Aharonov-Bohm Effect

Valentin Flunkert

Abstract

This report describes the occurrence and interpretation of the Aharonov-Bohm effect. Two simple examples are discussed and the connection to the Berry phase is pointed out. An exceptional case where the effect vanishes is shown. Finally a brief overview over the experimental situation is given.

Contents

1 Introduction

The Aharonov-Bohm effect (AB effect) is a purely quantum mechanical effect. It tells us something about the physical reality of electromagnetic fields and potentials. It was predicted in 1959 by Y. Aharonov and D. Bohm [1] and caused a controversial discussion. By now many experiments have been conducted that prove the existence of the effect with great precision.
When physicists tried to find a unified theory of interaction, the AB effect became even more important, since it gives evidence that gauge fields are physical reality.

To understand the relevance of the effect let us briefly recapitulate the basic

concept of a field.
Fields are introduced for example in electrodynamics to describe the forces between particles. The idea is that instead of having particles act on each other directly over long distances, the particles create a field that propagates through space and acts locally on the particles again. This *locality* is a very fundamental idea in physics and it is one of the most important properties of a field. So if a field has a "remote action" it is no longer a meaningful physical entity.

2 Potentials in Classical Mechanics

Classical mechanics with electromagnetic (EM) interaction can be formulated by using the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ (emphasis on forces) or by using the scalar potential V and the vector potential $\mathbf{A}$ (emphasis on energy). However the physically relevant objects seem to be the EM fields whereas the potentials have more of the character of mathematical tools. The reason for this is first that only the fields can be directly measured and second the potentials are only defined up to a gauge.

In the Lagrange formulation of classical mechanics only the electromagnetic potentials appear

$$L = \sum_i \left[\frac{m}{2} \dot{x}_i^2 - qV(\mathbf{x}) + q\, \dot{x}_i A_i(\mathbf{x}) \right] \quad . \tag{1}$$

However, since this formulation is equivalent to Newton's law, the motion of a charged particle is completely determined by the local electric and magnetic fields.
The potentials can be gauge transformed with an arbitrary function $\Lambda(\mathbf{r}, t)$ without any change of physics:

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda \ , \qquad V \longrightarrow V' = V - \frac{\partial}{\partial t}\Lambda \ , \tag{2}$$

because this changes the Lagrange function only by a total time derivative, which in turn does not change the equations of motion.

3 Quantum Theory

In quantum theory, we obtain the Schrödinger equation for a charged particle in EM fields

$$i\hbar \frac{\partial}{\partial t}\psi = \frac{1}{2m} \left[\frac{\hbar}{i} \nabla - q\mathbf{A} \right]^2 \psi + q\, V\, \psi \ . \tag{3}$$

The physics should of course still be invariant under a gauge transformation of the potentials as in (2). It is easy to calculate that this is true if the wavefunction transforms in the following way:

$$\psi(\mathbf{x}) \longrightarrow \psi'(\mathbf{x}) = \psi(\mathbf{x}) e^{iq\Lambda(\mathbf{x})/\hbar} \quad . \tag{4}$$

As it should be, the transformation has no measurable effect (see also chapter 6).

4 Aharonov-Bohm Effect

There are different realizations of the AB effect. In the following we will look at two simple examples and show the connection with the Berry phase.

4.1 Bound State AB Effect

Consider a particle with charge q constrained to move on a circle with radius R in the x-y plane, see Fig 1. Within the circle there is a magnetic flux Φ going up the z axis. Classically the motion of the particle should not be influenced whether there is a finite flux or not (as long as it is constant).

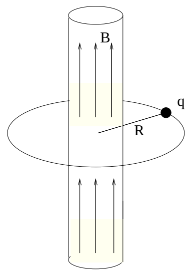

Figure 1: *Bound state AB effect*

The vector potential $\mathbf{A}$ outside the magnetic flux is in cylindrical coordinates given by (gauge $\nabla \cdot \mathbf{A} = 0$):

$$A_\rho = 0 \ , \qquad A_\phi = \frac{\Phi}{2\pi\rho} \ , \qquad A_z = 0 \ . \tag{5}$$

From classical mechanics we find the Hamilton function

$$H = \frac{1}{2mR^2}\left[p - qRA_\phi\right]^2 = \frac{1}{2mR^2}\left[p - \frac{q}{2\pi}\Phi\right]^2 \tag{6}$$

and the canonical (angular) momentum of the system [4]

$$p = mR^2\dot{\phi} + qRA_\phi = L_z + \frac{q}{2\pi}\Phi \ . \tag{7}$$

In order to quantize we must replace the canonical momentum p by $\frac{\hbar}{i}\frac{\partial}{\partial\phi}$: [1]

[1] There is a way to understand physically, why we have to replace the canonical and not the kinetic momentum by $\frac{\hbar}{i}\frac{\partial}{\partial\phi}$ [3]:
The wavefunction changes with time according to the Schrödinger equation (10). If we switch on the magnetic flux Φ in the circle very quickly the wavefunction will not change immediately, because only its rate of change changes. Therefore the gradient of the wavefunction is still the same, but when we switch on the flux an electric field is created that circulates around

$$H = \frac{1}{2mR^2}\left[\frac{\hbar}{i}\frac{\partial}{\partial\phi} - \frac{q}{2\pi}\Phi\right]^2 \quad . \tag{9}$$

The Schrödinger equation is then given by

$$i\hbar\frac{\partial}{\partial t}\psi(\phi) = \frac{1}{2mR^2}\left[\frac{\hbar}{i}\frac{\partial}{\partial\phi} - \frac{q}{2\pi}\Phi\right]^2 \psi(\phi) \quad . \tag{10}$$

For no magnetic flux within the circle ($\Phi = 0$) the Hamiltonian becomes

$$H = \frac{L_z^2}{2mR^2} = -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\phi^2} \quad . \tag{11}$$

The eigenfunctions and eigenvalues are

$$\psi_n(\phi) = \exp(in\phi) \quad , \tag{12}$$

$$E_n = \frac{\hbar^2 n^2}{2mR^2} \quad . \tag{13}$$

The discrete spectrum results from the condition that $\psi(\phi)$ must be single-valued and therefore $n \in \mathbf{Z}$.

What happens now if we have a finite magnetic flux Φ within the circle?

Then, we can expand the Hamiltonian (9) as

$$H = -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\phi^2} + \frac{i\hbar q\Phi}{2\pi mR^2}\frac{\partial}{\partial\phi} + \frac{q^2\Phi^2}{8\pi^2 mR^2} \tag{14}$$

and observe, that the eigenfunctions are still given by (12), but the energy eigenvalues (13) are now given by

$$E'_n = \frac{1}{2mR^2}\left[n\hbar - \frac{q}{2\pi}\Phi\right]^2, \qquad n \in \mathbf{Z} \quad . \tag{15}$$

We can understand this in the following way. Instead of the normal angular momentum L_z being multiples of $\hbar$ the canonical angular momentum is quantized in this way. Therefore with (7) the angular momentum takes on values

$$L_z = n\hbar - \frac{q}{2\pi}\Phi \quad . \tag{16}$$

4.2 Interference Experiment

The AB effect can also be measured in an interference experiment. Consider the following situation: We conduct a double slit experiment with charged particles, see Fig. 2. Between the holes we have placed a perfectly shielded magnetic flux Φ.

---

the flux.

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} \quad . \tag{8}$$

Integrating the Lorentz force on the particle over time, we find that the particle has picked up a momentum $-q\mathbf{A}$. So the kinetic momentum has changed, but the gradient is still the same. Thus, the gradient cannot measure the kinetic momentum. What has also not changed is the canonical momentum $\mathbf{p} = m\dot{\mathbf{x}} + q\mathbf{A}$. So we see that it is this momentum that has to be identified with $\frac{\hbar}{i}\nabla$ or in this case $\frac{\hbar}{i}\frac{\partial}{\partial\phi}$.

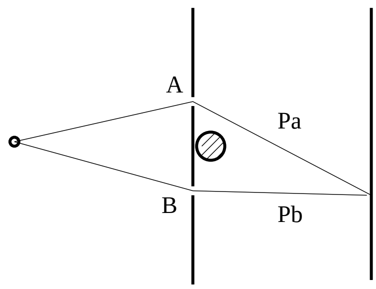

Figure 2: *Double slit*

Classically, we would expect no change in the motion of the particles whether we have a magnetic flux or not, since the particles are restricted to a field free region (of course, classically, we would also expect no interference pattern at all).

A very elegant way to approach the problem is by using Feynman path integrals (see the report of Christian Egli). The overall amplitude to find a particle at a point on the screen is the sum over all paths through A and all paths through B with the right phase factors. The phase factors are determined by the action S and are changed due to the magnetic flux by

$$\delta = \frac{q}{\hbar} \int_{\text{path}} \mathbf{A} \cdot \mathbf{dr} \ . \tag{17}$$

We note that all paths that go along above the flux have the same phase change, since the region above is simply connected, and thus, the line integral only depends on the starting and the end points ($\nabla \times \mathbf{A} = 0$). Also all paths that go along below the flux have the same change in phase.
If Ψ_A and Ψ_B are the amplitudes along the paths above and below without any flux, we can write the new amplitude as

$$\begin{aligned} \Psi &= \Psi_A \, exp\left(i\frac{q}{\hbar}\int_{Pa} \mathbf{A}\cdot\mathbf{dr}\right) + \Psi_B \, exp\left(i\frac{q}{\hbar}\int_{Pb} \mathbf{A}\cdot\mathbf{dr}\right) \\ &= \left[\Psi_A \, exp\left(i\frac{q}{\hbar}\oint \mathbf{A}\cdot\mathbf{dr}\right) + \Psi_B\right] exp\left(i\frac{q}{\hbar}\int_{Pb} \mathbf{A}\cdot\mathbf{dr}\right) \\ &= \left[\Psi_A \, exp\left(i\frac{q}{\hbar}\Phi\right) + \Psi_B\right] exp\left(i\frac{q}{\hbar}\int_{Pb} \mathbf{A}\cdot\mathbf{dr}\right) \ . \end{aligned} \tag{18}$$

The last factor in equation (18) is only an overall phase factor that does not change the interference pattern. We can therefore write the amplitude as:

$$\Psi = \left[\Psi_A \, exp\left(i\frac{q}{\hbar}\Phi\right) + \Psi_B\right] \ . \tag{19}$$

Therefore, we note that there is a relative phase change between the two paths, which is proportional to the magnetic flux Φ. This phase change results in a direct shift of the interference pattern.

4.3 AB Effect and Berry Phase

The AB effect is a good example of the Berry phase. Consider a charged particle that is kept in a box, see Fig. 3.

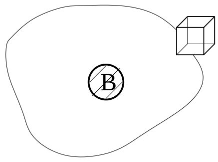

Figure 3: *Berry phase*

If we move the box around a magnetic flux Φ, the wavefunction of the particle in the box changes as in (17). So for a complete circuit we obtain for the change in phase

$$\delta = \frac{q}{\hbar} \oint \mathbf{A} \cdot \mathbf{dr} = \frac{q}{\hbar} \Phi \quad . \tag{20}$$

This is what is called a Berry or geometric phase.

5 Interpretation

The AB effect shows that charged particles, which are restricted to field free regions, can be influenced by distant EM-fields. In one example we saw that the energy levels of a system were changed and in another example we saw that an interference pattern was shifted (a phase was changed). So the magnetic and electric [2] fields have a remote action. If we demand locality from physically meaningful fields, as we did in the introduction, we can no longer regard the EM fields as such. Instead the potentials $\mathbf{A}$ and V become the physically "real" entities. This is a bit hard to digest, since they are still only defined up to a gauge.

6 Exceptional Case

Let us take a look at the energy levels in the bounded state AB effect (15)

$$E'_n = \frac{1}{2mR^2} \left[n\hbar - \frac{q}{2\pi} \Phi \right]^2 , \qquad n \in \mathbf{Z} \quad . \tag{21}$$

We note that the effect vanishes if the flux takes values

$$\Phi = k\,\Phi_0 \qquad \text{with} \qquad \Phi_0 := \frac{2\pi\hbar}{q} \qquad \text{and} \qquad k \in \mathbf{Z} \ , \tag{22}$$

because in this case all that happens is that the energy states obtain different "labels"

[2] Similar results can be obtained using time dependent electric fields. This is called the electric AB effect [2].

$$n \longrightarrow n' := n - k \ . \tag{23}$$

In the interference experiment we do not have an observable effect for this value of Φ, either. The relative phase change of the two paths is given by (19):

$$\delta = \frac{q}{\hbar}\Phi \ . \tag{24}$$

So for $\Phi = \frac{2\pi\hbar}{q}\,k$ (22) we find $\delta = 2\pi k$ (same for the Berry phase (17)).
We can explain this when we look at the Hamilton operators with and without an **A** field

$$H_0\psi_0 = \frac{1}{2m}\left[\frac{\hbar}{i}\nabla\right]^2 \psi_0 \ , \tag{25}$$

$$H\psi = \frac{1}{2m}\left[\frac{\hbar}{i}\nabla - qA\right]^2 \psi \ . \tag{26}$$

Let us define the following multiplication operator:

$$U(\mathbf{x}) := exp\left(-\frac{iq}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{A}\cdot\mathbf{dr}\right) \ . \tag{27}$$

This operator is only well defined if it is independent from the integration path. If this is the case, we obtain the following equations

$$\psi = U\psi_0 \ , \tag{28}$$

$$H = U\,H_0\,U^{-1} \ . \tag{29}$$

This means we have two unitary equivalent descriptions of the same system. Therefore, there is no physical difference and no observable effect from the **A** field.

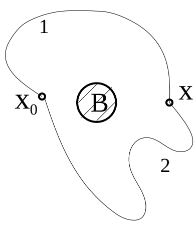

Figure 4: *Two paths*

Now let us investigate when the operator (27) is well defined. This is obviously the case when we have a simply connected region with $\nabla \times \mathbf{A} = 0$. In other words when there is no magnetic field at all and our vector potential can be written as $\mathbf{A} = \nabla\Lambda$.
Actually this is the only case where $\int_{x_0}^{x} \mathbf{A}\cdot\mathbf{dr}$ is independent from the integration

path. However, we have a less strong condition. We just demand that $U(\mathbf{x})$ (27) is independent from the integration path in the exponent.
In a region with one hole, the only different paths that we have to compare are one above and one below the magnetic flux [3], see Fig. 4. The difference of the integrals is of course given by $\Phi := \oint \mathbf{A} \cdot \mathbf{dr}$ and we note that $U(\mathbf{x})$ is well defined if Φ is an integer multiple of Φ_0. For paths that go around the hole a number of times we also obtain the same value for $U(\mathbf{x})$.
We can say in general that all observable phenomena depend only upon the magnetic flux Φ through the excluded region with period Φ_0.

7 Experimental Results

The first experiments gave evidence to the existence of the AB effect, but they could not proof the effect with absolute certainty. There are at least two reasons why the experiments are difficult to conduct. The first is that one has to observe interference of charged particles and thus one must have a very high resolution in the order of the de Broglie wavelength. The other reason is the so-called "stray field" problem. It is experimentally very difficult to obtain a magnetic field that is restricted to a certain region. One would in theory need an infinitely long solenoid.
Over the years the AB effect has been tested in a lot of different experiments and today there is no doubt about its existence [2].

References

[1] Y. Aharonov and D. Bohm: *Phys. Rev. 115*, 485 (1959)

[2] M. Peshkin and A. Tonomura: *The Aharonov-Bohm Effect*, Springer-Verlag, (1989)

[3] R. P. Feynman: *Lectures on Physics III*

[4] F. Scheck: *Mechanics*, Springer-Verlag

[5] B. H. Bransden and C. J. Joachain: *Quantum Mechanics*

[6] J. D. Jackson: *Classical Elektrodynamics*

[3] Same argumentation as for the interference experiment after (17).