}}(H_\omega) = \emptyset$ for every $\omega \in \Omega$.*

Proof. First, it follows from Keane's condition that we can assume (by possibly replacing T with T^{-1}) that ω^{\max} is not a discontinuity point of T^n , $n \geq 1$. Assume that (5.3) fails, and fix a discontinuity point ω^{disc} of T . Our goal is to show that f is constant, which of course contradicts the assumption that the maximum of f at ω^{\max} is non-degenerate. We do this by showing that $f'(\tilde{\omega}) = 0$ for every $\tilde{\omega} \in \Omega$. So let $\tilde{\omega} \in \Omega$ be given. By Proposition 3.3, there is a sequence $l_k \rightarrow \infty$ such that $T^{l_k} \omega^{\max} \rightarrow \tilde{\omega}$. By (5.5), we can find $\delta_k > 0$ such that $|f(\omega) - f(\omega^{\max})| \leq \delta_k$ implies

$$(5.6) \quad |\omega - \omega^{\max}| \leq \frac{1}{k}.$$

Applying Proposition 3.3 again, we can find a sequence $n_k \rightarrow \infty$ such that $|T^{n_k} \omega^{\text{disc}} - \omega^{\max}| \leq \frac{\delta_k}{2\|f'\|_\infty}$ and hence $|f(T^{n_k} \omega^{\text{disc}}) - f(\omega^{\max})| \leq \frac{\delta_k}{2}$ for every $k \geq 1$. Since (5.3) fails, we can find $\varepsilon_k > 0$ such that $|f(T^{n_k} \omega) - f(T^{n_k} \omega^{\text{disc}})| \leq \frac{\delta_k}{2}$ for every $\omega \in (\omega^{\text{disc}} - \varepsilon_k, \omega^{\text{disc}})$. Combining these two estimates, we find $|f(T^{n_k} \omega) - f(\omega^{\max})| \leq \delta_k$, $\omega \in (\omega^{\text{disc}} - \varepsilon_k, \omega^{\text{disc}})$. Thus, by (5.6), $|T^{n_k} \omega - \omega^{\max}| \leq \frac{1}{k}$, $\omega \in (\omega^{\text{disc}} - \varepsilon_k, \omega^{\text{disc}})$. Hence, for any choice of $\omega_k \in (\omega^{\text{disc}} - \varepsilon_k, \omega^{\text{disc}})$, we have $T^{n_k} \omega_k \rightarrow \omega^{\max}$ and $T^{n_k} \omega_k^{\text{disc}} \rightarrow \omega^{\max}$ as $k \rightarrow \infty$. Since T^m is continuous at ω^{\max} for every $m \geq 1$, we can achieve by passing to a subsequence of n_k (and hence also of ω_k and ε_k , but keeping l_k fixed) that

$$(5.7) \quad T^{n_k + l_k} \omega_k \rightarrow \tilde{\omega}, \quad T^{n_k + l_k} \omega_k^{\text{disc}} \rightarrow \tilde{\omega}$$

as $k \rightarrow \infty$. By possibly making ε_k smaller, it follows from Keane's condition that

$$(5.8) \quad \inf_{\omega \in (\omega^{\text{disc}} - \varepsilon_k, \omega^{\text{disc}})} |T^{n_k + l_k} \omega - T^{n_k + l_k} \omega^{\text{disc}}| \equiv c(k) > 0.$$

Indeed, for small enough ε_k , $T^{n_k + l_k - 1} : \overline{T(\omega^{\text{disc}} - \varepsilon_k, \omega^{\text{disc}})} \rightarrow [0, 1)$ is a continuous map and $T\omega^{\text{disc}} \notin \overline{T(\omega^{\text{disc}} - \varepsilon_k, \omega^{\text{disc}})}$. Hence their images under $T^{n_k + l_k - 1}$ are also disjoint. Now, since (5.3) fails and (5.8) holds, we can choose $\omega_k \in (\omega^{\text{disc}} - \varepsilon_k, \omega^{\text{disc}})$ such that $|f(T^{n_k + l_k} \omega_k) - f(T^{n_k + l_k} \omega_k^{\text{disc}})| \leq \frac{c(k)}{k}$ and

$|T^{n_k+l_k}\omega_k - T^{n_k+l_k}\omega^{\text{disc}}| \geq c(k)$ hold. By (5.7), $T^{n_k+l_k}\omega_k \rightarrow \tilde{\omega}$ and $T^{n_k+l_k}\omega^{\text{disc}} \rightarrow \tilde{\omega}$. It therefore follows that

$$|f'(\tilde{\omega})| = \lim_{k \rightarrow \infty} \frac{|f(T^{n_k+l_k}\omega_k) - f(T^{n_k+l_k}\omega^{\text{disc}})|}{|T^{n_k+l_k}\omega_k - T^{n_k+l_k}\omega^{\text{disc}}|} \leq \lim_{k \rightarrow \infty} \frac{1}{k} = 0.$$

This concludes the proof since $\tilde{\omega}$ was arbitrary. \square

Corollary 5.12. *For any T not a rotation and satisfying the Keane condition, and $\lambda > 0$, the Schrödinger operator with potential given by $V(n) = \lambda \cos(2\pi T^n(\omega))$ has $\sigma_{ac}(H_\omega) = \emptyset$*

Proof. Note that though $\cos(2\pi x)$ does not have a non-degenerate maximum (as a function on $[0, 1)$), it does have a non-degenerate minimum. The argument for this case is the same as in Theorem 5.11. \square

6. ABSENCE OF POINT SPECTRUM FOR SCHRÖDINGER OPERATORS GENERATED BY INTERVAL EXCHANGE TRANSFORMATIONS

A bounded map $V : \mathbb{Z} \rightarrow \mathbb{R}$ is called a *Gordon potential* if there exists a sequence $q_k \rightarrow \infty$ such that

$$(6.1) \quad \limsup_{k \rightarrow \infty} \left(\sup_{0 \leq j < q_k} |V(j) - V(j \pm q_k)| \exp(Cq_k) \right) = 0, \quad \omega \in \Omega_L(T)$$

for any $C > 0$. It is a well-known fact that Schrödinger operators with a Gordon potential have purely continuous spectrum; see [12] for the original Gordon result and [8] for the result in the form stated above and the history of criteria of this kind.

We will now identify a dense G_δ set of interval exchange transformations such that they generate Gordon potentials for suitable sampling functions. For the case of two intervals (i.e., rotations of the circle), this is a result due to Avron and Simon [3]. In analogy to the convention for rotations, we will refer to these as *Liouville interval exchange transformations*. Explicitly, we call an interval exchange transformation T Liouville if it satisfies the following property: There exists a sequence $q_k \rightarrow \infty$ and $\Omega_L(T) \subseteq \mathbb{T}$ of full measure such that

$$(6.2) \quad \limsup_{k \rightarrow \infty} \left(\sup_{0 \leq j < q_k} |T^j \omega - T^{j \pm q_k} \omega| \exp(Cq_k) \right) = 0, \quad \omega \in \Omega_L(T)$$

for any $C > 0$. We will show in the following that the set of Liouville interval exchange transformation is a dense G_δ set of zero measure.

If $f : \mathbb{T} \rightarrow \mathbb{R}$ is Hölder continuous, the above property implies that $V_\omega(n) = f(T^n \omega)$ is a Gordon potential. It was shown by Chulaevskii in [6] that the set of Liouville interval exchange transformations is dense and a G_δ set. In fact, we state without proof the following result, which clearly implies that Liouville interval exchange transformations are dense and G_δ :

Lemma 6.1. *Given a decreasing function $f : \mathbb{N} \rightarrow (0, \infty)$, we can find a dense G_δ set U in Δ^r such that for every interval exchange transformation in U the following holds.*

For almost every ω , there exists a sequence $q_k = q_k(\omega) \rightarrow \infty$ such that for $k \geq 1$,

$$(6.3) \quad \sup_{0 \leq j < q_k} |T^j \omega - T^{j \pm q_k} \omega| \leq f(q_k).$$

For the reader desiring further references, one could choose the interval exchange transformations to be in a small neighborhood of the primitive interval exchanges considered in [30] part II. These were introduced in [16]. Thus, we have the following theorem:

Theorem 6.2. *If T is a Liouville interval exchange transformation and f is Hölder continuous, then for every $\omega \in \Omega_L(T)$, H_ω has no eigenvalues.*

We remark that it is easy to show that the set of Liouville interval exchange transformations has zero measure.

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