RACKS, LEIBNIZ ALGEBRAS AND YETTER-DRINFEL'D MODULES

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ABSTRACT. A Hopf algebra object in Loday and Pirashvili's category of linear maps entails an ordinary Hopf algebra and a Yetter-Drinfel'd module. We equip the latter with a structure of a braided Leibniz algebra. This provides a unified framework for examples of racks in the category of coalgebras discussed recently by Carter, Crans, Elhamdadi and Saito.

CONTENTS

1. Introduction

The subject of the present paper is the relation between racks, Leibniz algebras and Yetter-Drinfel'd modules.

An augmented rack (or crossed G-module) can be defined as a Yetter-Drinfel'd module over a group G, viewed as a Hopf algebra object in the symmetric monoidal category $(\mathtt{Set}, \times)$. Explicitly, it is a right G-set X together with a G-equivariant map $p : X \to G$ where G carries the right adjoint action of G. A main application of racks is the construction of invariants of links and tangles, see e.g. [3, 6, 7] and the references therein.

Leibniz algebras are vector spaces equipped with a bracket that satisfies a form of the Jacobi identity, but which is not necessarily antisymmetric, see Definition 2 below. They were discovered by A.M. Blokh [2] in 1965, and then later rediscovered by J.-L. Loday in his search of an understanding for the obstruction to periodicity in algebraic K-theory [15]. In this context the problem of the integration of Leibniz algebras arose, that is, the problem of finding an object that is to a Leibniz algebra what a Lie group is to its Lie algebra. Lie racks provide one possible solution, see [4, 5, 12].

Analogously to augmented racks over groups, the Yetter-Drinfel'd modules M over a Hopf algebra H in $(\mathtt{Vect}, \otimes)$ form the Drinfel'd centre of the monoidal category of right H-modules, see Section 4.1. Taking in an H-tetramodule (bicovariant bimodule) M the invariant elements ${}^{\mathrm{inv}}M$ with respect to the left coaction defines an equivalence of categories between tetramodules and Yetter-Drinfel'd modules. Thus they are the coefficients in Gerstenhaber-Schack cohomology [8]. Another application is in the classification of pointed Hopf algebras, see e.g. [1].

Our aim here is to directly relate Leibniz algebras to Yetter-Drinfel'd modules, starting from the fact that the universal enveloping algebra of a Leibniz algebra gives rise to a Hopf algebra object in the category $\mathcal{LM}$ of linear maps [16], see Section 2.3. We extend some results from Woronowicz's theory of bicovariant differential calculi [23] which are dual to Hopf algebra objects in $\mathcal{LM}$. In particular, we show that one can construct braided Leibniz algebras as studied by V. Lebed [14] by generalising Woronowicz's quantum Lie algebras of finite-dimensional bicovariant differential calculi:

Theorem 1. *Let $f : M \to H$ be a Hopf algebra object in the category of linear maps $\mathcal{LM}$. Then f restricts to a morphism $\tilde{f} : {}^{\mathrm{inv}}M \to \ker \varepsilon$ of Yetter-Drinfel'd modules over the Hopf algebra H and*

$$x \lhd y = x\tilde{f}(y)$$

turns ${}^{\mathrm{inv}}M$ into a braided Leibniz algebra in the category of Yetter-Drinfel'd modules.

This allows us to study racks and Leibniz algebras in the same language, which provides in particular a unified approach to [3, Proposition 3.1] and [3, Proposition 3.5], see Examples 4 and 5 at the end of the paper.

The paper is structured as follows: Section 2 recalls basic facts and definitions about the category $\mathcal{LM}$ of linear maps and the construction of the universal enveloping algebra of a Leibniz algebra. In Section 3 we explore analogues in $\mathcal{LM}$ of functors relating groups and Lie algebras to Hopf algebras, with a view towards the integration problem of Lie algebras in $\mathcal{LM}$. In particular we point out that the linearisation $p : kX \to kG$ of an augmented rack $p : X \to G$ is not a Hopf algebra object in $\mathcal{LM}$, but instead a map of kG-modules and comodules, see Proposition 3. Section 4 recalls background on Yetter-Drinfel'd modules over bialgebras. The main section is Section 5 where we prove Theorem 1 and finish by discussing concrete examples.

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2. Algebraic objects in $\mathcal{LM}$

In this section we recall the neceesary background on the category of linear maps, algebraic objects therein, and the relevance of these for the theory of Leibniz algebras, mainly from [16, 17]. Throughout we work with vector spaces over a field k, although the results can be generalised to other base categories. An unadorned $\otimes$ denotes the tensor product over k.

2.1. The tensor categories $\mathcal{LM}$ and $\mathcal{LM}^\star$.

The following definition goes back to Loday and Pirashvili [16]:

Definition 1. The *category of linear maps* $\mathcal{LM}$ has linear maps $f : V \to W$ between vector spaces as objects, which are usually depicted by vertical arrows with V upstairs and W downstairs. A morphism ϕ between two linear maps $(f : V \to W)$ and $(f' : V' \to W')$ is a commutative square

$$\begin{array}{ccc} V & \xrightarrow{\phi_1} & V' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ W & \xrightarrow{\phi_0} & W' \end{array}$$

The *infinitesimal tensor product* between f and f' is defined to be

$$\begin{array}{c}(V\otimes W')\oplus(W\otimes V')\\ \Big\downarrow{\scriptstyle f\otimes\mathrm{id}_{W'}+\mathrm{id}_W\otimes f'}\\ W\otimes W'.\end{array}$$

The infinitesimal tensor product turns $\mathcal{LM}$ into a symmetric monoidal category with unit object being the zero map $0:\{0\}\to k$.

Remark 1. Alternatively, $\mathcal{LM}$ is the category of 2-term chain complexes with a truncated tensor product; one has just omitted the terms of degree two in the tensor product of complexes. One can analogously define categories $\mathcal{LM}_n$ of chain complexes of length n and a tensor product which is truncated in degree n, so in this sense $\mathcal{LM}=\mathcal{LM}_1$ and $\mathtt{Vect}=\mathcal{LM}_0$. Taking the inverse limit, one passes from these truncated versions to the category of chain complexes with the ordinary tensor product $\mathtt{Chain}=\mathcal{LM}_\infty$. △

Interpreting $\mathcal{LM}$ as the category of cochain rather than chain complexes of length 1 and depicting them consequently by arrows pointing upwards results in a different monoidal structure $\otimes^\star$ on $\mathcal{LM}$ in which

$$(f:V\to W)\otimes^\star(f':V'\to W')$$

is given by

$$\begin{array}{c}(V\otimes W')\oplus(W\otimes V')\\ {\scriptstyle \mathrm{id}_V\otimes f'+f\otimes\mathrm{id}_{V'}}\Big\uparrow\\ V\otimes V'.\end{array}$$

The resulting tensor category will be denoted $\mathcal{LM}^\star$.

2.2. **Algebraic objects in $\mathcal{LM}$.** In a symmetric monoidal tensor category, one can define associative algebra objects, Lie algebra objects and bialgebra objects. Loday and Pirashvili exhibit the structure of these in the tensor category $\mathcal{LM}$. For this, they use that the inclusion functor

$$\mathtt{Vect}\to\mathcal{LM},\quad W\mapsto(0:\{0\}\to W),$$

and the projection functor

$$\mathcal{LM}\to\mathtt{Vect},\quad (f:V\to W)\mapsto W$$

between the categories of vector spaces $\mathtt{Vect}$ and $\mathcal{LM}$ are tensor functors which compose to the identity functor on $\mathtt{Vect}$. This shows that for each of the above mentioned algebraic structures in $\mathcal{LM}$, the codomain W of $f:V\to W$ inherits the corresponding structure in the category of vector spaces. The linear map can be used to turn the vector space $V\oplus W$

into an abelian extension of W, in the sense discussed for example in [18, Section 12.3.2]. The domain V becomes an abelian ideal in $V \oplus W$.

More explicitly, Loday and Pirashvili show that in $\mathcal{LM}$:

- an associative algebra object $f : M \to A$ is the data of an associative algebra A, an A-bimodule M and a bimodule map $f : M \to A$,
- a Lie algebra object $f : M \to \mathfrak{g}$ is the data of a Lie algebra $\mathfrak{g}$, a (right) Lie module M and an equivariant map $f : M \to \mathfrak{g}$,
- a bialgebra object $f : M \to H$ is the data of a bialgebra H, of an H-*tetramodule* (or *bicovariant bimodule*) M, that is, an H-bimodule and H-bicomodule whose left and right coactions are H-bimodule maps, and of an H-bilinear coderivation $f : M \to H$,
- a Hopf algebra object in $\mathcal{LM}$ is a bialgebra object $f : M \to H$ in $\mathcal{LM}$ such that H admits an antipode.

Remark 2. While Loday and Pirashvili formulate their statement about Hopf algebra objects in $\mathcal{LM}$ rather as a definition, see [16, Seciton 5.1], these really are the Hopf algebra objects in $\mathcal{LM}$ in the categorical sense: it is straightforward to verify that if H has an antipode $S : H \to H$, then the bialgebra object $f : M \to H$ has an antipode given by

$$\begin{array}{ccc} M & \xrightarrow{T} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ H & \xrightarrow{S} & H \end{array}$$

with T given in Sweedler notation by $T(x) = -S(m_{(-1)})m_{(0)}S(m_{(1)})$. Thus T is uniquely determined by the antipode S on H and is not additional data.

Remark 3. Dually, a bialgebra object $f : H \to M$ in $\mathcal{LM}^\star$ consists of a bialgebra H in `Vect` and an H-tetramodule M such that f is a derivation and bicolinear. If $M = \mathrm{span}_k\{gf(h) \mid g, h \in H\}$, this structure is referred to as a *first order bicovariant differential calculus* over H [23], see e.g. [13] for a pedagogical account. Linear duality $F : V \mapsto V^*$ yields a (weakly) monoidal functor $F : \mathcal{LM} \to (\mathcal{LM}^\star)^{\mathrm{op}}$, which is strongly monoidal on the subcategory of finite-dimensional vector spaces. In Remark 7 below we will describe the class of bialgebras in $\mathcal{LM}$ that is under F dual to first order bicovariant differential calculi. △

2.3. **Universal enveloping algebras in $\mathcal{LM}$.** Loday and Pirashvili furthermore construct in [16] a pair of adjoint functors P (primitives) and U (universal enveloping algebra) associating a Lie algebra object in $\mathcal{LM}$ to a Hopf algebra object in $\mathcal{LM}$, and vice versa, and prove an analogue of the classical Milnor-Moore theorem in this context. For a given Lie algebra object $f : M \to \mathfrak{g}$, the enveloping algebra is $\phi : U\mathfrak{g} \otimes M \to U\mathfrak{g}$, $u \otimes m \mapsto uf(m)$.

The underlying $U\mathfrak{g}$-tetramodule structure on $U\mathfrak{g} \otimes M$ is as follows: the right $U\mathfrak{g}$-action on $U\mathfrak{g} \otimes M$ is induced by

$$(u \otimes m) \cdot x \;=\; ux \otimes m + u \otimes m \cdot x$$

for all $x \in \mathfrak{g}$, all $u \in U\mathfrak{g}$ and all $m \in M$. The left action is by multiplication on the left-hand factor. The left and right $U\mathfrak{g}$-coactions are given by the coproduct on the left-hand factor, that is, for $x \in \mathfrak{g}, m \in M$ they are

$$(x \otimes m) \mapsto 1 \otimes (x \otimes m) + x \otimes (1 \otimes m), \quad (x \otimes m) \mapsto (1 \otimes m) \otimes x + (x \otimes m) \otimes 1.$$

2.4. **Leibniz algebras.** We finally recall from [16] that a particular class of Lie algebra objects in $\mathcal{LM}$ arises in a canonical way from Leibniz algebras:

Definition 2. A k-vector space $\mathfrak{g}$ together with a bilinear map

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is called a (right) *Leibniz algebra*, in case for all $x, y, z \in \mathfrak{g}$

$$[[x, y], z] \;=\; [x, [y, z]] + [[x, z], y]$$

holds.

In particular, any Lie algebra is a Leibniz algebra. Conversely, for any Leibniz algebra $\mathfrak{g}$ the quotient by the Leibniz ideal generated by the squares $[x, x]$ for $x \in \mathfrak{g}$ is a Lie algebra $\mathfrak{g}_{\mathrm{Lie}}$, and the right adjoint action of $\mathfrak{g}_{\mathrm{Lie}}$ on itself lifts to a well-defined right action on $\mathfrak{g}$. So by construction, the canonical quotient map $\pi : \mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$ is a Lie algebra object in $\mathcal{LM}$. The universal enveloping algebra of $\mathfrak{g}$ as defined in [17] is exactly the abelian extension of the associative algebra $U\mathfrak{g}_{\mathrm{Lie}}$ in `Vect` that is defined by the universal enveloping algebra $U(\mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}})$, see [16, Theorem 4.7].

3. The problem of integrating Lie algebras in $\mathcal{LM}$

In this section we discuss the direct analogues in $\mathcal{LM}$ of some functorial constructions that relate groups to Lie algebras, with a view towards the problem of integrating Leibniz algebras to some global structure. Augmented racks and their linearisations are one possible framework for these, so we end by recalling some background on racks.

3.1. **From Lie algebras to groups.** Consider the following diagram of functors:

$$\begin{array}{ccc} \mathtt{Lie} & \xrightarrow{U} & \mathtt{ccHopf} \\ \downarrow & & \downarrow {\scriptstyle \_^{\circ}} \\ \mathtt{Grp} & \xleftarrow[\chi]{} & \mathtt{cHopf} \end{array}$$

Here `Lie` is the category of Lie algebras over the field k, `Grp` is the category of groups, `Hopf` is the category of k-Hopf algebras, and `ccHopf` and `cHopf` are its subcategories of cocommutative respectively commutative Hopf algebras. The functor U is that of the enveloping algebra, and χ is the functor of characters, while H° is the Hopf dual of a Hopf algebra H, that is, the Hopf algebra of matrix coefficients of finite-dimensional representations, see e.g. [13, 20].

An affine algebraic group G over an algebraically closed field k of characteristic 0 can be recovered in this way from its Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ as $\chi(U\mathfrak{g}^\circ)$ provided G is perfect, i.e. $G = [G, G]$. More generally, if G has unipotent radical, then G is isomorphic to the characters on the subalgebra of basic representative functions on $U\mathfrak{g}$, see [10] for details.

3.2. **Characters of Hopf algebra objects in $\mathcal{LM}$.** The functor $\chi(-)$ (characters) can be extended to Hopf algebra objects in $\mathcal{LM}$, hence one might attempt to use it to integrate Lie algebras in $\mathcal{LM}$ and in particular Leibniz algebras. By definition, a character χ of a Hopf algebra object $f : M \to H$ is an algebra morphism in $\mathcal{LM}$ from $f : M \to H$ to the unit of the tensor category $\mathcal{LM}$ which is simply $0 : \{0\} \to k$. This amounts to a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\chi_1} & \{0\} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} \\ H & \xrightarrow{\chi_0} & k. \end{array}$$

One therefore obtains just characters χ_0 of H, because χ_1 is supposed to be the zero map. The same applies to Hopf algebra objects in $\mathcal{LM}^\star$, that is, the component of the character associated to the tetramodule vanishes. Thus we have:

Proposition 1. *The functor $\chi(-)$ (characters), applied to a Hopf object in $\mathcal{LM}$ or $\mathcal{LM}^\star$, results just in characters of the underlying Hopf algebra H.*

Hence the integration of Lie algebra objects in $\mathcal{LM}$ (and thus in particular Leibniz algebras) along the lines outlined in the previous section must fail. One can associate to a Lie algebra object in $\mathcal{LM}$ its universal enveloping algebra, and then by duality some commutative Hopf algebra object in $\mathcal{LM}^\star$, but characters of this object will always be only characters of the underlying Hopf algebra.

3.3. **Formal group laws in $\mathcal{LM}$.** Another approach to the integration of Lie algebras is that of formal group laws, see [22]. Here one studies a continuous dual of $U\mathfrak{g}$.

Recall that a *formal group law* on a vector space V is a linear map $F : S(V \oplus V) \to V$ which is unital and associative, i.e. its extension to a coalgebra morphism $F' : S(V) \otimes S(V) \to S(V)$ is an associative product on the symmetric algebra $S(V)$.

Mostovoy [21] transposes this definition into the realm of $\mathcal{LM}$. Namely, a formal group law in $\mathcal{LM}$ is a map

$$G : S\big((V \oplus V) \to (W \oplus W)\big) \to (V \to W),$$

whose extension to a morphism of coalgebra objects

$$G' : S(V \to W) \otimes S(V \to W) \to (V \to W)$$

is an algebra object in $\mathcal{LM}$. Starting with a Lie algebra object $M \to \mathfrak{g}$ in $\mathcal{LM}$, the product in the universal enveloping algebra $U(M \to \mathfrak{g})$ composed with the projection onto the primitive subspace yields a formal group law using the identification of $U(M \to \mathfrak{g})$ with $S(M \to \mathfrak{g})$ provided by the analogue of the Poincaré-Birkhoff-Witt theorem for Lie algebra objects in $\mathcal{LM}$. Explicitly, one gets a diagram

$$\begin{array}{ccc} S(\mathfrak{g}) \otimes M \otimes S(\mathfrak{g}) \oplus S(\mathfrak{g}) \otimes S(\mathfrak{g}) \otimes M & \xrightarrow{G^1 + G^2} & M \\ \downarrow & & \downarrow \\ S(\mathfrak{g}) \otimes S(\mathfrak{g}) & \xrightarrow{\quad F \quad} & \mathfrak{g} \end{array}$$

Mostovoy [21] shows then:

Proposition 2. *The functor that assigns to a Lie algebra object $M \to \mathfrak{g}$ in $\mathcal{LM}$ the primitive part of the product in $U(M \to \mathfrak{g})$ is an equivalence of categories of Lie algebra objects in $\mathcal{LM}$ and of formal group laws in $\mathcal{LM}$.*

An interesting problem that arises is to specify what this framework gives for the Lie algebra objects in $\mathcal{LM}$ coming from a Leibniz algebra, i.e. for those of the form $\pi : \mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$. Furthermore, one should clarify what the global objects associated to these formal group laws are. The results in the present paper are meant to motivate why augmented racks are a natural candidate, by going the other way and studying the Hopf algebra objects in $\mathcal{LM}$ that are obtained by linearisation from augmented racks.

3.4. **Augmented racks.** The set-theoretical version of $\mathcal{LM}$ is the category $\mathcal{M}$ of all maps $X \to Y$ between sets X and Y. One defines an analogue of the infinitesimal tensor product in which the disjoint union of sets takes the place of the sum of vector spaces, and the cartesian product replaces the tensor product. This defines a monoidal category structure on $\mathcal{M}$ with unit object $\varnothing \to \{*\}$. However, the latter is not terminal in $\mathcal{M}$, thus one cannot define inverses, and a fortiori group objects.

One way around this "no-go" argument is to consider augmented racks:

Definition 3. Let X be a set together with a binary operation denoted $(x, y) \mapsto x \lhd y$ such that for all $y \in X$, the map $x \mapsto x \lhd y$ is bijective and for all $x, y, z \in X$,

$$(x \lhd y) \lhd z \;=\; (x \lhd z) \lhd (y \lhd z).$$

Then we call X a (right) *rack*. In case the invertibility of the maps $x \mapsto x \lhd y$ is not required, it is called a *shelf*.

The guiding example of a rack is a group together with its conjugation map $(g, h) \mapsto g \lhd h := h^{-1}gh$. Augmented racks are generalisations of these in which the rack operation results from a group action:

Definition 4. Let G be a group and X be a (right) G-set. Then a map $p : X \to G$ is called an *augmented rack* in case p satisfies the augmentation identity, i.e. for all $g \in G$ and all $x \in X$

$$p(x \cdot g) \;=\; g^{-1}\, p(x)\, g. \tag{1}$$

In other words p is equivariant with respect to the G-action on X and the adjoint action of G on itself. The G-set X in an augmented rack $p : X \to G$ carries a canonical structure of a rack by setting

$$x \lhd y \;:=\; x \cdot p(y).$$

Remark 4. Any rack X can be turned into an augmented rack as follows: let $\mathrm{As}(X)$ be the *associated group* (see for example [6]) of X, which is the quotient of the free group on the set X by the relations $y^{-1}xy = x \lhd y$ for all $x, y \in X$. Then there is a canonical map $p : X \to \mathrm{As}(X)$ assigning to $x \in X$ the class of x in $\mathrm{As}(X)$ which turns X into an augmented rack. △

A more conceptual point of view goes back to Yetter, confer [7]: a group is the same as a Hopf algebra object in the symmetric monoidal category `Set` with $\times$ as monoidal structure. In this sense, right G-modules are just right G-sets while right G-comodules are just sets X equipped with a map $p : X \to G$. The augmentation identity (1) becomes the Yetter-Drinfel'd condition that we will discuss in detail in the next section. Thus augmented racks are the same as Yetter-Drinfel'd modules over G in `Set`, or, in other words, the category of augmented racks over G is the Drinfel'd centre of the category of right G-sets.

3.5. **Linearised augmented racks.** By linearisation, one obtains the group algebra kG of a group G which consequently is a Hopf algebra in `Vect`, see e.g. [11, p.51, Example 2]. Hence one might ask whether a linearisation of an augmented rack $p : X \to G$ defines a Hopf algebra object in $\mathcal{LM}$. The functor $k-$ (k-linearisation of a set) sends $p : X \to G$ to a linear map

$p : kX \to kG$. Consider kX as a kG-bimodule where kG acts on kX on the right via the given action and on the left via the trivial action. Consider further the two linear maps

$$\triangle_l : kX \to kG \otimes kX, \quad \triangle_r : kX \to kX \otimes kG$$

given for $x \in X$ by

$$\triangle_l x \;=\; p(x) \otimes x \;\; \text{and} \;\; \triangle_r x \;=\; x \otimes p(x).$$

Then we have:

Proposition 3. *The maps $\triangle_l, \triangle_r$ turn kX into a kG-bicomodule such that $p : kX \to kG$ is a morphism of bicomodules and bimodules, where kG carries the left and right coaction given by the coproduct, the trivial left action, and the adjoint right action.*

Proof. The augmentation identity

$$p(x \cdot g) \;=\; g^{-1} p(x) g, \quad \forall x \in X, g \in G$$

shows that p is a morphism of bimodules. We have

$$(p \otimes 1)(\triangle_r x) \;=\; p(x) \otimes p(x) \;\; \text{and} \;\; (1 \otimes p)(\triangle_l x) \;=\; p(x) \otimes p(x)$$

for all $x \in X$, thus p is a morphism of bicomodules. □

In particular, $p : kX \to kG$ is not a Hopf algebra object in $\mathcal{LM}$ in general.

3.6. **Regular functions on augmented racks.** Taking the coordinate ring $k[X]$ of an algebraic set X is a contravariant functor, so applying it to an algebraic augmented rack $p : X \to G$ gives rise to an algebra map $p^* : k[G] \to k[X]$ which is most naturally considered in $\mathcal{LM}^\star$.

The right G-action on X induces a right $k[G]$-comodule structure on $k[X]$. Together with the trivial left comodule structure, $k[X]$ becomes a $k[G]$-bicomodule. On $k[G]$ itself, we consider the bicomodule structure obtained from the trivial left coaction and the right adjoint coaction given in Sweedler notation by $f \mapsto f_{(2)} \otimes S(f_{(1)}) f_{(3)}$, and then obtain:

Proposition 4. *$p^* : k[G] \to k[X]$ is a morphism of bimodules and bicomodules.*

Proof. For the augmented rack $p : X \to G$, we have the following commutative diagram:

$$\begin{array}{ccc} X \times G & \longrightarrow & X \\ \downarrow{\scriptstyle p \times \mathrm{id}_G} & & \downarrow{\scriptstyle p} \\ G \times G & \longrightarrow & G \end{array}$$

which reads explicitly as

$$\begin{array}{ccc} (x,g) & \longrightarrow & x\cdot g \\ \Big\downarrow{\scriptstyle p\times \mathrm{id}_G} & & \Big\downarrow{\scriptstyle p} \\ (p(x),g) & \longrightarrow & p(x\cdot g) \;=\; g^{-1}p(x)g \end{array}$$

Applying the functor $k[-]$ to this diagram yields

$$\begin{array}{ccc} k[X] & \longrightarrow & k[X]\otimes k[G] \\ {\scriptstyle p^*}\Big\uparrow & & {\scriptstyle p^*\otimes \mathrm{id}_{k[G]}}\Big\uparrow \\ k[G] & \longrightarrow & k[G]\otimes k[G] \end{array}$$

This means exactly that p^* is a morphism of right comodules. As the left coactions on $k[G]$ and $k[X]$ are trivial, it is a map of bicomodules. □

3.7. **The Yetter-Drinfel'd braiding.** It is well-known (see for example [11] p. 319) that the category of augmented racks over a fixed group G carries a braiding:

Proposition 5. *Define for augmented racks $p_1 : X \to G$ and $p_2 : Y \to G$ with respect to a fixed group G their tensor product $X \otimes Y$ by $X \times Y$ with the action $(x, y)\cdot g := (x\cdot g, y\cdot g)$ and the equivariant map $p : X \times Y \to G$ being $p(x, y) := p_1(x)p_2(y)$. Then the formula*

$$c_{X,Y} : X \otimes Y \to Y \otimes X, \quad c_{X,Y}(x, y) := (y, x \cdot p(y))$$

defines a braiding on the category of augmented racks over G.

This is just a special case of the Yetter-Drinfel'd braiding that we are going to study in detail next.

4. YETTER-DRINFEL'D MODULES

In this section we recall definitions and facts about Yetter-Drinfel'd modules over Hopf algebras in `Vect` that we need. For more information, the reader is referred to [11, 13, 19, 20].

4.1. **Yetter-Drinfel'd modules.** Let $H = (H, \mu, \eta, \triangle, \varepsilon)$ be a bialgebra over k. To every right module and right comodule M over H, one functorially associates a bimodule and bicomodule M^H over H which is $H \otimes M$ as a vector space with left and right action given by

$$g(h \otimes x) := gh \otimes x, \quad (h \otimes x)g := hg_{(1)} \otimes xg_{(2)}$$

and left and right coaction given in Sweedler notation by

$$(h \otimes x)_{(-1)} \otimes (h \otimes x)_{(0)} := h_{(1)} \otimes (h_{(2)} \otimes x),$$

$$(h \otimes x)_{(0)} \otimes (h \otimes x)_{(1)} := (h_{(1)} \otimes x_{(0)}) \otimes h_{(2)} x_{(1)}.$$

These coactions and actions are compatible in the sense that M^H is a Hopf tetramodule if and only if M is a Yetter-Drinfel'd module:

Definition 5. A *Yetter-Drinfel'd module* over H is a right module and right comodule M for which we have

$$(xh_{(2)})_{(0)} \otimes h_{(1)}(xh_{(2)})_{(1)} = x_{(0)} h_{(1)} \otimes x_{(1)} h_{(2)} \tag{2}$$

for all $x \in M$ and $h \in H$.

Remark 5. If H is a Hopf algebra with antipode S, then the Yetter-Drinfel'd condition (2) is easily seen to be equivalent to

$$(xh)_{(0)} \otimes (xh)_{(1)} = x_{(0)} h_{(2)} \otimes S(h_{(1)}) x_{(1)} h_{(3)}. \tag{3}$$

△

More precisely, H is a Hopf algebra if and only if $M \mapsto M^H$ defines an equivalence between the categories of Yetter-Drinfel'd modules and that of Hopf tetramodules. In this case, the inverse functor is given by taking the invariants with respect to the left coaction,

$$N \mapsto {}^{\mathrm{inv}}N := \{x \in N \mid x_{(-1)} \otimes x_{(0)} = 1 \otimes x\}.$$

This is an equivalence of monoidal categories, where the tensor product of Hopf tetramodules is $\otimes_H$.

Example 1. Let G be a group and M be a kG-Yetter-Drinfel'd module. Then M is in particular a kG-module, i.e. a G-module. The comodule structure of M is a G-grading of this G-module:

$$M = \bigoplus_{g \in G} M_g.$$

The Yetter-Drinfel'd compatibility condition now reads for $u \in kG$ and $m \in M$

$$(um)_{(-1)} \otimes (um)_{(0)} = u_{(1)} m_{(-1)} S(u_{(2)}) \otimes u_{(3)} m_{(0)}$$

which means for a group element $g = u \in G$ and a homogeneous element $m \in M_h$

$$(gm)_{(-1)} \otimes (gm)_{(0)} = ghg^{-1} \otimes g \cdot m.$$

This means that the action of $g \in G$ on M maps M_h to $M_{ghg^{-1}}$.

When the module M is a permutation representation of G, that is, is obtained by linearisation from a (right) G-set X, $M \simeq kX$, then M is Yetter-Drinfel'd precisely when X carries the structure of an augmented rack. The full subcategory of the category of all Yetter-Drinfel'd modules over kG of these permutation modules has been studied first by Freyd and Yetter, see [7, Definition 4.2.3].

Example 2. Recall from Section 2.3 that if $f : M \to \mathfrak{g}$ is any Lie algebra object in $\mathcal{LM}$, then the universal enveloping algebra construction in $\mathcal{LM}$ yields the $U\mathfrak{g}$-tetramodule $U\mathfrak{g} \otimes M$. In this case, M is recovered as the Yetter-Drinfel'd module of left invariant elements, with trivial right coaction and right action being induced by the right $\mathfrak{g}$-module structure on M.

More generally, every right module over a cocommutative bialgebra H becomes a Yetter-Drinfel'd module with respect to the trivial right coaction.

4.2. **The Yetter-Drinfel'd braiding revisited.** Every right H-module and right H-comodule M carries a canonical map

$$\tau : M \otimes M \to M \otimes M, \quad x \otimes y \mapsto y_{(0)} \otimes xy_{(1)} \tag{4}$$

The following well-known fact characterises when τ is a braiding:

Proposition 6. *The map (4) is a braiding on M if and only if M is a Yetter-Drinfel'd module.*

Remark 6. While (3) is maybe easier to memorise, (2) makes sense for all bialgebras and is directly the condition that occurs when testing whether or not τ satisfies the braid relation. More generally, τ can be extended to braidings $N \otimes M \to M \otimes N$ between any right H-module N and a Yetter-Drinfel'd module M, and this identifies the category of Yetter-Drinfel'd modules with the Drinfel'd centre of the category of right H-modules. △

4.3. **The Yetter-Drinfel'd module** $\ker \varepsilon$**.** The following example of a Yetter-Drinfel'd module is of particular importance to us:

Proposition 7. *If H is any Hopf algebra, then the kernel $\ker \varepsilon$ of its counit is a Yetter-Drinfel'd module with respect to the right adjoint action*

$$g \blacktriangleleft h := S(h_{(1)})gh_{(2)}$$

and the right coaction

$$\tilde{\triangle} : \ker \varepsilon \to \ker \varepsilon \otimes H, \quad k \mapsto h_{(1)} \otimes h_{(2)} - 1 \otimes h.$$

One can view $\ker \varepsilon$ as a bicomodule with respect to the trivial left coaction $h \mapsto 1 \otimes h$, and then the inclusion map $\iota : \ker \varepsilon \to H$ is a coderivation. This is universal in the sense that every coderivation factors through ι:

Lemma 1. *Let H be a bialgebra, M be an H-bicomodule, and $f : M \to H$ be a coderivation.*

(1) *We have* $\operatorname{im} f \subseteq \ker \varepsilon$.
(2) *The restriction of f to $\tilde{f} : {}^{\mathrm{inv}}M \to \ker \varepsilon$ is right H-colinear with respect to the coaction $\tilde{\triangle}$ on $\ker \varepsilon$.*
(3) *If M is a tetramodule and f is H-bilinear, then $\tilde{f}$ is a morphism of Yetter-Drinfel'd modules.*

Proof. (1) Applying $\varepsilon \otimes \varepsilon$ to the coderivation condition

$$(f(m))_{(1)} \otimes (f(m))_{(2)} \;=\; m_{(-1)} \otimes f(m_{(0)}) + m_{(0)} \otimes f(m_{(1)})$$

yields $\varepsilon(f(m)) = 2\varepsilon(f(m))$, so $\varepsilon(f(m)) = 0$.

(2) For left invariant $m \in M$, we have $m_{(-1)} \otimes m_{(0)} = 1 \otimes m$, so subtracting $1 \otimes f(m)$ from the coderivation condition yields

$$\tilde{\triangle}(f(m)) \;=\; (f(m))_{(1)} \otimes (f(m))_{(2)} - 1 \otimes f(m) \;=\; m_{(0)} \otimes f(m_{(1)}).$$

(3) The right action on ${}^{\mathrm{inv}}M$ respectively $\ker \varepsilon$ is obtained from the bimodule structure on M respectively H by passing to the right adjoint actions, so $\tilde{f}(m \blacktriangleleft h) = f(S(h_{(1)})mh_{(2)}) = S(h_{(1)}f(m)h_{(2)} = \tilde{f}(m) \blacktriangleleft h$. □

Remark 7. In Remark 7 we mentioned that first order bicovariant differential calculi in the sense of Woronowicz are formally dual to certain bialgebras in $\mathcal{LM}$. We can explain this now in more detail: given a first order bicovariant differential calculus over a Hopf algebra A, that is, a bicolinear derivation $d : A \to \Omega$ with values in a tetramodule Ω which is minimal in the sense that $\Omega = \mathrm{span}_k\{adb \mid a, b \in A\}$, one defines

$$\mathcal{R}_{(\Omega,d)} := \{a \in \ker \varepsilon \mid S(a_{(1)})da_{(2)} = 0\}.$$

It turns out that $(\Omega, d) \mapsto \mathcal{R}_{(\Omega,d)}$ establishes a one-to-one correspondence between first order bicovariant differential calculi and right ideals in $\ker \varepsilon$ that are invariant under the right adjoint coaction $a \mapsto a_{(2)} \otimes S(a_{(1)})a_{(3)}$ of A, see [13, Proposition 14.1 and Proposition 14.7]. When $A = k[G]$ is the coordinate ring of an affine algebraic group, Ω are the Kähler differentials and da is the differential of a regular function a, then $\mathcal{R}_{(\Omega,d)}$ is just $(\ker \varepsilon)^2$ and $\ker \varepsilon / \mathcal{R}_{(\Omega,d)}$ is the cotangent space of G in the unit element.

Motivated by this example, one introduces the *quantum tangent space*

$$\mathcal{T}_{(\Omega,d)} := \{\phi \in A^* \mid \phi(1) = 0, \phi(a) = 0 \,\forall\, a \in \mathcal{R}_{(\Omega,d)}\},$$

where $A^* = \mathrm{Hom}_k(A, k)$ denotes the dual algebra of A. Provided that Ω is finite-dimensional in the sense that $\dim_k {}^{\mathrm{inv}}\Omega < \infty$, the quantum tangent space belongs to the Hopf dual $H := A^\circ$ of A and uniquely characterises the calculus up to isomorphism, see [13, Proposition 14.4] and the subsequent discussion. By definition, $\mathcal{T}_{(\Omega,d)}$ is then a subspace of $\ker \varepsilon \subset H$ which is by [13, (14)] invariant under the right coaction $\tilde{\triangle}$ and as a consequence of [13, Proposition 14.7] it is also invariant under the right adjoint action of H on itself; in other words, the quantum tangent space is a Yetter-Drinfel'd submodule of $\ker \varepsilon$, and if we equip $M := H \otimes \mathcal{T}_{(\Omega,d)}$ with the corresponding H-tetramodule structure we can extend the inclusion of the quantum tangent space into $\ker \varepsilon$ to a Hopf algebra object $f : M \to H$ in $\mathcal{LM}$. Thus first order bicovariant differential calculi should be viewed as structures dual to

Hopf algebra objects $f : M \to H$ in $\mathcal{LM}$ for which the induced map $\tilde{f}$ is injective. $\triangle$

5. Braided Leibniz algebras

The definition of a Leibniz algebra extends straightforwardly from `Vect` to other additive braided monoidal categories [14]. In this final section we discuss the construction of such generalised Leibniz algebras from Hopf algebra objects in $\mathcal{LM}$ which is the main objective of our paper.

5.1. **Definition.** The following structure is meant to generalise both racks and Leibniz algebras in their role of domains of objects in $\mathcal{LM}$:

Definition 6. A *braided Leibniz algebra* is a vector space M together with linear maps

$$\lhd : M \otimes M \to M, \quad x \otimes y \mapsto x \lhd y$$

and

$$\tau : M \otimes M \to M \otimes M, \quad x \otimes y \mapsto y_{\langle 1 \rangle} \otimes x_{\langle 2 \rangle}$$

satisfying

$$(x \lhd y) \lhd z = x \lhd (y \lhd z) + (x \lhd z_{\langle 1 \rangle}) \lhd y_{\langle 2 \rangle} \quad \forall x, y, z \in M. \tag{5}$$

Remark 8. We do not assume that τ maps elementary tensors to elementary tensors, the notation $y_{\langle 1 \rangle} \otimes x_{\langle 2 \rangle}$ should be understood symbolically like Sweedler's notation $\triangle(h) = h_{(1)} \otimes h_{(2)}$ for the coproduct of an element h of a coalgebra H which is also in general not an elementary tensor. $\triangle$

Remark 9. It is natural to ask for τ to satisfy the braid relation (Yang-Baxter equation), so that M is just a braided Leibniz algebra as studied e.g. in [14]. Instead of assuming this a priori we rather characterise this case in the examples that we study below, and later we investigate the consequences of this condition. $\triangle$

Example 3. When τ is the tensor flip, $y_{\langle 1 \rangle} \otimes x_{\langle 2 \rangle} = y \otimes x$, we recover Definition 2 from Section 2.4 with $x \lhd y =: [x, y]$, as the Leibniz rule (5) becomes the (right) Jacobi identity in the form

$$[[x, y], z] = [x, [y, z]] + [[x, z], y].$$

$\triangle$

5.2. **Leibniz algebras from modules-comodules.** The following proposition allows one to construct Leibniz algebras from modules-comodules:

Proposition 8. *Let M be a right module and a right comodule over a bialgebra H, $q : M \to H$ be a k-linear map, and define*

$$x \lhd y := xq(y).$$

Then $(M, \tau, \lhd)$ is a braided Leibniz algebra with respect to

$$\tau : M \otimes M \to M \otimes M, \quad x \otimes y \mapsto y_{(0)} \otimes xy_{(1)}$$

from (4) provided that

$$h_{(1)}q(xh_{(2)}) = q(x)h \tag{6}$$

and

$$q(x)_{(1)} \otimes q(x)_{(2)} = 1 \otimes q(x) + q(x_{(0)}) \otimes x_{(1)} \tag{7}$$

holds for all $x \in M$ and $h \in H$.

Proof. Straightforward computation gives

$$\begin{aligned} (x \lhd y) \lhd z &= (xq(y))q(z) = x(q(y)q(z)) \\ &= x(q(z)_{(1)}q(yq(z)_{(2)})) \\ &= xq(yq(z)) + xq(z_{(0)})q(yz_{(1)}) \\ &= x \lhd (y \lhd z) + (x \lhd z_{\langle 1\rangle}) \lhd y_{\langle 2\rangle} \end{aligned}$$

as had to be shown. □

Remark 10. Observe that applying $\mathrm{id}_H \otimes \varepsilon$ to (7) implies

$$q(x) = \varepsilon(q(x)) + q(x),$$

so this condition necessarily requires $\mathrm{im}\, q \subseteq \ker \varepsilon \subset H$. If H is a Hopf algebra, then (6) is equivalent to the right H-linearity of q with respect to the right adjoint action of H on $\ker \varepsilon$. Furthermore, the condition (7) can be stated also as saying that $q : M \to \ker \varepsilon$ is right H-colinear with respect to the right coaction $\tilde{\triangle}$ on $\ker \varepsilon$ from Section 4.3. △

Thus we can restate the above proposition also as follows:

Corollary 1. *Let M be a right module and right comodule over a Hopf algebra H and $q : M \to \ker \varepsilon$ be an H-linear and H-colinear map. Then*

$$\tau(x \otimes y) := y_{(0)} \otimes xy_{(1)}, \quad x \lhd y := xq(y)$$

turns M into a braided Leibniz algebra.

5.3. **Leibniz algebras from Hopf algebra objects in $\mathcal{LM}$.** Altogether, the above results provide a proof of our main theorem:

Proof of Theorem 1. From the description of Hopf algebra objects in the category of linear maps $\mathcal{LM}$ in Section 2.1, it follows that $f : M \to H$ is the data of a Hopf algebra H, a tetramodule M and a morphism of bimodules f which is also a coderivation. Hence Lemma 1 proves the first part of the theorem. Now Corollary 1 applied to $q := \tilde{f}$ yields the structure of a braided Leibniz algebra on ${}^{\mathrm{inv}}M$. □

Now we see that classical Leibniz algebras can be viewed as a special case of the constructions from this subsection:

Example 4. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a (right) Leibniz algebra in the category of k-vector spaces with the flip as braiding as in Example 3. We have recalled in Section 2.2 how to regard $\mathfrak{g}$ as a Lie algebra object in $\mathcal{LM}$, and in Section 2.3 how to associate to it its universal enveloping algebra, which is a Hopf algebra object $\phi : U\mathfrak{g}_{\mathrm{Lie}} \otimes \mathfrak{g} \to U\mathfrak{g}_{\mathrm{Lie}}$ in $\mathcal{LM}$. The canonical quotient map $\pi : \mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$ is given by $\pi(x) = \phi(1 \otimes x)$.

Recall now from Example 2 that $\mathfrak{g}$ is recovered as ${}^{\mathrm{inv}}(U\mathfrak{g}_{\mathrm{Lie}} \otimes \mathfrak{g})$ (with trivial right coaction), and in this sense, π coincides with $\tilde{\phi}$. The Yetter-Drinfel'd braiding thus becomes the tensor flip, and the generalised Leibniz bracket $\lhd$ on $\mathfrak{g}$ is the original one.

This generalises the corresponding example for Lie algebras [19] p. 63, [3] Proposition 3.5, to Leibniz algebras. △

The above example should be viewed as an infinitesimal variant of the following one:

Example 5. Let X be a finite rack and $G := \mathrm{As}(X)$ be its associated group [6]. Then $p : X \to G$ is an augmented rack, see Remark 4 above. We have seen in Proposition 3 that the linearisation $p : kX \to kG$ is not a Hopf algebra object in $\mathcal{LM}$, so we cannot apply Theorem 1 in this situation in order to obtain a Leibniz algebra structure on kX.

However, recall from Example 1 that kX is by the very definition of an augmented rack a Yetter-Drinfel'd module over the group algebra kG, and we obtain a morphism $q : kX \to \ker\varepsilon \subset kG$, $x \mapsto p(x) - 1$ of Yetter-Drinfel'd modules. Now we can apply Corollary 1 to obtain a braided Leibniz algebra structure $x \lhd y = x(p(y) - 1)$. This construction works for all augmented racks, so augmented racks can be converted into special examples of braided Leibniz algebras. In this way, we recover [3, Proposition 3.1]. △

Example 6. If $\mathcal{T} \subset H := A^\circ$ is the quantum tangent space of a finite-dimensional first order bicovariant differential calculus over a Hopf algebra

A and $f : H \otimes \mathcal{T} \to H$ is the corresponding Hopf algebra object in $\mathcal{LM}$ (recall Remark 3), then the generalised Leibniz bracket from Theorem 1 becomes

$$x \lhd y = x\tilde{f}(y) = S(y_{(1)})xy_{(2)}.$$

That is, the generalised Leibniz algebra structure is precisely the quantum Lie algebra structure of $\mathcal{T}$, compare [13, Section 14.2.3].

Example 7. We end by explicitly computing the R-matrix representing the Yetter-Drinfel'd braiding from Example 4 for the Heisenberg-Voros algebra $\mathfrak{g}$. This is the 3-dimensional Leibniz algebra spanned by x, y, z such that the only non-trivial brackets are

$$[x,x] \;=\; z, \qquad [y,y] \;=\; z, \qquad [x,y] \;=\; z, \qquad [y,x] \;=\; -z$$

This Leibniz algebra can also be described as a 1-dimensional central extension of the abelian 2-dimensional Lie/Leibniz algebra, but rather than being antisymmetric, the cocycle has a symmetric and an antisymmetric part (in contrast to the Heisenberg Lie algebra).

The shelf structure on $\mathfrak{g}$ is given for constants $a, b, c, d, a', b', c', d' \in k$ by

$$\begin{aligned} & (a + bx + cy + dz) \lhd (a' + b'x + c'y + d'z) \\ = \;\; & aa' + a'bx + a'cy + z(a'd + bb' + bc' - cb' + cc'). \end{aligned}$$

One computes the R-matrix to be

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Observe the 13th line. This matrix does not square to 1.

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