THE METHOD OF FOKAS FOR SOLVING LINEAR PARTIAL DIFFERENTIAL EQUATIONS

BERNARD DECONINCK, TOM TROGDON AND VISHAL VASAN
DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WA 98195-2420

Abstract. The classical methods for solving initial-boundary-value problems for linear partial differential equations with constant coefficients rely on separation of variables, and specific integral transforms. As such, they are limited to specific equations, with special boundary conditions. Here we review a method introduced by Fokas, which contains the classical methods as special cases. However, this method allows also for the equally explicit solution of problems for which no classical approach exists. In addition, it is easy to elucidate which boundary-value problems are well posed and which are not. We provide examples of problems posed on the positive half-line and on the finite interval. Some of these examples have solutions obtainable using classical methods, others do not. For the former examples, it is illustrated how the classical methods may be recovered from the more general approach of Fokas.

1. Introduction. The canonical first course in partial differential equations (PDEs) focuses on explicit solution methods for problems for which such a solution can be found. The solution techniques presented include the method of separation of variables, Fourier series and transforms, Laplace and other integral transforms, Green's functions, *etc.* Standard textbooks for such courses include [4, 11, 12, 13, 17], each with their own amount of rigor. These textbooks focus almost all of their attention on scalar equations that are first- or second-order in any independent variable. In this context, an initial- or boundary-value problem (IVP or BVP) is considered solved if an explicit[1] expression is constructed for the dependent variable as a function of the independent variables and of the given initial and boundary conditions. This expression is typically given as an infinite series or an integral, or a combination of such.

In this paper, we advocate a different method from the ones taught in any course. The method is relatively new. It was discovered by A. S. Fokas in his quest to generalize the method of inverse scattering, which solves the IVP for $x \in \mathbb{R}$ for so-called soliton equations, to BVPs posed either on the half line $x \geq 0$ or on the finite interval $x \in [0, L]$ [6, 7, 9]. It was observed immediately [8, 10] that the method produces interesting results for linear equations as well, which is our focus here. Just like the classical methods, the method of Fokas produces an explicit solution for the dependent variable $u(x, t)$. The solution formula is in terms of one or more integrals along paths in the complex plane of an auxiliary variable k. All x and t dependence in the solution formula is entirely explicit. Although the method is more general, we restrict our attention to scalar problems with one spatial independent variable x and one temporal independent variable t. We present the method of Fokas as a valuable addition to the standard methods taught, for several reasons.

- It encompasses the standard methods. For those cases where a standard method produces an explicit solution, Fokas's method does so as well. In fact, as we show in several examples below, the resulting solution formulas are equivalent, as they should be.
- It is more general than the standard methods: we are able to produce solution formulas for many problems where the classical methods are unable to do so. This is particularly clear for problems containing higher than second-order derivatives.
- Where the standard methods are a collection of situation-specific approaches, tailored to specific equations and boundary conditions, the method of Fokas produces a solution using the same ideas for all these different problems, with the differences appearing only in the calculational details.
- In addition to producing an explicit formula for the solution, the method allows one to determine in a straight-forward way how many and which boundary conditions result in a well-posed problem. Especially for BVPs for equations with more than second-order derivatives, this is a nontrivial issue.

[1] For our purposes, an explicit expression is one whose evaluation is the topic of a lower-level course. For instance, the ordinary differential equation $y' + 2xy = 1$ has an explicit solution since it is possible to express the solution in terms of operations whose evaluation is the topic of calculus, even though not all anti-derivatives can be written in terms of elementary functions. Similarly, Fourier series, integrals in the complex plane, *etc*, give rise to explicit solutions of partial differential equations.

- The background required for the method is limited to the knowledge of the Fourier and inverse Fourier transform pair, the Residue Theorem and Jordan's Lemma, see [1] or [3], for instance. In other words, even if no course on complex analysis precedes the first course on partial differential equations, the necessary material is easily introduced in three or four lectures.

Our introduction to the method of Fokas proceeds mostly by example. More details and general arguments can be found in [9]. We start by revisiting the IVP on the whole line in the framework of the new method, see Section 2. This results merely in the construction of the familiar Fourier transform expression for the solution, but it does allow us to generalize the approach to the half-line and finite interval problems more naturally. This is done in Sections 3 and 4, respectively. These sections comprise the bulk of the paper. The reader will find examples of problems that can be solved using the standard methods, in which case we demonstrate the equivalence of the classical results with those obtained using Fokas's method. Other examples illustrate the use of the method in situations where the standard methods fail or are unable to proceed. Our emphasis is not on rigor and no comments are made about function spaces. Rather, in the vein of a first course on PDEs (see, for instance, [4, 12]), we assume whatever conditions are necessary for the calculations to proceed: for instance, functions are at least as differentiable as dictated by the equation and boundary and initial conditions are compatible at $(x,t)=(0,0)$ and, for the finite interval, at $(x,t)=(L,0)$.

2. The initial-value problem on the whole line. In contrast to a traditional first course on PDEs we begin by considering the IVP for linear constant coefficient equations on the entire real line. In order to start with a concrete example, consider the IVP for the heat equation:

$$q_t = q_{xx}, \qquad x\in\mathbb{R},\ t\in(0,T], \tag{2.1a}$$

$$q(x,0) = q_0(x), \qquad x\in\mathbb{R}, \tag{2.1b}$$

where subscripts denote partial differentiation. Here T is a positive real number, and we impose the boundary condition that $q(x,t)\to 0$ sufficiently fast as $|x|\to\infty$, for all $t\geq 0$. The solution to this problem is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{q}_0(k)e^{ikx-\omega(k)t}dk, \tag{2.2}$$

where $\omega(k)=k^2$ is the dispersion relation[2] for the heat equation, and $\hat{q}_0(k)$ is the Fourier transform of the initial condition:

$$\hat{q}_0(k) = \int_{-\infty}^{\infty} q_0(x)e^{-ikx}dx. \tag{2.3}$$

Further, letting $T\to\infty$ does not affect the validity of the solution. In a traditional text, this result is obtained by taking the Fourier transform of the original PDE (2.1a). This results in a one-parameter family of ordinary differential equations (ODEs) with parameter k. Solving this ODE gives the Fourier transform of the solution at any time t. Using the inverse transform, the above result is obtained.

In the new approach, we start by rewriting the heat equation as a one-parameter family of PDEs in divergence form

$$\partial_t\rho(x,t,k)+\partial_x j(x,t,k)=0, \tag{2.4}$$

where

$$\rho(x,t,k)=e^{-ikx+\omega(k)t}q,$$

with $k\in\mathbb{C}$. We refer to (2.4) as the *local relation*. The explicit form of j is easily obtained using the reverse

[2]For consistency with the literature on Fokas's method [9], we adopt the convention that the dispersion relation $\omega(k)$ is found by substitution of $\exp(ikx-\omega(k)t)$ in the PDE.

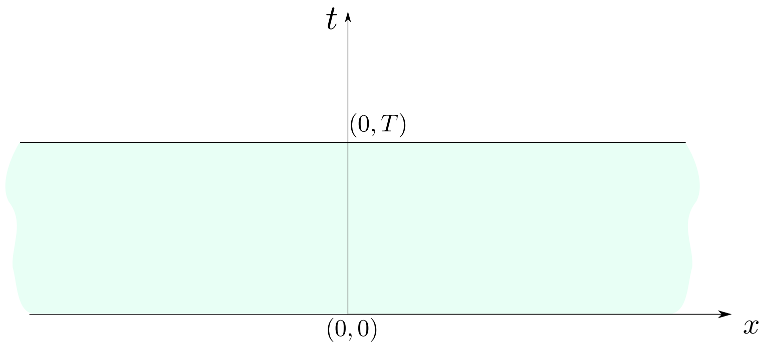

Fig. 2.1: The region of integration in the (x,t) plane for problems posed on the whole-line.

product rule:

$$\begin{aligned}
\left(e^{-ikx+\omega(k)t}q\right)_t &= \omega(k)e^{-ikx+\omega(k)t}q + e^{-ikx+\omega(k)t}q_t\\
&= \omega(k)e^{-ikx+\omega(k)t}q + e^{-ikx+\omega(k)t}q_{xx}\\
&= \omega(k)e^{-ikx+\omega(k)t}q + \left(e^{-ikx+\omega(k)t}q_x\right)_x + ike^{-ikx+\omega(k)t}q_x\\
&= \omega(k)e^{-ikx+\omega(k)t}q + \left(e^{-ikx+\omega(k)t}q_x\right)_x + \left(ike^{-ikx+\omega(k)t}q\right)_x - k^2e^{-ikx+\omega(k)t}q\\
&= \left(e^{-ikx+\omega(k)t}(q_x+ikq)\right)_x, \qquad (2.5)
\end{aligned}$$

so that $j = -e^{-ikx+\omega(k)t}(q_x+ikq)$. Note that this calculation also determines $\omega(k)$.

We are now in a position to apply Green's Theorem in the (x,t) plane. Consider (2.5) on an infinite horizontal strip $\mathcal{D}$ of height T (see Figure 2.1). In what follows s is a temporal dummy variable. Using the decay properties of the solution where necessary, we have

$$\begin{aligned}
&\iint_{\mathcal{D}}\left(\left[e^{-ikx+\omega(k)s}q(x,s)\right]_s - \left[e^{-ikx+\omega(k)s}(q_x(x,s)+ikq(x,s))\right]_x\right)dsdx = 0\\
\overset{\text{(Green's Theorem)}}{\Rightarrow}\quad &\int_{\partial\mathcal{D}}\left(e^{-ikx+\omega(k)s}q(x,s)dx + e^{-ikx+\omega(k)s}(q_x(x,s)+ikq(x,s))ds\right) = 0\\
\Rightarrow\quad &\int_{-\infty}^{\infty}e^{-ikx}q_0(x)dx - \int_{-\infty}^{\infty}e^{-ikx+\omega(k)T}q(x,T)dx = 0\\
\Rightarrow\quad &\int_{-\infty}^{\infty}e^{-ikx}q(x,T)dx = e^{-\omega(k)T}\int_{-\infty}^{\infty}e^{-ikx}q_0(x)dx. \qquad (2.6)
\end{aligned}$$

The contribution of the second integrand on the second line vanishes due to the decay properties of the solution as $|x|\to\infty$. In later sections, such terms bring in the contributions from the boundary data. The last line relates the Fourier transform of the solution at time T to the Fourier transform of the initial value $q_0(x)$, as expected. Inverting the transform we obtain the solution (2.2).

Let us take a step back to review our work. It appears we have obtained the traditional result in a roundabout way. However, progress has been made. The procedure used here is suited for solving problems other than those posed on the whole line. By working with different domains (as those used below) for the application of Green's Theorem to the local relation we obtain the integral expressions for solutions to the corresponding BVP.

Remarks.

- The procedure above works equally well for equations other than the heat equation. In fact, the solution formula (2.2) stands as is, as long as the appropriate dispersion relation is filled in. We do

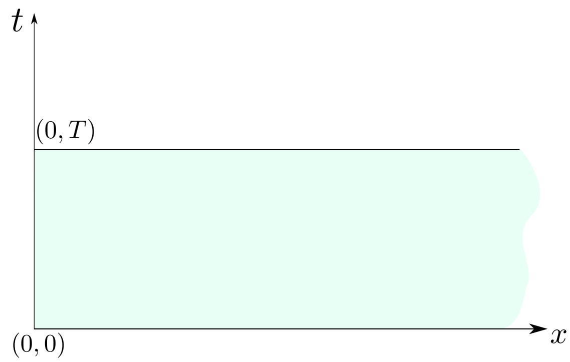

Fig. 3.1: The region of integration in the (x,t) plane for boundary-value problems posed on the positive half-line.

not elaborate on this here, details are found, for instance, in the Appendix of [2]. For BVPs, more detail is found below.

- The conversion of the PDE to the local relation (2.4) is always possible for linear constant coefficient PDEs [9]. The explicit form of $j(x,t,k)$ in terms of $\omega(k)$, avoiding the reverse product rule, is given in (3.33). See Section 3.5 for more detail.

3. The problem on the half line.

3.1. The heat equation with Dirichlet boundary conditions. In section 2 the heat equation on the whole real line was solved. The success of this method was due to the properties of the Fourier transform which incorporates the "boundary condition" of decay at infinity. In this section we solve the heat equation on the half line with Dirichlet boundary data:

$$q_t = q_{xx}, \qquad x > 0,\ t \in (0,T], \tag{3.1a}$$
$$q(x,0) = q_0(x), \qquad x \geq 0, \tag{3.1b}$$
$$q(0,t) = g_0(t), \qquad t \in [0,T]. \tag{3.1c}$$

This is a standard textbook problem, easily solved using classical techniques. As mentioned, its inclusion here serves the purpose of allowing the reader to see how problems solvable using the classical methods can be solved using the method of Fokas. We begin by considering the local relation of the heat equation (2.5), which holds independent of the solution domain and the boundary conditions, as it is a local statement. Applying Green's Theorem to this equation with the domain of integration $\mathcal{R} = \{x \geq 0,\ 0 < t \leq T\}$ (see Figure 3.1) we obtain

$$\begin{aligned}
&\int_{\partial\mathcal{R}} \left(e^{-ikx+\omega(k)s} q(x,s)dx + e^{-ikx+\omega(k)s}(q_x(x,s) + ikq(x,s))ds\right) = 0 \\
\Rightarrow\quad &\int_0^\infty e^{-ikx} q_0(x)dx - \int_0^\infty e^{-ikx+\omega(k)T} q(x,T)dx - \int_0^T e^{\omega(k)s}(q_x(0,s) + ikq(0,s))ds = 0 \\
\Rightarrow\quad &\int_0^\infty e^{-ikx} q_0(x)dx - \int_0^T e^{\omega(k)s}(q_x(0,s) + ikq(0,s))ds = e^{\omega(k)T} \int_0^\infty e^{-ikx} q(x,T)dx \\
\Rightarrow\quad &\hat{q}_0(k) - [\tilde{g}_1(\omega(k),T) + ik\tilde{g}_0(\omega(k),T)] = e^{\omega(k)T}\hat{q}(k,T),
\end{aligned} \tag{3.2}$$

where $\partial\mathcal{R}$ denotes the boundary of the domain $\mathcal{R}$, oriented so that the domain $\mathcal{R}$ is on the left when the boundary is traversed. Further, $\hat{q}_0, \hat{q}$ are the Fourier transforms of the initial condition and the solution at time T respectively. Similarly, $\tilde{g}_0$ and $\tilde{g}_1$ are defined in terms of the boundary data as

$$\tilde{g}_0(\omega,T) = \int_0^T e^{\omega s} q(0,s)ds, \quad \tilde{g}_1(\omega,T) = \int_0^T e^{\omega s} q_x(0,s)ds.$$

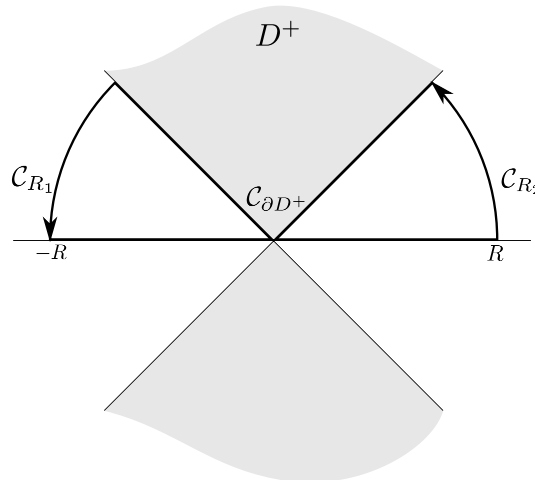

Fig. 3.2: The domain D for the heat equation is indicated in gray. The contour $\mathcal{C}$, shown in bold, is used to justify the deformation of the line-integral along the real line up to the boundary of D^+.

We refer to (3.2) as the global relation for the heat equation on the half line. The time transforms $\tilde{g}_0, \tilde{g}_1$ arise because of the presence of the boundary at $x = 0$. For the problem under consideration, the Dirichlet data are given, thus $\tilde{g}_0$ is determined, whereas $\tilde{g}_1$ is not.

The spectral parameter k associated with the Fourier transform is typically real. However, the terms in the global relation (3.2) are analytic for $\text{Im}(k) < 0$. Indeed, the Fourier transforms may be analytically continued into the lower-half of the complex k plane due to the exponential decay there, assuming sufficient decay of $q(x,t)$ for large x and all t. Further, the time transforms $(\tilde{g}_0, \tilde{g}_1)$ are entire functions (analytic and bounded for all finite k).

Remark. The continuation of the global relation into the lower half plane, for the equation on the half line, should be contrasted with the "global relation" for the whole-line case (2.6) where such an extension to complex k is not possible.

The global relation (3.2) is equally valid for any $t \in (0, T]$. Replacing T by t in the global relation (3.2) and inverting the Fourier transform we arrive at an integral expression for $q(x,t)$:

$$\hat{q}_0(k) - [\tilde{g}_1(\omega(k), t) + ik\tilde{g}_0(\omega(k), t)] = e^{\omega(k)t}\hat{q}(k,t)$$

$$\Rightarrow \qquad q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}[\tilde{g}_1(\omega(k), t) + ik\tilde{g}_0(\omega(k), t)]dk. \qquad (3.3)$$

Let $D = \{k \in \mathbb{C} \,:\, \text{Re}(\omega(k)) < 0\}$. Further, let $D^+ = D \cap \mathbb{C}^+$ where $\mathbb{C}^+ = \{k \in \mathbb{C} : \text{Im}(k) > 0\}$. The integrand of the second integral in (3.3) is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \backslash D^+$. Consider a contour $\mathcal{C} = [-R, R] \cup \mathcal{C}_{R_2} \cup \mathcal{C}_{\partial D^+} \cup \mathcal{C}_{R_1}$ as shown in bold in Figure 3.2. Let $\mathcal{C}_{\partial D^+}$ be the part of $\mathcal{C}$ on the boundary of D^+ and $\mathcal{C}_{R_1}$, $\mathcal{C}_{R_2}$ be circular arcs of radius R. Using the analyticity of the integrand

$$\int_{\mathcal{C}} e^{ikx-\omega(k)t}\tilde{g}(\omega(k), t)dk = \left(\int_{-R}^{R} + \int_{\mathcal{C}_{R_2}} + \int_{\mathcal{C}_{\partial D^+}} + \int_{\mathcal{C}_{R_1}}\right) e^{ikx-\omega(k)t}\tilde{g}(\omega(k), t)dk = 0, \qquad (3.4)$$

where $\tilde{g}(\omega(k), t) = \tilde{g}_1(\omega(k), t) + ik\tilde{g}_0(\omega(k), t)$. Taking the limit $R \to \infty$ of the above expression, we observe that the contour $\mathcal{C}_{\partial D^+}$ becomes the contour $-\partial D^+$. The negative sign arises from the convention that the positive orientation of a boundary is defined so that the region is to the left as the boundary is traversed. Further, an application of Jordan's Lemma in the wedge-like regions shows that for large R, the contribution of the integrals along $\mathcal{C}_{R_1}$ and $\mathcal{C}_{R_2}$ vanishes. We obtain that the integral of $\exp(ikx - \omega(k)t)\tilde{g}(\omega(k), t)$ along

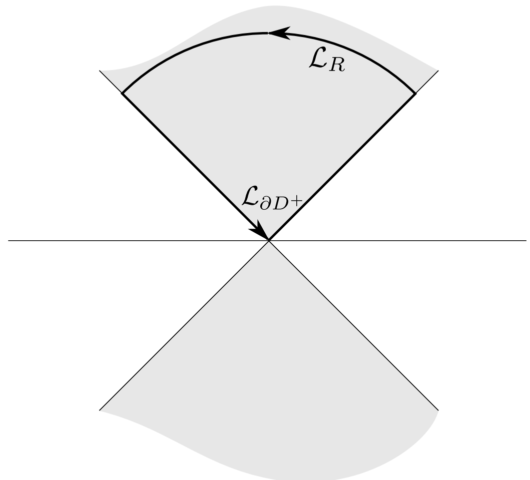

Fig. 3.3: The contour $\mathcal{L}$ is shown in bold. Application of Cauchy's Integral Theorem using this contour allows one to eliminate the contribution of $\hat{q}(-k,t)$ from the integral expression (3.7).

the real line may be replaced by one along ∂D^+. Thus

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-\omega(k)t}\left[\tilde{g}_1(\omega(k),t) + ik\tilde{g}_0(\omega(k),t)\right]dk. \tag{3.5}$$

Summarizing the argument, the contour $\mathcal{C}$ may be replaced by $\partial(\mathbb{C}^+\backslash D^+)$ as $R\to\infty$. The integral of $\exp(ikx-\omega(k)t)\tilde{g}(\omega(k),t)$ along $\partial(\mathbb{C}^+\backslash D^+)$ is zero due to the analyticity and decay properties of the integrand. Adding this integral to the right-hand side of (3.3) we obtain (3.5).

Although (3.5) is an expression for $q(x,t)$, it does not present a solution since it depends on boundary data we have not prescribed through $\tilde{g}_1$, the transform of the Neumann data. To resolve this problem, we could solve the global relation (3.2) for $\tilde{g}_1$. This results in an expression for $\tilde{g}_1$ valid in $\mathbb{C}^- = \{k : \mathrm{Im}(k) < 0\}$, whereas (3.5) requires an expression for $\tilde{g}_1$ valid along ∂D^+. To this end we seek a transform that maps the contour ∂D^+ to a contour in the lower-half plane $\mathbb{C}^-$ but leaves $\tilde{g}_1(\omega(k),t)$ invariant. Thus we turn to the discrete symmetries of $\omega(k) = k^2$. The dispersion relation $\omega(k)$ is invariant under the transform $k\to -k$. Applying this transformation to the global relation (3.2) (and replacing T by t) we have

$$\hat{q}(-k,t) = e^{-\omega(k)t}\hat{q}_0(-k) - e^{-\omega(k)t}[\tilde{g}_1(\omega(k),t) - ik\tilde{g}_0(\omega(k),t)], \quad \mathrm{Im}(k)\geq 0. \tag{3.6}$$

Solving this version of the global relation yields an expression for $\tilde{g}_1(\omega(k),t)$ which *is* valid along ∂D^+. The integral expression for $q(x,t)$ becomes

$$\begin{aligned} q(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-\omega(k)t}\left[2ik\tilde{g}_0(\omega(k),t) + \hat{q}_0(-k)\right]dk \\ &+ \frac{1}{2\pi}\int_{\partial D^+} e^{ikx}\hat{q}(-k,t)dk. \end{aligned} \tag{3.7}$$

The above expression does not depend on the unknown boundary data. However, the function we wish to solve for, $q(x,t)$, also appears in the third integral on the right-hand side. Using analyticity, this problem is resolved as follows. The function $\hat{q}(-k,t)$ is bounded and analytic in $\mathbb{C}^+$ with $\hat{q}(-k,t)\to 0$ uniformly as $k\to\infty$. This implies that the integral of $\exp(ikx)q(-k,t)$ along a closed, bounded curve in $\mathbb{C}^+$ vanishes. In particular, we consider a closed curve $\mathcal{L} = \mathcal{L}_{\partial D^+}\cup\mathcal{L}_R$ where $\mathcal{L}_{\partial D^+} = \partial D^+\cap\{k : |k| < R\}$ and $\mathcal{L}_R = \{k\in D^+ : |k| = R\}$ (see Figure 3.3). We have

$$\int_{\mathcal{L}} e^{ikx}\hat{q}(-k,t)dk = \int_{\mathcal{L}_{\partial D^+}} e^{ikx}\hat{q}(-k,t)dk + \int_{\mathcal{L}_R} e^{ikx}\hat{q}(-k,t)dk = 0. \tag{3.8}$$

If it can be shown that the integral along $\mathcal{L}_R$ vanishes for large R, then the third integral on the right-hand side of (3.7) must also vanish since the contour $\mathcal{L}_{\partial D^+}$ becomes ∂D^+ as $R \to \infty$. From Jordan's Lemma, the uniform decay of $\hat{q}(-k, t)$ for large k is precisely the condition required for the integral to vanish. Thus

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-\omega(k)t} \hat{q}_0(k)dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-\omega(k)t} \left[\hat{q}_0(-k) + 2ik\tilde{g}_0(\omega(k), t)\right] dk \tag{3.9}$$

is the solution to the Dirichlet problem for the heat equation on the half line. The contours in the complex plane may be parameterized as desired.

Let us summarize the steps involved in this method.

1. Using the dispersion relation $\omega(k)$ we define the regions $D = \{k : \mathrm{Re}(\omega(k)) < 0\}$, $D^+ = D \cap \mathbb{C}^+$ and $D^- = D \cap \mathbb{C}^-$.
2. The PDE is rewritten as a one-parameter family of equations in divergence form. Applying Green's Theorem in the (x, t) plane, we obtain the global relation. By considering complex values for the spectral parameter k, we extend the domain of definition of the global relation. For problems posed on the half line, the global relation is valid in $\mathrm{Im}(k) \leq 0$.
3. The global relation is solved for $\hat{q}(k, t)$. The integral expression for $q(x, t)$ is constructed by inverting the Fourier transform. The integral involving the boundary terms is deformed off the real line. For problems on the half line, we deform up to ∂D^+.
4. The discrete symmetries $\nu(k)$ of $\omega(k)$, $\omega(k) = \omega(\nu(k))$, are used to obtain additional versions of the global relation valid for k in certain regions of $\mathbb{C}^+$. These additional global relations are solved simultaneously for the transforms of the unknown boundary data. The expressions thus obtained are substituted into the integral expression for $q(x, t)$.
5. The integral expression for $q(x, t)$ now depends on $\hat{q}(\nu(k), t)$. Analyticity considerations are used to evaluate the contribution of $\hat{q}(\nu(k), t)$. Typically this contribution vanishes. The functions $\hat{q}(\nu(k), t)$ and $\exp(ikx)$ are bounded and analytic in the upper-half plane. If the coefficient of $\exp(ikx)\hat{q}(\nu(k), t)$ in the integral expression is also analytic in D^+, then the contribution of this term is zero. For instance, this happens for the Dirichlet problem for the heat equation posed on the half line where this coefficient is a constant. Let us now consider the case when this coefficient has a simple pole at some point $k = k_0$ in D^+. An application of the Residue Theorem shows that we need the value of $\hat{q}(\nu(k_0), t)$. At this point, the global relation is used once more. The global relation connects the transform of the solution at time t to the transforms of both the known initial-boundary conditions and the unknown boundary conditions. Evaluating the global relation at $k = \nu(k_0)$, we obtain an expression for $\hat{q}(\nu(k_0), t)$. If this expression depends only on known initial-boundary conditions (*i.e.* the coefficients of the unknown terms add up to zero) we have solved the problem. The presence of unknown boundary conditions hints at an ill-posed problem. Note that we select only those transformations $\nu(k)$ whose image lies in the lower-half plane. Consequently $\hat{q}(\nu(k_0), t)$ is defined.

Remarks.

- For the Neumann problem, when $q_x(0, t)$ is supplied as the boundary condition, we can just as easily solve the global relation for $\tilde{g}_0$. The integral involving $\hat{q}(-k, t)$ vanishes for the same reason as for the Dirichlet problem [9].
- The classical solution in terms of the sine transform may be recovered from the solution to the heat equation (3.9). Note that $\hat{q}_0(-k)$ is analytic and bounded in the upper-half plane and $\exp(-\omega(k)t)k\tilde{g}_0(\omega(k), t)$ is bounded and analytic in the region between D^+ and the real line. Thus the contour ∂D^+ may be deformed back to the real line:

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-\omega(k)t} \hat{q}_0(k)dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-\omega(k)t} (\hat{q}_0(-k) + 2ik\tilde{g}_0(\omega(k), t))dk.$$

  By splitting the integral along the real line into integrals from $-\infty$ to 0 and from 0 to ∞, and using the definition of sine in terms of exponentials, we obtain

$$q(x,t) = \frac{2}{\pi} \int_0^{\infty} e^{-\omega(k)t} \sin(kx) \left[\int_0^{\infty} \sin(ky)q(y,0)dy - k\tilde{g}_0(\omega(k), t)\right] dk.$$

This deformation back to the real line is not possible for all PDEs. In particular the deformation is possible only when a classical transform exists. Also note, that unlike the integral representation (3.9), the sine transform solution is not uniformly convergent at $x = 0$. Further, using methods like steepest descent or stationary phase, the contribution due to the boundary can be evaluated much more efficiently in (3.9) than in the classical sine transform solution, see [5].

- An alternative to the new method is the use of Laplace transforms in t. This results in multivalued integral kernels due to the presence of radicals. Consequently, the inversion of the Laplace transform involves contour integrals along branch cuts, *e.g* see example on pg. 358 of [17]. For spatial derivatives of order greater than two, this procedure quickly becomes far more involved. Further, the Laplace transform involves an integration over all time $t \geq 0$. This seems to contradict causality for evolution problems. For evolution equations we do not expect the solution at time $t = T$ to depend on times greater than T.

3.2. A third-order PDE with Dirichlet boundary conditions. As a second example consider the following problem, posed again on the half line:

$$q_t + q_{xxx} = 0, \qquad x \geq 0,\ t \in (0, T], \tag{3.10a}$$

$$q(x,0) = q_0(x), \qquad x \geq 0, \tag{3.10b}$$

$$q(0,t) = g_0(t), \qquad t \in (0, T]. \tag{3.10c}$$

As before, we assume that $q(x,t)$ decays sufficiently fast as $x \to \infty$, for all $t \geq 0$. We follow the steps outlined in the previous section.

1. *Dispersion relation.* The dispersion relation for the PDE is $\omega(k) = -ik^3$. Using

$$\omega(k) = -i|k|^3[\cos(3\arg k) + i\sin(3\arg k)],$$

we define the region D (where $\mathrm{Re}(\omega(k)) < 0$) as

$$D = \left\{k : \arg k \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{4\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right)\right\},$$

so that

$$D^+ = \left\{k : \arg k \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)\right\},$$

and

$$D^- = D_1^- \cup D_2^-,$$

where

$$D_1^- = \left\{k : \arg k \in \left(\pi, \frac{4\pi}{3}\right)\right\}, \quad D_2^- = \left\{k : \arg k \in \left(\frac{5\pi}{3}, 2\pi\right)\right\}.$$

See Figure 3.4 for a depiction of these regions.

2. *Global relation.* Using the same method as before, we find the local relation

$$\left(e^{-ikx+\omega(k)t}q\right)_t + \left(e^{-ikx+\omega(k)t}(q_{xx} + ikq_x + (ik)^2 q)\right)_x = 0,$$

which is easily verified. Integrating the above equation over the domain $\mathcal{R} = \{0 \leq x < \infty, 0 < t \leq T\}$

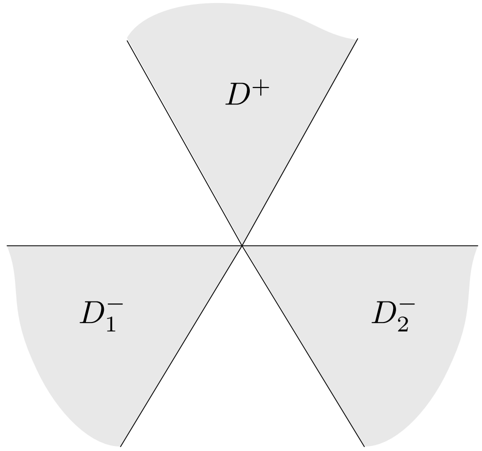

Fig. 3.4: Domain $D = \{k : \mathrm{Re}(\omega(k)) < 0\}$ for the third-order PDE (3.10a).

and applying Green's Theorem, we obtain

$$\begin{aligned}
&\iint_{\mathcal{R}} \left(\left[e^{-ikx+\omega(k)s} q(x,s) \right]_s - \left[e^{-ikx+\omega(k)t} (-q_{xx} - ikq_x - (ik)^2 q) \right]_x \right) ds dx = 0 \\
\Rightarrow &\int_{\partial\mathcal{R}} \left(e^{-ikx+\omega(k)s} q(x,s) dx + e^{-ikx+\omega(k)s} (-q_{xx}(x,s) - ikq_x(x,s) - (ik)^2 q(x,s)) dt \right) = 0 \\
\Rightarrow &\int_0^\infty e^{-ikx} q(x,0) dx - \int_0^T e^{\omega(k)s} (-q_{xx}(0,s) - ikq_x(0,s) - (ik)^2 q(0,s)) ds \\
&\qquad = \int_0^\infty e^{-ikx+\omega(k)T} q(x,T) dx \\
\Rightarrow &\hat{q}_0(k) - [k^2 \tilde{g}_0(\omega(k),T) - ik\tilde{g}_1(\omega(k),T) - \tilde{g}_2(\omega(k),T)] = e^{\omega(k)T} \hat{q}(k,T), \qquad (3.11)
\end{aligned}$$

where, as before, $\hat{q}_0(k)$ and $\hat{q}(k,T)$ represent the Fourier transform of the solution at time $t = 0$ and time $t = T$. The time transforms of the boundary data are given by

$$\tilde{g}_i(\omega, T) = \int_0^T e^{\omega s} \partial_x^i q(0,s) ds, \; i = 0, 1, 2.$$

Equation (3.11) is the global relation for the third-order PDE (3.10a) posed on the positive half-line. Note that it is valid for $\mathrm{Im}(k) \leq 0$.

3. *Integral expression.* Let

$$\tilde{g}(k,t) = k^2 \tilde{g}_0(\omega(k),t) - ik\tilde{g}_1(\omega(k),t) - \tilde{g}_2(\omega(k),t).$$

Replacing T by t in the global relation (3.11) and applying the inverse Fourier transform, we obtain

$$\begin{aligned}
&e^{\omega(k)t} \hat{q}(k,t) = \hat{q}_0(k) - \tilde{g}(k,t) \\
\Rightarrow\ &\hat{q}(k,t) = e^{-\omega(k)t} \hat{q}_0(k) - e^{-\omega(k)t} \tilde{g}(k,t) \\
\Rightarrow\ &q(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx-\omega(k)t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx-\omega(k)t} \tilde{g}(k,t) dk \\
\Rightarrow\ &q(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx-\omega(k)t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-\omega(k)t} \tilde{g}(k,t) dk,
\end{aligned}$$

where the second integral on the last line has been deformed into the upper-half complex plane up to the boundary of D^+.

4. *Solving for the unknown boundary data.* As for the heat equation, the integral expression for $q(x,t)$ does not represent a solution due to presence of unknown boundary terms, here $\tilde{g}_1, \tilde{g}_2$. As before we use the discrete symmetries of $\omega(k)$. In this case, these are $\nu_1(k) = e^{\frac{2\pi i}{3}}k$ and $\nu_2(k) = e^{\frac{4\pi i}{3}}k$.
We observe that for $k \in D^+$, $\nu_1(k) \in D_1^-$ and $\nu_2(k) \in D_2^-$. Applying these transformations to the global relation, for $k \in D^+$ with $\alpha = e^{\frac{2\pi i}{3}}$ we find

$$\hat{q}_0(\alpha k) - \alpha^2 k^2 \tilde{g}_0(\omega(k),t) + i\alpha k \tilde{g}_1(\omega(k),t) + \tilde{g}_2(\omega(k),t) = e^{\omega(k)t}\hat{q}(\alpha k,t),$$
$$\hat{q}_0(\alpha^2 k) - \alpha k^2 \tilde{g}_0(\omega(k),t) + i\alpha^2 k \tilde{g}_1(\omega(k),t) + \tilde{g}_2(\omega(k),t) = e^{\omega(k)t}\hat{q}(\alpha^2 k,t).$$

Given $q(x,0)$ and $q(0,t)$ (or $\hat{q}_0(k)$ and $\tilde{g}_0(\omega,t)$) we may solve the above two equations for the two unknowns $\tilde{g}_1$ and $\tilde{g}_2$, for k in D^+ to obtain

$$\begin{aligned}
ik\tilde{g}_1(\omega(k),t) = -k^2\tilde{g}_0(\omega(k),t) &+ \frac{1}{\alpha(1-\alpha)}\left[\hat{q}_0(\alpha^2 k) - \hat{q}_0(\alpha k)\right] \\
&+ \frac{e^{\omega(k)t}}{\alpha(1-\alpha)}\left[\hat{q}(\alpha k,t) - \hat{q}(\alpha^2 k,t)\right], \\
\tilde{g}_2(\omega(k),t) = -k^2\tilde{g}_0(\omega(k),t) &+ \frac{1}{\alpha(1-\alpha)}\left[\alpha^2\hat{q}_0(\alpha k) - \alpha\hat{q}_0(\alpha^2 k)\right] \\
&+ \frac{e^{\omega(k)t}}{\alpha(1-\alpha)}\left[-\alpha^2\hat{q}(\alpha k,t) + \alpha\hat{q}(\alpha^2 k,t)\right].
\end{aligned}$$

These expressions are substituted into the integral expression for $q(x,t)$, resulting in

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} H(k,x,t)dk, \tag{3.12}$$

$$\begin{aligned}
H(k,x,t) = &e^{ikx-\omega(k)t}\left[3k^2\tilde{g}_0(\omega(k),t) - \alpha\hat{q}_0(\alpha k) - \alpha^2\hat{q}_0(\alpha^2 k)\right] \\
&+ e^{ikx}\left[-\alpha\hat{q}(\alpha k,t) - \alpha^2\hat{q}(\alpha^2 k,t)\right].
\end{aligned}$$

We have used that α is the cube root of unity and thus $\alpha^3 = 1$ and $1+\alpha+\alpha^2 = 0$.

5. *Contribution of $\hat{q}(\nu(k),t)$.* The functions $\hat{q}(\alpha k,t)$ and $\hat{q}(\alpha^2 k,t)$ are bounded and analytic in D^+ and decay to zero uniformly as $k \to \infty$. Once again Jordan's Lemma implies that these terms do not contribute to the final solution, which is given by

$$\begin{aligned}
q(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} 3k^2 e^{ikx-\omega(k)t}\tilde{g}_0(\omega(k),t)dk \\
&+ \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-\omega(k)t}\left[\alpha\hat{q}_0(\alpha k) + \alpha^2\hat{q}_0(\alpha^2 k)\right]dk.
\end{aligned} \tag{3.13}$$

The classical *method of images* approach, by which one obtains the sine transform solution to the heat equation, cannot be applied to this third-order PDE. Indeed it is not possible to obtain solutions via the method of images for any PDE which involves odd order derivatives in x. For the third-order prolem considered here, since $\hat{q}(\alpha k)$ is not bounded in the region $\{\arg(k) \in [0, \pi/3]\}$, Jordan's Lemma may not be applied in order to justify the deformation of the contour integral back to the real line. This apparently implies an integral transform pair using integrals along the real line does not exist for this third-order PDE.

3.3. A (slightly) different third-order PDE. The two problems considered so far are well posed when only $q(0,t)$ is specified at the left boundary. Using a slight modification of the previous example we illustrate what changes when more boundary conditions are required. Consider the following third-order PDE on the positive half-line:

$$q_t - q_{xxx} = 0. \tag{3.14}$$

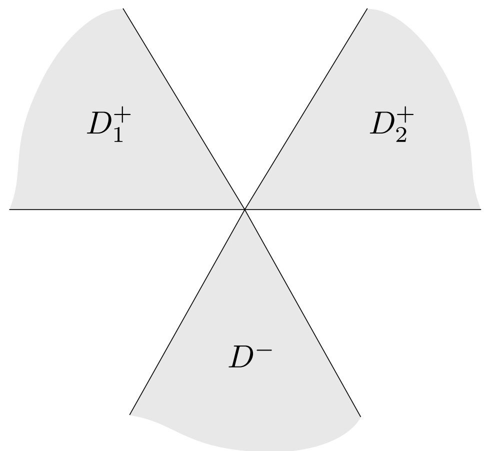

Fig. 3.5: Domain $D = \{k : \mathrm{Re}(\omega(k)) < 0\}$ for the third-order PDE (3.14).

We assume that an initial condition $q(x,0)$ has been given. The analysis of the global relation indicates the number and type of the boundary conditions that need to be prescribed in order for the problem to be well posed. The dispersion relation is

$$\omega(k) = ik^3, \tag{3.15}$$

and

$$D = \left\{k : \arg k \in \left(0, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right) \cup \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)\right\}. \tag{3.16}$$

Let $D^+ = D_1^+ \cup D_2^+$, where

$$D_1^+ = \left\{k : \arg k \in \left(\pi, \frac{\pi}{3}\right)\right\}, \quad D_2^+ = \left\{k : \arg k \in \left(\frac{2\pi}{3}, \pi\right)\right\}, \tag{3.17}$$

and

$$D^- = \left\{k : \arg k \in \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)\right\},$$

see Figure 3.5.

The local relation is given by

$$(e^{-ikx+\omega(k)t}q)_t - \left(e^{-ikx+\omega(k)t}(q_{xx} + ikq_x - k^2 q)\right)_x = 0, \tag{3.18}$$

leading to the global relation (by integrating over the region $\mathcal{R} = \{x \geq 0,\ 0 < t \leq T\}$ and applying Green's theorem)

$$\hat{q}_0(k) - [-k^2\tilde{g}_0(\omega(k),t) + ik\tilde{g}_1(\omega(k),t) + \tilde{g}_2(\omega(k),t)] = e^{\omega(k)t}\hat{q}(k,t), \quad \mathrm{Im}(k) \leq 0. \tag{3.19}$$

The integral expression for the solution is

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D_1^+} e^{ikx-\omega(k)t}\tilde{g}(k,t)dk - \frac{1}{2\pi}\int_{\partial D_2^+} e^{ikx-\omega(k)t}\tilde{g}(k,t)dk, \tag{3.20}$$

where

$$\tilde{g}(k,t) = -k^2\tilde{g}_0(\omega(k),t) + ik\tilde{g}_1(\omega(k),t) + \tilde{g}_2(\omega(k),t).$$

The "deformation" from the real line to ∂D^+ is obtained by adding to the path of integration the closed contour that is the boundary of the wedge between D_1^+ and D_2^+ (see Figure 3.5) traversed in a clockwise sense. The contribution of the integrands along this contour is zero (Cauchy's Theorem), and the contribution from the arc at infinity vanishes due to Jordan's Lemma.

The symmetries of the global relation are as in the case of the previous example. Hence if $k \in D_1^+, l \in D_2^+$ then $\alpha k, \alpha^2 l \in D^-$, where $\alpha = e^{\frac{2\pi i}{3}}$. We obtain the following versions of the global relation

$$\hat{q}_0(\alpha k) + \alpha^2 k^2 \tilde{g}_0(\omega(k), t) - i\alpha k \tilde{g}_1(\omega(k), t) - \tilde{g}_2(\omega(k), t) = e^{\omega(k)t}\hat{q}(\alpha k, t), \qquad k \in D_1^+, \tag{3.21}$$

$$\hat{q}_0(\alpha^2 k) + \alpha k^2 \tilde{g}_0(\omega(k), t) - i\alpha^2 k \tilde{g}_1(\omega(k), t) - \tilde{g}_2(\omega(k), t) = e^{\omega(k)t}\hat{q}(\alpha^2 k, t), \qquad k \in D_2^+. \tag{3.22}$$

Hence in each region, D_1^+ and D_2^+, there is one relation between the three quantities $\tilde{g}_i(\omega(k), t), i = 0, 1, 2$. Thus two boundary conditions are required at the left boundary in order to be able to solve the global relation for the unspecified boundary condition. For instance, given $q(0, t)$ and $q_x(0, t)$ we can calculate $\tilde{g}_0(\omega(k))$ and $\tilde{g}_1(\omega(k))$. The above relations can be used to obtain two expressions for $\tilde{g}_2(\omega(k))$, one valid for $k \in D_1^+$ and the other for $k \in D_2^+$. Substituting the resulting expressions into the integral expression for $q(x, t)$ we find the solution to (3.14) posed on the positive half-line. The difference in the number of boundary conditions required for the seemingly very similar boundary value problems with the equations (3.10a) and (3.14) is understood as follows. For (3.10a), the phase velocity $-i\omega/k = -k^2$ is negative and the left boundary acts as a sink. Using (3.14), the phase velocity $-i\omega/k = k^2$ is positive, and the left boundary is a source. The same conclusion is reached using the group velocity which is $3\times$ the phase velocity.

3.4. A multi-term third-order PDE. For the problems above, the symmetries of the dispersion relation are easily found. With the present example we illustrate the use of Fokas's method if the symmetries are somewhat more complicated and the introduction of a branch cut is necessary. This method illustrates how to proceed in general, even if the symmetries cannot be written out explicitly. Consider

$$q_t = q_x + q_{xxx} = 0, \qquad x > 0, \tag{3.23}$$

where an initial condition $q(x, 0)$ is given. The previous example indicates that two boundary conditions are required on the left boundary, which we will verify in the process of solving the equation. As in the previous example, we skip some calculation details, which are similar to previous examples.

The dispersion relation is

$$\omega(k) = -ik + ik^3, \tag{3.24}$$

and

$$D = \left\{k : k_I(1 + k_I^2 - 3k_R^2) < 0\right\}, \tag{3.25}$$

where k_R and k_I denote the real and imaginary parts of k, respectively. The region D is shown on the left in Figure 3.6. The boundaries of the region consist of a hyperbola with asymptotes $k_I = \pm\sqrt{3}k_R$ and the real line. As above, we define D^+ (D^-) to be the intersection of D with the upper (lower) half plane, and $D^+ = D_1^+ \cup D_2^+$, where

$$D_1^+ = \left\{k : D^+ \cap \text{second quadrant}\right\}, \quad D_2^+ = \left\{k : D^+ \cap \text{first quadrant}\right\}, \tag{3.26}$$

as indicated in Figure 3.6.

The local relation is given by

$$(e^{-ikx+\omega(k)t}q)_t - \left(e^{-ikx+\omega(k)t}(q_{xx} + ikq_x + (1 - k^2)q)\right)_x = 0, \tag{3.27}$$

integrating over the region $\mathcal{R} = \{x \geq 0,\ 0 < t \leq T\}$ and using Green's theorem gives the global relation

$$\hat{q}_0(k) - \tilde{g}(k, t) = e^{\omega(k)t}\hat{q}(k, t), \quad \text{Im}(k) \leq 0. \tag{3.28}$$

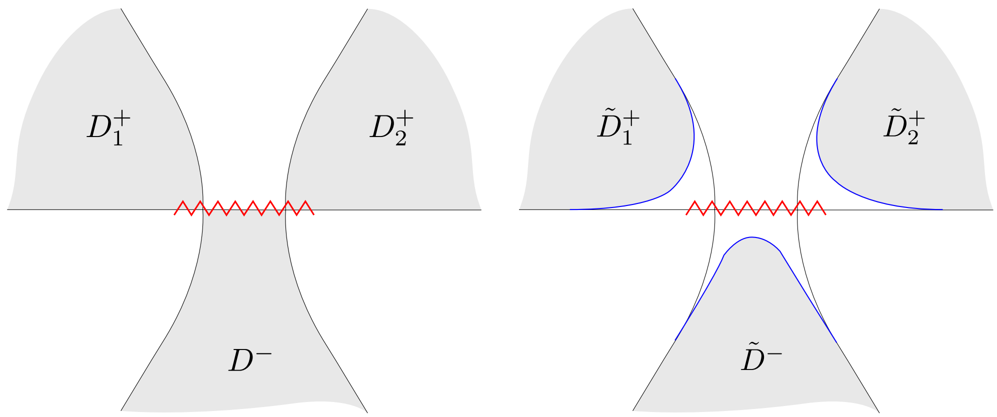

Fig. 3.6: Domain $D = \{k : \mathrm{Re}(\omega(k)) < 0\}$ for the third-order PDE (left), and its deformation $\tilde{D}$ (right), for the third-order PDE (3.23). The branch cut $[-2/\sqrt{3}, 2/\sqrt{3}]$ is indicated by the jagged line.

where we have defined $\tilde{g}(k,t) = (1-k^2)\tilde{g}_0(\omega(k),t) + ik\tilde{g}_1(\omega(k),t) + \tilde{g}_2(\omega(k),t)$. At this point, we might write an integral expression for $q(x,t)$ involving a contour integral over the boundary of D^+. Before we proceed this way, we examine the symmetries of the dispersion relation, as their functional form will influence what follows.

The discrete symmetries of $\omega(k)$ are found by solving $-ik + ik^3 = -i\nu(k) + i\nu^3(k)$ for $\nu(k)$. Eliminating the solution $\nu(k) = k$, we find

$$\nu = -\frac{k}{2} \pm \sqrt{1 - \frac{3k^2}{4}}.$$

This is a two-sheeted expression with branch points at $\pm\sqrt{3}/2$, leading to a choice of branch cut along $[-\sqrt{3}/2, \sqrt{3}/2]$. Define ν_1 to be the branch of ν which limits to $(-1/2 + i\sqrt{3}/2)k = k\exp(2\pi i/3)$, and let ν_2 be the other branch, limiting to $k\exp(4\pi i/3)$. Our standard procedure would be to deform the integration over the real k axis to one along ∂D^+, leading to the solution expression

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D_1^+} e^{ikx-\omega(k)t}\tilde{g}(k,t)dk - \frac{1}{2\pi}\int_{\partial D_2^+} e^{ikx-\omega(k)t}\tilde{g}(k,t)dk,$$

where

$$\tilde{g}(k,t) = (1-k^2)\tilde{g}_0(\omega(k),t) + ik\tilde{g}_1(\omega(k),t) + \tilde{g}_2(\omega(k),t).$$

This is problematic, since the integration paths ∂D_1^+ and ∂D_2^- contain the branch points and part of the branch cut. Instead, we deform D to $\tilde{D}$ with its constituent parts $\tilde{D}_1^+$, $\tilde{D}_2^+$ and $\tilde{D}^-$, so that the boundary of these domains is separated from the branch cut. Although the integrands are growing as $k \to \infty$ in D, they are analytic in any bounded region, and this deformation has no overall effect, due to Cauchy's Theorem. The deformation from D to $\tilde{D}$ can be chosen in many ways, as long as $\tilde{D}$ has the same asymptotic form as D. Specifically, we may deform D_1^+ to $\tilde{D}_1^+$ and induce deformations on D_2^+ and D^- using $\tilde{D}_2^+ = \nu_2(\tilde{D}_1^+)$, $D^- = \nu_1(\tilde{D}_1^+)$, respectively. This leads to the solution formula

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial \tilde{D}_1^+} e^{ikx-\omega(k)t}\tilde{g}(k,t)dk - \frac{1}{2\pi}\int_{\partial \tilde{D}_2^+} e^{ikx-\omega(k)t}\tilde{g}(k,t)dk.$$

Next, we eliminate the dependence on unnecessary boundary conditions. In addition to (3.28), we have the following versions of the global relation:

$$\hat{q}_0(\nu_1(k)) - (1-\nu_1(k)^2)\tilde{g}_0(\omega(k),t) - i\nu_1(k)\tilde{g}_1(\omega(k),t) - \tilde{g}_2(\omega(k),t) = e^{\omega(k)t}\hat{q}(\nu_1(k),t), \quad k \in \tilde{D}_1^+, \tag{3.29}$$

$$\hat{q}_0(\nu_2(k)) - (1-\nu_2(k)^2)\tilde{g}_0(\omega(k),t) - i\nu_2(k)\tilde{g}_1(\omega(k),t) - \tilde{g}_2(\omega(k),t) = e^{\omega(k)t}\hat{q}(\nu_2(k),t), \quad k \in \tilde{D}_2^+. \tag{3.30}$$

As for the previous example, in each region, $\tilde{D}_1^+$ and $\tilde{D}_2^+$, there is one relation between the three quantities $\tilde{g}_i(\omega(k),t), i = 0,1,2$. Thus two boundary conditions are required at the left boundary in order to be able to solve the global relation for whichever boundary condition is unspecified.

3.5. A general evolution PDE. Fokas's method is also applicable to the general constant coefficient linear evolution PDE

$$q_t + \omega(-i\partial_x)q = 0, \quad x \geq 0,\ t \in (0,T]. \tag{3.31}$$

Here $\omega(k)$ is a polynomial of degree n. To ensure that solutions do not grow in time, we impose that $\mathrm{Re}(\omega(k)) \geq 0$, for real k. Let

$$\omega(k) = \alpha_n k^n + \alpha_{n-1}k^{n-1} + \ldots + \alpha_0. \tag{3.32}$$

The large k limit of the condition $\mathrm{Re}(\omega(k)) \geq 0, k \in \mathbb{R}$ implies that if n is odd then $\alpha_n = \pm i$. For n even, $\mathrm{Re}(\alpha_n) \geq 0$. Using the dispersion relation we define the following regions in the complex k plane

$$D = \{k : \mathrm{Re}(\omega(k)) < 0\},$$

and

$$D^+ = D \cap \mathbb{C}^+, \quad D^- = D \cap \mathbb{C}^-.$$

The local relation is given by

$$\partial_t\left(e^{-ikx+\omega(k)t}q(x,t)\right) - \partial_x\left(e^{-ikx+\omega(k)t}\sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t)\right) = 0,$$

where

$$\sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t) = i\left(\frac{\omega(k)-\omega(l)}{k-l}\right)\Bigg|_{l=-i\partial_x} q(x,t). \tag{3.33}$$

The proof is straightforward [9]. Equation (3.33) implies the global relation

$$e^{\omega(k)T}\hat{q}(k,T) = \hat{q}_0(k) - \sum_{j=0}^{n-1} c_j(k)\tilde{g}_j(\omega(k),T), \quad \mathrm{Im}(k) \leq 0,$$

where

$$\tilde{g}_j(\omega,T) = \int_0^T e^{\omega s}\partial_x^j q(0,s)ds.$$

Applying the inverse Fourier transform to the global relation we obtain the integral expression for the solution

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx+\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+}\left(e^{-ikx+\omega(k)t}\sum_{j=0}^{n-1} c_j(k)\tilde{g}_j(\omega(k),t)\right)dk. \tag{3.34}$$

In order to obtain a solution, we require expressions for the time transforms of the unknown boundary data valid for $k \in \partial D^+$. As with the previous examples, we use the discrete symmetries of the equation

$$\omega(k) = \omega_0.$$

This relation is a polynomial or order n and thus has n roots in the complex plane. The mappings from one root to another are precisely the transformations $k \to \nu(k)$ which leave $\omega(k)$ invariant. We employ the induced versions of the global relation to solve for the transforms of the unknown boundary data. The solution proceeds as before. Having eliminated the unknown boundary data, we obtain an expression that depends on $\hat{q}(\nu(k), t)$. The transformation ν is such that $\hat{q}(\nu(k), t)$ is analytic and bounded in the region D^+. Hence we may use the Cauchy Integral Theorem to eliminate the contribution of the term involving $\hat{q}(\nu(k), t)$ to the integral along ∂D^+.

It is possible to predict how many boundary conditions are required for a well-posed problem by considering the large k behavior of $\omega(k)$. For large values of k, $\omega(k) \sim \alpha_n k^n$ and the region D approaches

$$D_R = \left\{ k : \arg \alpha_n + n \arg k \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) + 2m\pi, \quad m = 0, \ldots, n-1 \right\}.$$

Indeed for large k, $\mathrm{Re}(\omega(k)) \sim |k|^n \cos(\arg \alpha_n + n \arg k)$ which is negative for k in D_R. The region D_R consists of n unbounded equal-angled sectors in the complex-k plane. Let N represent the number of unbounded sectors of D_R in the upper-half plane. It is easily seen that

$$N = \begin{cases} n/2 & n \text{ even;} \\ (n+1)/2 & n \text{ odd and } \alpha_n = -i; \\ (n-1)/2 & n \text{ odd and } \alpha_n = i. \end{cases} \tag{3.35}$$

The reader is encouraged to verify the formulas for D_R and N for the examples presented above. Indeed Figures 3.2, 3.4 and 3.5 are examples of the three possible cases for N. In these examples, the regions D_R and D coincide.

In the integral expression for $q(x, t)$, the contour ∂D^+ may be deformed to $\partial D_R^+ = \partial D_R \cap \mathbb{C}^+$, since the integrands are entire functions. Thus we require expressions for the time transform of the n boundary data, valid in the unbounded sectors of D_R in the upper-half plane.

Assume we are given p boundary conditions at the boundary at $x = 0$. This implies we require, for each of the N sectors in $\mathbb{C}^+$, expressions for the transforms of the $n - p$ unknown boundary data. For each of the N sectors in $\mathbb{C}^+$, the discrete symmetries of $\omega(k)$ allow us to choose $n - N$ transformations $\nu_i(k), i = 0, 1 \ldots n - N - 1$ which map that sector to the $n - N$ remaining sectors. Hence, by substituting $\nu_i(k)$ for k, there are $n - N$ versions of the global relation valid in each of the N sectors in $\mathbb{C}^+$. Thus we have $n - p$ unknowns with $n - N$ equations in each sector. Thus we need N boundary conditions at $x = 0$ for a well-posed problem or in other words a well-posed problem on the positive half-line requires as many boundary conditions as there are sectors of D_R in the upper-half plane.

Remarks.

- A canonical problem is one for which $q(0, t)$ and its first $N - 1$ derivatives are provided as boundary conditions. An example of a non-canonical problem is the Neumann problem for the heat equation. Providing N linear combinations with constant coefficients of a subset of the boundary values is another example (Robin problem). In this case, versions of the global relation valid in D_R^+ and the N linear combinations form a system of equations which can be solved for the boundary data, provided a certain determinant is not identically zero. If the determinant has zeros in D_R^+, then the contribution of these zeros to the final solution is computed via the Residue Theorem.
- For the class of the PDEs discussed in this paper, the dispersion relation $\omega(k)$ is a polynomial of degree n, see (3.32). Thus, by the fundamental theory of algebra, in addition to the identity, $n - 1$ additional symmetries $\nu(k)$ exist, even if typically they cannot be written out in closed form. In this sense, the problem has been reduced to one of polynomial algebra.

4. The problem on the finite interval.

4.1. The general method. Having studied the problem posed on the half line in some detail, we turn to the problem on the finite interval. We use the same notation as before, extending it where necessary to incorporate the boundary on the right-hand side.

$$\begin{aligned} q_t + \omega(-i\partial_x)q = 0, &\qquad (x,t) \in [0,L] \times [0,T], \\ q(x,0) = q_0(x), &\qquad x \in [0,L], \\ \partial_x^j q(0,t) = g_j(t), &\qquad j = 0,\ldots,n-1, \\ \partial_x^j q(L,t) = h_j(t), &\qquad j = 0.\ldots,n-1, \end{aligned}$$

Here $g_j(t)$ and $h_j(t)$ denote the relevant boundary values at $x = 0$ and $x = L$, respectively. We assume that n of the functions $g_j(t)$ and $h_j(t)$ are given. In what follows, we will see that for a well-posed problem N of these specified boundary functions should be specified at $x = 0$ and the other $n-N$ at $x = L$, where N is as in (3.35). As for the half-line problem, the different incarnations of the global relation (see below) obtained by using the symmetries of the dispersion relation are used to eliminate unspecified boundary conditions.

As above, we define

$$\tilde{g}_j(k,t) = \int_0^t e^{ks} g_j(s)ds, \quad \tilde{h}_j(k,t) = \int_0^t e^{ks} h_j(s)ds, \quad k \in \mathbb{C},$$

where we have introduced the time transforms of both the left and right boundary values.

Employing the divergence form (3.33), we use Green's Theorem on the domain $[0,L] \times [0,t]$. This is an easier domain to work with since it is bounded. The convergence of integrals over the spatial domain is not an issue, as it might be for problems posed on the half line. We obtain the following global relation:

$$\hat{q}_0(k) - \tilde{g}(k,t) + e^{-ikL}\tilde{h}(k,t) = e^{\omega(k)t}\int_0^L e^{-ikx}q(x,t)dx = e^{\omega(k)t}\hat{q}(k,t), \quad k \in \mathbb{C}, \tag{4.1}$$

with

$$\tilde{h}(k,t) = \sum_{j=0}^{n-1} c_j(k)h_j(k,t), \quad \tilde{g}(k,t) = \sum_{j=0}^{n-1} c_j(k)g_j(k,t). \tag{4.2}$$

As before, the global relation is also valid if we replace t by T. In contrast to the global relation for a problem on the half line, the global relation (4.1) on the finite interval is valid for all $k \in \mathbb{C}$. It is clear this has important implications, as is discussed below.

To formally obtain a solution, we apply the inverse Fourier transform to (4.1) :

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{q}(k,t)dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\tilde{g}(k,t)dk \\ &\qquad + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik(L-x)-\omega(k)t}\tilde{h}(k,t)dk. \end{aligned} \tag{4.3}$$

There are two major differences between the half line problem and the problem on the finite interval. The most obvious is the addition of $\tilde{h}(k,t)$ due to the presence of the boundary on the right side of the domain. This provides extra algebraic complications for obtaining the solution. This complication is counteracted by the validity of the global relation in all of $\mathbb{C}$, which is the second major difference. This allows us to use D^+ to eliminate the unknowns in $\tilde{h}(k,t)$, in addition to using D^- to eliminate the unknowns in $\tilde{g}(k,t)$ as we did earlier for the half line. Proceeding with this intuition, we deform the contour of integration for the integral involving $\tilde{g}(k,t)$ up to ∂D^+ as in the half line case.

The third integral in (4.3) does not appear in the half line case. The exponential $e^{-ik(L-x)}$ is bounded in the lower-half plane since $x \in [0,L]$. This observation allows us analytically extend the integrand of the third term in (4.3) to the lower-half plane, at which point the path $(-\infty, \infty)$ can be deformed to the boundary of D^-, using Jordan's lemma in $\mathbb{C}^- \setminus D^-$, in the same way the deformation to the boundary of D^+ was justified earlier in the half-line case. Taking our orientation convention into account we obtain

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-\omega(k)t}\tilde{g}(k,t)dk - \frac{1}{2\pi}\int_{\partial D^-} e^{-ik(L-x)-\omega(k)t}\tilde{h}(k,t)dk. \tag{4.4}$$

At this point, we have obtained the solution in terms of $2n$ boundary values. We can prescribe only n boundary conditions. The boundary values that are not given are eliminated by using the symmetries of the dispersion relation as for the half-line case. As before, we have, $n-1$ transformations that leave $\omega(k)$ invariant, *i.e.*, there exist functions $\{\nu_1, \dots \nu_{n-1}\}$, such that $\omega(k) = \omega(\nu_j(k))$. This provides n equations with $2n$ unknowns. If these equations may be solved for n of the unknown boundary functions (or their t-transforms), then these functions may be eliminated in terms of the given boundary conditions. In [9] the conditions under which this system of equations can be solved in terms of a set of boundary functions to be eliminated is discussed. For instance, the system cannot be solved for the boundary functions on one boundary in terms of the boundary functions on the other boundary. Indeed, a well-posed problem on the finite-interval requires boundary conditions at both boundaries. Of course, as for the half-line problem, using these different incarnations of the global relation introduces the quantities $\hat{q}(\nu_j(k),t)$, $j = 1, \dots, n-1$ on the right-hand side of (4.4). This dependence is eliminated as before, using contour deformation and Jordan's Lemma. These different steps are illustrated on a variety of examples below.

4.2. The heat equation with homogeneous Dirichlet boundary conditions. Consider the problem

$$q_t = q_{xx}, \qquad x \in (0,L),\ t \in (0,T], \tag{4.5a}$$
$$q(x,0) = q_0(x), \qquad x \geq 0, \tag{4.5b}$$
$$q(0,t) = 0, \qquad t \in [0,T], \tag{4.5c}$$
$$q(L,t) = 0, \qquad t \in [0,T]. \tag{4.5d}$$

As we have already seen, $\omega(k) = k^2$. Also, $\tilde{g}(k,t) = ik\tilde{g}_0(k^2,t) + \tilde{g}_1(k^2,t) = \tilde{g}_1(k^2,t)$ and $\tilde{h}(k,t) = ik\tilde{h}_0(k^2,t) + \tilde{h}_1(k^2,t) = \tilde{h}_1(k^2,t)$. The transformation that leaves $\omega(k) = k^2$ invariant is $k \to -k$, thus the two equations for the two unknowns $\tilde{g}_1(k^2,T)$ and $\tilde{h}_1(k^2,T)$ are

$$\hat{q}_0(k) - \tilde{g}_1(k^2,t) + e^{ikL}\tilde{h}_1(k^2,t) = e^{k^2t}\hat{q}(k,t),$$
$$\text{and} \quad \hat{q}_0(-k) - \tilde{g}_1(k^2,t) + e^{-ikL}\tilde{h}_1(k^2,t) = e^{k^2t}\hat{q}(-k,t).$$

These two equations are rewritten as

$$\begin{bmatrix} -1 & e^{ikL} \\ -1 & e^{-ikL} \end{bmatrix} \begin{bmatrix} \tilde{g}_1(k^2,t) \\ \tilde{h}_1(k^2,t) \end{bmatrix} = \begin{bmatrix} e^{k^2t}\hat{q}(k,t) - \hat{q}_0(k) \\ e^{k^2t}\hat{q}(-k,t) - \hat{q}_0(-k) \end{bmatrix},$$

from which

$$\begin{bmatrix} \tilde{g}_1(k^2,t) \\ \tilde{h}_1(k^2,t) \end{bmatrix} = \frac{1}{e^{ikL} - e^{-ikL}} \begin{bmatrix} e^{-ikL} & -e^{ikL} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{k^2t}\hat{q}(k,t) - \hat{q}_0(k) \\ e^{k^2t}\hat{q}(-k,t) - \hat{q}_0(-k) \end{bmatrix}.$$

Defining $\Delta(k) = e^{ikL} - e^{-ikL}$, we find

$$\tilde{g}_1(k^2,t) = \frac{1}{\Delta(k)}[\hat{q}_0(-k)e^{-ikL} - \hat{q}_0(k)e^{ikL}] - \frac{1}{\Delta(k)}[e^{k^2t}\hat{q}(-k,t)e^{-ikL} - e^{k^2t}\hat{q}(k,t)e^{ikL}],$$
$$\tilde{h}_1(k^2,t) = \frac{1}{\Delta(k)}[\hat{q}_0(-k) - \hat{q}_0(k)] - \frac{1}{\Delta(k)}[e^{k^2t}\hat{q}(-k,t) - e^{k^2t}\hat{q}(k,t)].$$

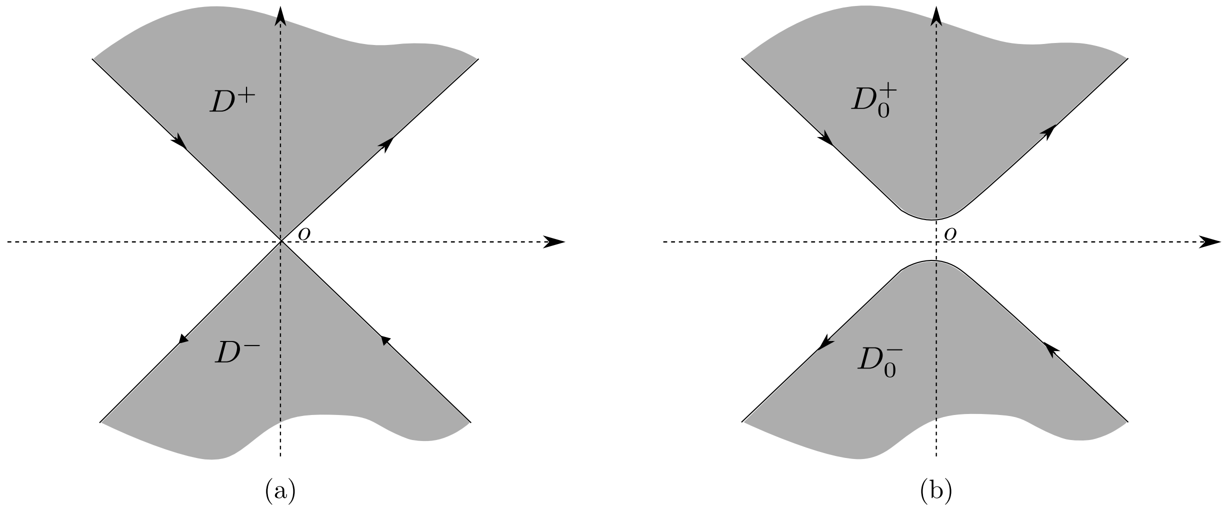

Fig. 4.1: (a) The domains D^+, D^-, and their respective boundaries. (b) The deformed domains D_0^+ and D_0^-, used to avoid the root of $\Delta(k)$ at the origin.

The final step is to justify neglecting the terms involving $\hat{q}(\pm k, t)$. For $\tilde{g}$, this is analogous to the case of the half line, and it is not repeated here. However, we do demonstrate this for $\tilde{h}$. Consider the integral

$$\int_{\partial D^-} e^{-ik(L-x)-k^2t}\frac{1}{\Delta(k)}[e^{k^2t}\hat{q}(-k,t)-e^{k^2t}\hat{q}(k,t)]dk=\int_{\partial D^-} e^{-ik(L-x)}\frac{1}{\Delta(k)}[\hat{q}(-k,t)-\hat{q}(k,t)]dk, \tag{4.6}$$

where both $(\hat{q}(-k,t)-\hat{q}(k,t))/\Delta(k) = \exp(-ikL)(\hat{q}(-k,t)-\hat{q}(k,t))/(1-\exp(-2ikL))$ and $\exp(-ik(L-x))$ decay to zero uniformly as $k \to \infty$ in the lower-half plane. Another application of Jordan's Lemma shows that

$$\int_{\partial D^-} e^{-ik(L-x)-k^2t}\frac{1}{\Delta(k)}[e^{k^2t}\hat{q}(-k,t) - e^{k^2t}\hat{q}(k,t)]dk = 0,$$

which implies

$$\begin{aligned} q(x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)dk \\ & - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{1}{\Delta(k)}\left[e^{ikL}\hat{q}_0(k) - e^{-ikL}\hat{q}_0(-k)\right]dk \\ & - \frac{1}{2\pi}\int_{\partial D^-} e^{-ik(L-x)-k^2t}\frac{1}{\Delta(k)}\left[\hat{q}_0(k) - \hat{q}_0(-k)\right]dk. \end{aligned} \tag{4.7}$$

It should be noted that $\Delta(0) = 0$ and $0 \in \partial D$, thus it is convenient to deform the paths of integration, as shown in Figure 4.1. Having obtained the solution of (4.5a-d) we demonstrate that it may be rewritten in the more familiar form of an infinite sum over a set of fundamental modes. It should be noted that this is not always possible. Indeed, for many problems a countable set of fundamental modes does not exist. The third-order equation of Section 4.4 provides an example of this. Even in that case, the method of Fokas still provides a solution.

To reduce the integral representation (4.7) of the solution to an infinite series, we start by noticing that $\Delta(k)$ has simple zeros at a countable number of roots $k_n = 2\pi n/L$. The idea is to deform the paths of integration in (4.7) to a sequence of small loops around these roots, leading to a residue contribution for all of these loops, from which a series representation of the solution is obtained. In order for this to happen, all remaining integral contributions must vanish. For the heat equation, this is the case. In fact this occurs for all problems where a series solution exists. For other problems (again, see Section 4.4) the integral contribution to the solution does not vanish and little is gained by separating out a series part.

To ease the notation, define

$$f^+(x,t,k) = \frac{1}{\Delta(k)}\left[e^{ikL}\hat{q}_0(-k) - e^{-ikL}\hat{q}_0(k)\right],$$
$$f^-(x,t,k) = \frac{1}{\Delta(k)}\left[\hat{q}_0(-k) - \hat{q}_0(k)\right],$$

where $f^+(x,k)$ and $f^-(x,k)$) appear in the integrands along ∂D^+ and ∂D^-, respectively. We compute the residues at each k_n for the integrand in the integral along ∂D^+. All poles are simple, thus

$$\begin{aligned} c_n(x,t) &= \operatorname{Res}\{e^{ikx-k^2t}f^+(x,t,k), k=k_n\} \\ &= \frac{1}{\Delta'(k_n)}e^{ik_nx-k_n^2t}\left[e^{ik_nL}\hat{q}_0(-k_n) - e^{-ik_nL}\hat{q}_0(k_n)\right]. \end{aligned}$$

Using $e^{ik_nL}=1$ and $\Delta'(k_n)=2iL$ so that

$$c_n(x,t) = \frac{1}{2iL}e^{ik_nx-k_n^2t}\left[\hat{q}_0(-k_n) - \hat{q}_0(k_n)\right]. \tag{4.8}$$

This can be simplified using the explicit form of $\hat{q}_0(k)$. We have

$$\hat{q}_0(k_n) - \hat{q}_0(-k_n) = -2i\int_0^L \sin(k_nx)q_0(x)dx, \tag{4.9}$$

which allows us to obtain a simple expression for the residue:

$$c_n(x,t) = -\frac{1}{2}e^{ik_nx-k_n^2t}q_n, \quad q_n = \frac{2}{L}\int_0^L \sin(k_nx)q_0(x)dx. \tag{4.10}$$

It remains to compute the residues of the integrand on ∂D^-. This is similar to the calculation just detailed, and we end up with the residues being equal to those already computed.

$$\begin{aligned} \operatorname{Res}\{e^{ikx-w(k)t}f^-(x,t,k), k=k_n\} &= e^{ik_nx-w(k_n)t}\frac{1}{\Delta'(k_n)}\left[\hat{q}_0(-k_n) - \hat{q}_0(k_n)\right] \\ &= \frac{1}{2iL}e^{ik_nx-w(k_n)t}\left[-2i\int_0^L \sin(k_nx)q_0(x)dx\right] \\ &= \operatorname{Res}\{e^{ikx-w(k)t}f^+(x,t,k), k=k_n\}. \end{aligned}$$

All the poles are on the real line. We deform the paths $\partial D^\pm$ back to the real line in order to collect their contributions. Additionally, in order to obtain a series solution the first Fourier transform integral in (4.4) needs to be addressed. Using Jordan's Lemma, we may deform back to the real line, to end up with contours in Figure 4.2. The solution can be written as

$$\begin{aligned} q(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)dk \\ &-\frac{1}{2\pi}\int_{\partial D_\epsilon^+} e^{ikx-k^2t}f^+(x,t,k)dk - \frac{1}{2\pi}\int_{\partial D_\epsilon^-} e^{-ik(L-x)-k^2t}f^-(x,t,k)dk. \end{aligned} \tag{4.11}$$

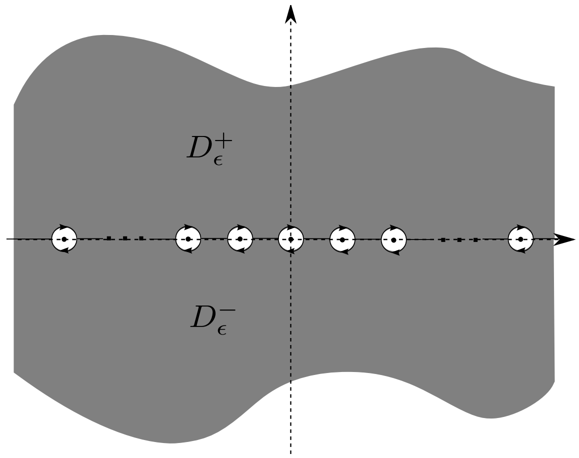

Fig. 4.2: Deformation to obtain Fourier series

Replacing the integrals with principal-value integrals and sums

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)dk \\ &- \left(\frac{1}{2\pi}⨍_{-\infty}^{\infty} e^{ikx-k^2t}f^+(x,t,k)dk - \frac{\pi i}{2\pi}\sum_{n=-\infty}^{\infty}\text{Res}\{f^+(x,t,k), k=k_n\}\right) \\ &+ \left(\frac{1}{2\pi}⨍_{-\infty}^{\infty} e^{-ik(L-x)-k^2t}f^-(x,t,k)dk + \frac{\pi i}{2\pi}\sum_{n=-\infty}^{\infty}\text{Res}\{f^-(x,t,k), k=k_n\}\right). \end{aligned} \tag{4.12}$$

The two principal-value integrals add to cancel the first integral, leaving us with

$$q(x,t) = i\sum_{n=-\infty}^{\infty} c_n(x,t) = i\sum_{n=-\infty}^{\infty}\frac{1}{2}e^{ik_nx-k_n^2t}(-q_n). \tag{4.13}$$

Using $q_n = -q_{-n}$ and $q_0 = 0$ so that

$$q(x,t) = \sum_{i=1}^{\infty} q_n e^{-k_n^2t}\sin(k_nx),$$

which is the classical sine series solution.

4.3. The heat equation with Robin boundary conditions. Consider the somewhat more complicated problem

$$q_t = q_{xx}, \qquad (x,t) \in [0,L]\times[0,T], \tag{4.14a}$$

$$q(x,0) = q_0(x), \qquad x \in [0,L], \tag{4.14b}$$

$$q_x(0,t) - \gamma q(0,t) = 0, \qquad t \in [0,T] \tag{4.14c}$$

$$q_x(L,t) - \gamma q(L,t) = 0, \qquad t \in [0,T], \tag{4.14d}$$

where γ is real. The heat equation problem with Dirichlet or Neumann boundary data is straightforward using classical techniques and contains no surprises. The same is not true for the Robin problem, which contains an exponentially growing mode. We demonstrate how easily the method of Fokas extends to Robin boundary conditions, and how naturally the growing mode is captured . We work with homogeneous

Robin conditions, to simplify the algebra involved. As before, $w(k) = k^2$, $\tilde{g}(k) = ik\tilde{g}_0(k^2) + \tilde{g}_1(k^2)$ and $\tilde{h}(k) = ik\tilde{h}_0(k^2) + \tilde{h}_1(k^2)$. From the boundary conditions,

$$\tilde{h}_1(k^2) = \gamma \tilde{h}_0(k^2), \quad \tilde{g}_1(k^2) = \gamma \tilde{g}_0(k^2). \tag{4.15}$$

Using $k \mapsto -k$ the two global relations are

$$\hat{q}_0(k) - [ik\tilde{g}_0(k^2) + \tilde{g}_1(k^2)] + e^{ikL}[ik\tilde{h}_0(k^2) + \tilde{h}_1(k^2)] = e^{k^2 t}\hat{q}(k,t), \tag{4.16}$$

$$\hat{q}_0(-k) - [-ik\tilde{g}_0(k^2) + \tilde{g}_1(k^2)] + e^{-ikL}[-ik\tilde{h}_0(k^2) + \tilde{h}_1(k^2)] = e^{k^2 t}\hat{q}(-k,t). \tag{4.17}$$

The equations (4.15-4.17) form a system of four linear equations for the four unknowns $\tilde{g}_0(k^2)$, $\tilde{g}_1(k^2)$, $\tilde{h}_0(k^2)$, and $\tilde{h}_1(k^2)$. These equations are valid in the whole complex k-plane, all integrals involved are proper. Substitution of (4.15) in (4.16-4.17) gives

$$\begin{aligned}
\tilde{g}_0(k^2) - e^{ikL}\tilde{h}_0(k^2) &= \frac{\hat{q}_0(k) - e^{k^2 t}\hat{q}(k,t)}{\gamma + ik}, \\
\tilde{g}_0(k^2) - e^{-ikL}\tilde{h}_0(k^2) &= \frac{\hat{q}_0(-k) - e^{k^2 t}\hat{q}(-k,t)}{\gamma - ik},
\end{aligned}$$

so that

$$\begin{aligned}
g_0(k^2) &= \frac{1}{\Delta(k)}\left[e^{ikL}\frac{\hat{q}_0(-k) - e^{k^2 t}\hat{q}(-k,t)}{\gamma - ik} - e^{-ikL}\frac{\hat{q}_0(k) - e^{k^2 t}\hat{q}(k,t)}{\gamma + ik}\right], \\
h_0(k^2) &= \frac{1}{\Delta(k)}\left[\frac{\hat{q}_0(-k) - e^{k^2 t}\hat{q}(-k,t)}{\gamma - ik} - \frac{\hat{q}_0(k) - e^{k^2 t}\hat{q}(k,t)}{\gamma + ik}\right],
\end{aligned}$$

where, as before, $\Delta(k) = e^{ikL} - e^{-ikL}$. The solution formula (4.3) requires us to find $\tilde{g}$ and $\tilde{h}$:

$$\begin{aligned}
\tilde{g}(k) = {} & \frac{1}{\Delta(k)}\left[e^{ikL}\hat{q}_0(-k)\frac{\gamma + ik}{\gamma - ik} - e^{-ikL}\hat{q}_0(k)\right] \\
& - \frac{1}{\Delta(k)}\left[e^{ikL}e^{k^2 t}\hat{q}(-k,t)\frac{\gamma + ik}{\gamma - ik} - e^{-ikL}e^{k^2 t}\hat{q}(k,t)\right], \\
\tilde{h}(k) = {} & \frac{1}{\Delta(k)}\left[\hat{q}_0(-k)\frac{\gamma + ik}{\gamma - ik} - \hat{q}_0(k)\right] - \frac{1}{\Delta(k)}\left[e^{k^2 t}\hat{q}(-k,t)\frac{\gamma + ik}{\gamma - ik} - e^{k^2 t}\hat{q}(k,t)\right].
\end{aligned}$$

There is an added complication. Since $\gamma \in \mathbb{R}$ the individual terms in the above expressions have a pole in D at $k = -i\gamma$, while $\tilde{g}(k)$ and $\tilde{h}(k)$ are analytic. Depending on the sign of γ the pole at $k = -i\gamma$ is in D^+ or in D^-. Either way it affects the use of Jordan's Lemma which is necessary to remove the terms involving $\hat{q}(k,t)$ and $\hat{q}(-k,t)$. Using the above expressions, the second and third integrals in (4.4) are split term-by-term. The resulting integrands are not analytic, and the contour deformation required to eliminate the contribution of $\hat{q}(-k,t)$ necessarily accounts for the pole singularity. For concreteness, let us assume that $\gamma < 0$, thus $k = -i\gamma \in D^+$. We have two equivalent options. We can deform ∂D^+ to pass above $k = -i\gamma$ or we can calculate a residue. If we pursue this second option we use the fact that $e^{ikx - k^2 t}\tilde{g}(k)$ is entire, so that

$$\text{Res}\{e^{ikx - k^2 t}\tilde{g}(k), k = -i\gamma\} = 0.$$

Therefore

$$\begin{aligned}
& \text{Res}\left\{\frac{-1}{\Delta(k)}\left[e^{ikL}e^{k^2 t}\hat{q}(-k,t)\frac{\gamma + ik}{\gamma - ik} - e^{-ikL}e^{k^2 t}\hat{q}(k,t)\right], k = -i\gamma\right\} \\
&= \text{Res}\left\{\frac{-1}{\Delta(k)}\left[e^{ikL}\hat{q}_0(-k)\frac{\gamma + ik}{\gamma - ik} - e^{-ikL}\hat{q}_0(k)\right], k = -i\gamma\right\} = \frac{-2i\gamma}{\Delta(-i\gamma)}e^{\gamma L}\hat{q}_0(i\gamma).
\end{aligned}$$

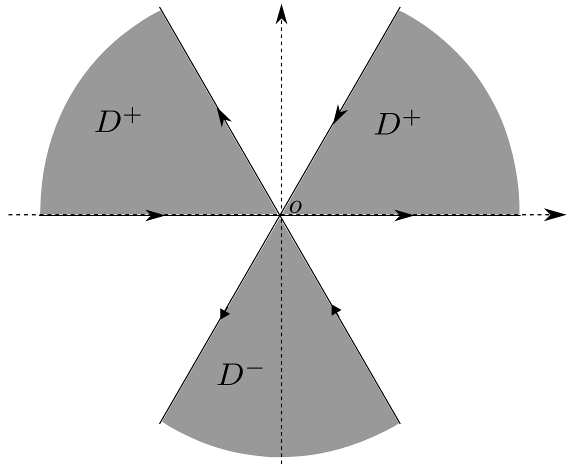

Fig. 4.3: D for $q_t = q_{xxx}$

The final expression for the solution $q(x,t)$ is

$$
\begin{aligned}
q(x,t) = {} & \frac{2\gamma}{1-e^{-2\gamma L}}\hat{q}_0(i\gamma)e^{\gamma x+\gamma^2 t} + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{q}_0(k)dk \\
& -\frac{1}{2\pi}\int_{\partial D_0^+} \frac{e^{ikx-k^2 t}}{\Delta(k)}\left[e^{ikL}\hat{q}_0(-k)\frac{\gamma+ik}{\gamma-ik} - e^{-ikL}\hat{q}_0(k)\right]dk \\
& -\frac{1}{2\pi}\int_{\partial D_0^-} \frac{e^{-ik(L-x)-k^2 t}}{\Delta(k)}\left[\hat{q}_0(-k)\frac{\gamma+ik}{\gamma-ik} - \hat{q}_0(k)\right]dk,
\end{aligned}
$$

which displays the mode that is exponentially growing in time ,which is present in this problem (see [15]), explicitly.

Remark. As in the case of the Dirichlet problem, we deform ∂D^+ to ∂D_0^+ to avoid the removable singularity at the origin.

4.4. A third-order equation with mixed boundary data. To demonstrate the true power of the Fokas method we consider a difficult problem whose explicit solution cannot be obtained using classical methods. Consider the problem

$$
\begin{aligned}
q_t &= q_{xxx}, & (x,t) &\in [0,L]\times[0,T], & \text{(4.18a)}\\
q(x,0) &= q_0(x), & x &\in [0,L], & \text{(4.18b)}\\
q(0,t) - \alpha q_x(0,t) &= 0, & t &\in [0,T], & \text{(4.18c)}\\
q(L,t) - \alpha q_x(L,t) &= 0, & t &\in [0,T], & \text{(4.18d)}\\
q_{xx}(L,t) &= \beta, & t &\in [0,T], & \text{(4.18e)}
\end{aligned}
$$

where α and β are real parameters. Classical methods [17] would proceed using separation of variables. This approach fails since a complete basis of discrete fundamental modes does not exist for this problem, as can be verified with some effort [9].

To find an explicit solution, we use the general solution (4.4), with D as in Figure (4.3), since $\omega(k) = -ik^3$. Using the formulas (4.2) for $\tilde{g}$ and $\tilde{h}$,

$$\tilde{g}(w(k),t) = (ik)^2\tilde{g}_0(w(k),t) + ik\tilde{g}_1(w(k),t) + \tilde{g}_2(w(k),t), \tag{4.19}$$

$$\tilde{h}(w(k),t) = (ik)^2\tilde{h}_0(w(k),t) + ik\tilde{h}_1(w(k),t) + \tilde{h}_2(w(k),t). \tag{4.20}$$

The relations that leave $\omega(k)$ invariant are $k \mapsto \alpha k$ and $k \mapsto \alpha^2 k$ where $\alpha = e^{2i\pi/3}$. We have introduced six boundary functions $\tilde{h}_j$ and $\tilde{g}_j$, on which the solution $q(x,t)$ depends, according to (4.4). The boundary-value problem provides $h_2(k,t) = \beta \int_0^t e^{ks} ds$ and relations between $g_0(k,t)$ and $g_1(k,t)$, and between $h_0(k,t)$ and $h_1(k,t)$, respectively. It remains to find extra relations for the remaining boundary functions, using the invariances of the dispersion relation. As stated, the boundary function $h_2(k,t)$ is known. To determine the others we set up a system of five equations for five unknowns. Define $M(k,t) = \hat{q}_0(k) - e^{\omega(k)t}\hat{q}(k,t) - e^{ikL}\tilde{h}_2(\omega(k),t)$. Representing the system of equations in matrix form, we obtain

$$\begin{bmatrix} k^2 & -ik & -1 & -k^2 e^{ikL} & ike^{ikL} \\ \gamma^2 k^2 & -i\gamma k & -1 & -\gamma^2 k^2 e^{i\gamma kL} & i\gamma k e^{i\gamma kL} \\ \gamma k^2 & -i\gamma^2 k & -1 & -\gamma k^2 e^{i\gamma^2 kL} & i\gamma^2 k e^{i\gamma^2 kL} \\ 1 & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\alpha \end{bmatrix} \begin{bmatrix} \tilde{g}_0(\omega(k),t) \\ \tilde{g}_1(\omega(k),t) \\ \tilde{g}_2(\omega(k),t) \\ \tilde{h}_0(\omega(k),t) \\ \tilde{h}_1(\omega(k),t) \end{bmatrix} = \begin{bmatrix} -M(k,t) \\ -M(\gamma k,t) \\ -M(\gamma^2 k,t) \\ 0 \\ 0 \end{bmatrix}.$$

Once this system is solved, its solution is substituted into (4.4). At this point, the contributions from any poles are collected and Jordan's Lemma is used to remove terms involving $\hat{q}(k,t)$, yielding the desired explicit solution. Although the algebra involved may be messy, the process is clear, and is easily completed.

5. Conclusion. We have presented the application of Fokas's method to BVPs for linear evolution PDEs with constant coefficients on both the half line and the finite interval. The main ideas behind the method are the use (i) of the analyticity properties of various functions, inherited from the global relation, and (ii) of the discrete symmetries of the dispersion relation. We have illustrated that the method is more general than the standard methods used in the sense that it reproduces the results they do, while being applicable to situations where the standard methods fail or are not applicable. We have illustrated that the method is suitable to be taught in the classroom, at the undergraduate level.

Many other topics can be discussed at this point. Perhaps our biggest omission is not having mentioned Lax pairs. Indeed, it is possible to write a linear PDE with constant coefficients as the compatibility condition of two first-order ordinary differential equations for an auxiliary function $\psi(x,t)$: one of these equations dictates how $\psi(x,t)$ changes as a function of x, with t as a parameter. The other equation has the roles of x and t reversed. Both equations may depend on $u(x,t)$ and its derivatives, thus they are not autonomous. As shown in [9], this approach leads to the solution of the PDE via a scalar Riemann-Hilbert problem, which may be solved explicitly using the Plemelj formula [1]. Using this route, it is not necessary to introduce even the Fourier transform! Admittedly, the background in complex analysis required for this approach is far more extensive.

The method is far more general than we have discussed. For instance, it can be extended to apply to evolution equations with periodic boundary conditions [16], to evolution equations with more than one spatial dimension, or to elliptic equations. It is applicable also to some linear PDEs with nonconstant coefficients, and, as already stated, to so-called integrable nonlinear equations. These topics are beyond the scope of this article, but the interested reader can find more details and additional references in [9]. The method continues to be extended. For instance, it was recently applied to elliptic PDEs in polar coordinates [14].

REFERENCES

[1] M. J. Ablowitz and A. S. Fokas. *Complex variables: introduction and applications.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, second edition, 2003.

[2] M. J. Ablowitz and H. Segur. *Solitons and the inverse scattering transform*, volume 4 of *SIAM Studies in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA., 1981.

[3] N. H. Asmar. *Applied Complex Analysis with Partial Differential Equations.* Prentice-Hall Inc., Englewood Cliffs, N.J., 2002.

[4] S. J. Farlow. *Partial differential equations for scientists and engineers.* Dover Publications Inc., New York, NY, 1993.

[5] N. Flyer and A. S. Fokas. A hybrid analytical-numerical method for solving evolution partial differential equations. I. The half-line. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 464(2095):1823–1849, 2008.

[6] A. S. Fokas. A unified transform method for solving linear and certain nonlinear PDEs. *Proc. Roy. Soc. London Ser. A*, 453(1962):1411–1443, 1997.

[7] A. S. Fokas. Integrable nonlinear evolution equations on the half-line. *Comm. Math. Phys.*, 230(1):1–39, 2002.

[8] A. S. Fokas. A new transform method for evolution partial differential equations. *IMA J. Appl. Math.*, 67(6):559–590, 2002.

[9] A. S. Fokas. *A unified approach to boundary value problems*, volume 78 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[10] A. S. Fokas and B. Pelloni. A transform method for linear evolution PDEs on a finite interval. *IMA J. Appl. Math.*, 70(4):564–587, 2005.

[11] R. B. Guenther and J. W. Lee. *Partial differential equations of mathematical physics and integral equations*. Dover Publications Inc., Mineola, NY, 1996.

[12] R. Haberman. *Elementary applied partial differential equations*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1983.

[13] J. Kevorkian. *Partial differential equations*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990.

[14] E. A. Spence and A. S. Fokas. A new transform method II: the global relation and boundary-value problems in polar coordinates. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 466(2120):2283–2307, 2010.

[15] W. A. Strauss. *Partial differential equations*. John Wiley & Sons Ltd., Chichester, 2008.

[16] T. Trogdon and B. Deconinck. The solution of linear constant-coefficient evolution pdes with periodic boundary conditions. *Applicable Analysis*, 91:529–544, 2011.

[17] H. F. Weinberger. *A first course in partial differential equations with complex variables and transform methods*. Dover Publications Inc., New York, NY, 1995.