

# A quick proof of the classification of surfaces

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## Abstract

This note contains a very short and elegant proof of the classification of surfaces that is due to Zeeman.

The classification of surfaces is one of the cornerstones of low-dimensional topology. The goal of this brief note is to explain an amazingly efficient proof of this result that is due to Zeeman [Z]. Most textbook sources for the classification are aimed at beginning students and include so much detail that it is difficult to see the main ideas. In contrast, this note is aimed at reasonably mature topologists and leaves many technical details to the reader.

I wish to thank Benson Farb for showing this proof to me.

**Remark.** For the history of the classification of surfaces and its many proofs, I recommend the book [GX].

**Remark.** Our proof will depend on the fact that surfaces can be triangulated. A wonderful proof of this that is based on Kirby's torus trick (and somehow manages to avoid almost all point-set topology) is in [H].

The version of the classification of surfaces we will prove is as follows. Let  $\Sigma_g$  denote a closed oriented genus  $g$  surface.

**Theorem 1.** *Let  $X$  be a closed oriented surface. Then  $X \cong \Sigma_g$  for some  $g \geq 0$ .*

**Remark.** It is an easy exercise to extend this proof to deal with non-orientable surfaces and surfaces with boundary.

*Proof of Theorem 1.* Fix a triangulation of  $X$ . We begin by proving the “Poincaré conjecture”, namely that if  $\chi(X) = 2$ , then  $X$  is homeomorphic to a 2-sphere. A byproduct of our argument will be that  $\chi(X) \leq 2$  for all  $X$ . Choose a maximal tree  $T$  in the 1-skeleton of  $X$ . Next, let  $\Gamma$  be the dual graph to  $T$ , i.e. the graph with one vertex in each triangle of  $X$  and where two vertices are connected by an edge exactly when their associated triangles share an edge of the triangulation which does not lie in  $T$ . Embed  $\Gamma$  in  $X$  in the obvious way. Since  $T$  contains every vertex of  $X$  but does not disconnect  $X$  (easy exercise!), the graph  $\Gamma$  is connected.

Assume that  $X$  has  $v$  vertices,  $e$  edges, and  $t$  triangles. The tree  $T$  thus has  $v$  vertices and  $e_0$  edges for some  $e_0 \leq e$ . Also, the graph  $\Gamma$  has  $t$  vertices and  $e_1$  edges for some  $e_1 \leq e$ , and  $e = e_0 + e_1$ . Therefore

$$\chi(X) = v - e + t = (v - e_0) + (t - e_1) = \chi(T) + \chi(\Gamma).$$

Recall that if  $G$  is a connected graph, then  $\chi(G) \leq 1$  with equality exactly when  $G$  is a tree (another easy exercise!). Hence  $\chi(T) = 1$  and

$$\chi(X) = 1 + \chi(\Gamma) \leq 2.$$

Moreover,  $\chi(X) = 2$  if and only if  $\Gamma$  is a tree. Assume that this holds. We can then find disjoint closed regular neighborhoods  $U$  and  $V$  of  $T$  and  $\Gamma$ , respectively, such that  $U$  and  $V$  are both homeomorphic to discs. Letting  $U$  and  $V$  expand until they touch, we see that  $X$  has been written as the union of two discs joined along a common boundary circle, so  $X$  is a 2-sphere, as desired.

Assume now that  $\chi(X) < 2$ . Letting  $T$  and  $\Gamma$  be as above, we get that  $\Gamma$  is not a tree, so  $\Gamma$  contains an embedded loop  $\gamma$ . The loop  $\gamma$  does not separate the surface; indeed, using the fact that  $T$  is a tree containing every vertex of  $X$ , we can construct a path that starts on one side of  $\gamma$ , goes in  $X \setminus \Gamma$  to  $T$ , follows a path in  $T$ , and then goes through a path in  $X \setminus \Gamma$  to end up on the other side of  $\gamma$ . Since  $X$  is orientable, the loop  $\gamma$  is 2-sided, so cutting  $X$  open along  $\gamma$  yields a surface with two boundary components. Capping these two boundary components off, we obtain a closed oriented surface  $Y$  with  $\chi(Y) = \chi(X) + 2$ . Since  $\chi(Y)$  is strictly greater than  $\chi(X)$ , we can appeal to our already-proven cases of the theorem to conclude that  $Y \cong \Sigma_{g-1}$  for some  $g \geq 1$ . By construction,  $X$  is the connect sum of  $Y$  with a torus, so  $X \cong \Sigma_g$ , as desired.  $\square$

## References

- [GX] J. Gallier and D. Xu, *A guide to the classification theorem for compact surfaces*, Geometry and Computing, 9, Springer, Heidelberg, 2013.
- [H] A. Hatcher, The Kirby torus trick for surfaces, preprint 2013, [arXiv:1312.3518](https://arxiv.org/abs/1312.3518)
- [Z] E. C. Zeeman, Introduction to topology, unpublished notes (Warwick, 1966); available at <http://www.maths.ed.ac.uk/~aar/surgery/ecztop.pdf>.

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