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Mathematics Workshops

Rods for Multiplication and Division

THE GARDEN OF ARCHIMEDES.
A Museum for Mathematics

Rods for Multiplication and Division

Introduction

The way we count is certainly one of the most powerful and fully developed systems ever invented. But it is also one of the most complex and difficult to learn. Other strategies, preliminary or alternative, other viewpoints, more primitive but in some cases no less effective, give us a better understanding of some aspects of counting, and help us to overcome some difficulties, to grasp the potentials of the way we count, and also provide us with some details of the fascinating history of counting.

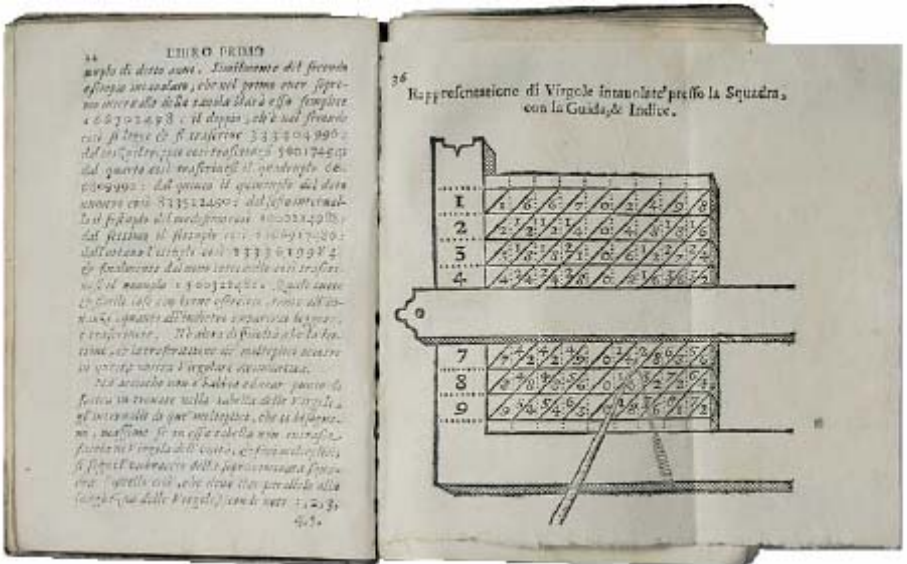
With this in mind, Garden of Archimedes laboratories were set up, dedicated to numbering systems and designed for all types of schools and grades of students, and to some of these ancient ways of counting. It is an experimental activity for classes in which our operators work.

The purpose of this booklet, dedicated to rods used for calculations, is to provide teachers who want to carry out this activity with some theory and many practical hints about how to carry out these laboratories.

1. Historical Remarks

The seventeenth century saw the beginning of the development of instruments to aid calculation, making the process gradually more and more automatic. One of the forefathers of calculating machines were Napier’s bones (aka Neper’s rods). Neper was the Latin name used for John Napier, 8th Laird of Merchistoun (1550 – 1617).

Napier’s bones are described in the book *Rabdologiae seu Numerationis per virgulas libri duo*, published in 1617.



In the Preface to the book, Napier declares that the purpose of his work was to eliminate the difficulty and length of time involved in making calculations, turning many against the study of mathematics. The same was true of the invention of logarithms, for which Napier is chiefly remembered.

Napier’s bones are the first of three devices described in

the book. He calls them “virgulae numeratrices” and “**rabdologia**” (Rabdology) is the way they are used for calculation. Using multiplication tables embedded in the rods multiplication is reduced to addition.

Napier describes how the rods are constructed and used. The version presented here is slightly different from the original in which the rods appeared four by four on the faces of a rod with square cross-section. The method is the same.

Napier demonstrates multiplication and other applications of the rods, which we will not go into here, limiting ourselves to showing how, with the use of an additional ruler, the rods can be used to calculate square roots.

One interesting variant on the rods for multiplication, eliminating even addition (for multiplications of single-digit numbers) was put forward in 1885 by the Frenchmen Henri Genaille, a civil engineer, and Edouard Lucas. These rods were described in a book entitled *Les reglettes multiplicatrices*.

In the same work by Genaille and Lucas, the two authors also describe rods called *reglettes multisectrices* which can be used for the division of single-digit numbers. The practical application of

the rods used for division was in the special rods *des financières* relating to recurrent values in the calculation of interest, used for accounting purposes. Here, we will describe only the first, used for ordinary divisions.

Compared to Napier’s bones, Genaille and Lucas’s rods for multiplication eliminate carry-overs for the products of single-digit numbers. Another variant, simplifying the products of multiple-digit numbers, was suggested by Napier himself in an appendix to *Rabdologia*. This was the *Multiplicationis Promptuarium*, comprising two types of rod. Choosing them and suitably placing one on top of the other, the calculation becomes a simple matter of adding numbers in diagonal strips. According to the length of the rods, the factors can have a large number of digits, at least in theory. Napier gives an example of a 10-digit number multiplied by another 10-digit number.

2. Multiplication Using Napier’s Bones

To multiply using Napier’s bones ten different “bones” or rods are used, numbered across the top from 0 to 9, and a further rod we shall call “ruler”, marked by an \times . Each rod is divided into nine squares bearing numbers. Apart from the ruler, the squares are divided by a diagonal making two triangles. The diagonal goes from the bottom left to the top right.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | \times |
|---|---|---|---|---|---|---|---|---|---|----------|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9 |

2.1. How They Are Used

2.1.1. Multiplying by Single-Digit Numbers

To multiply a multi-digit number by a single-digit number, the multi-digit number is put together by placing side by side the rods marked with the corresponding numbers and then the ruler marked \times is placed on the right. Only the line with the second factor is taken into consideration and to find the answer the two diagonal numbers are added together.

For example to multiply 725×3 , we put next to each other the rods numbered 7, 2, 5 making the number 725 and we put the ruler to the right of these, as shown below:

| 7 | 3 | 5 | x |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 1 | 3 |
| 2 | 0 | 2 | 4 |
| 3 | 1 | 2 | 5 |
| 4 | 1 | 3 | 6 |
| 4 | 1 | 4 | 7 |
| 5 | 1 | 4 | 8 |
| 6 | 1 | 4 | 9 |

Because we want to multiply by 3 we read off only the third line, shown by the number 3 on the ruler:

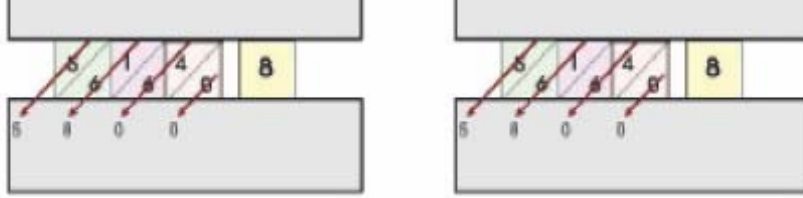
| 7 | 3 | 5 | x |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| | | | |
| 2 | 0 | 1 | 3 |
| | | | |
| 5 | 0 | 5 | 0 |
| 4 | 1 | 3 | 6 |
| 4 | 1 | 3 | 7 |
| 5 | 1 | 4 | 8 |
| 6 | 1 | 4 | 9 |

Now we add up from right to left, following the diagonals, the numbers in the triangles of the rods, and we obtain: 5 , $6+1=7$, $0+1=1$, 2 .

Reading from left to right we get the result: 2175.

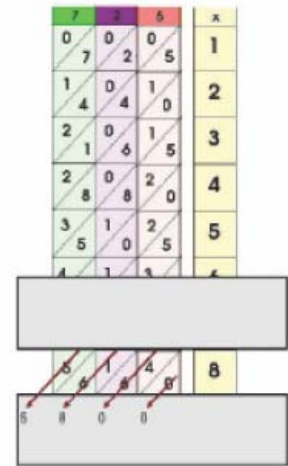
| 7 | 3 | 5 | x |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| | | | |
| 2 | 0 | 1 | 3 |
| | | | |
| 5 | 0 | 5 | 0 |
| 4 | 1 | 3 | 6 |
| 4 | 1 | 3 | 7 |
| 5 | 1 | 4 | 8 |
| 6 | 1 | 4 | 9 |

2 1 7 5



2.1.2. Multiplications with the Carry-Over

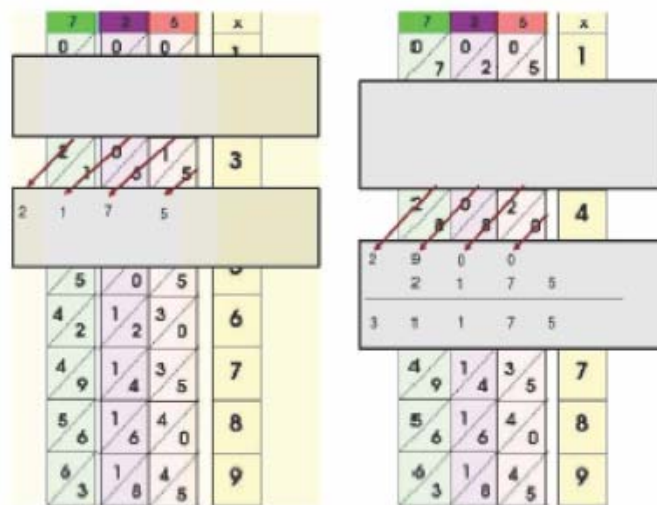
Sometimes, adding the two diagonal numbers will produce a number to be carried over. This is done as usual, adding the carry-over to the next diagonal. For example, multiplying 725 by 8 I obtain: 0, 4+6=10 so I write 0 and carry over 1, 1+6=7 plus 1 carried over = 8, 5. So the result is 5800.



2.1.3. Multiplying by Multi-Digit Numbers.

As Napier himself showed, it is possible to multiply two multi-digit numbers, using the rods to calculate partial products. For example, if we want to multiply 725 by 43, first we multiply 725 \times 3, as described above, noting the partial result of 2175. Then we drop down to the fourth line to calculate 725 \times 4 and note the second partial product of 2900, together with the first partial result, but shifting the number to the left by one column, i.e. writing one 0 under the 7 and the other 0 under the 1, the 9 under the 2 and then the 2.

The result, 31175, is obtained by adding up the two partial results.



Instead of calculating the last product explicitly, it is also possible to add it diagonally with the partial products obtained, making sure the numbers are in the correct columns.

2.2. How They Are Constructed

Looking at one of the rods, how they are constructed can be seen immediately: each of the nine rods carries the multiple of the number it represents, so rod 5 carries 5, 10, 15, 20, 25, 30, 35, 40, 45.

For each number, the units are written in the lower triangle and the tens in the upper triangle. What is the reason for this unusual positioning?

When the bars are put next to each other, the tens are on the same diagonal as the units on the rod to the left. In this way diagonal strips are formed (in a parallelogram) in which there is one triangle from one rod and one triangle from another rod. Therefore the numbers in the same parallelogram are of the same decimal order (and can be added together).

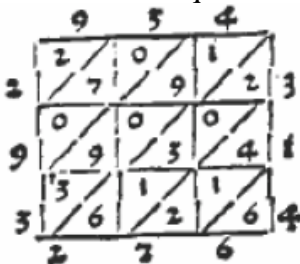
Let's look again at the first example: 725×3 . On the right-hand side, on line 3 we find 15 on the rod for 5, 06 on the rod for 2, and 21 on the rod for 7, i.e. we find 5×3 , 2×3 and 7×3 . When we look at the diagonal strips, in the first strip on the right we have just the number 5 (only one triangle), and in the next strip we have 1+6 (in the parallelogram formed by the pink triangle with 1 and the purple triangle with 6). The 1 of the first rod (the tens) are added to the 6 of the second. Exactly as it should be! Because the 6 is not in the units column but the tens column: when we multiply 2×3 we are multiplying 2 tens by 3 units and hence we obtain 6 tens, which are added to the tens obtained from 5×3 . In a similar way, 21 obtained from 7×3 are hundreds, or rather 1 hundred and 2 thousands and therefore should go into the next diagonal strips.

In other words, the diagonal disposition of the multiples of each rod and the sum inside the parallelograms are the graphic representation of the distribution properties of the product by which 725×3 is of course $5 \times 3 + 20 \times 3 + 700 \times 3$.

2.3. Remarks

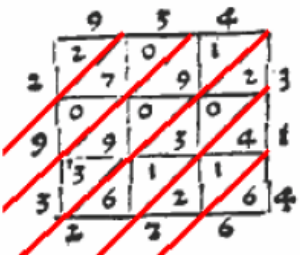
The basic idea of Napier's bones is exactly the same as the multiplication algorithm known as "multiplication by **jealousie**", a method used by the Arabs from the thirteenth century and very common in **abacus treatises**.

To find the product of a number with m digits by a number with n digits, a grid can be designed with $m \times n$ squares.



In the example, taken from *Aritmetica di Treviso* dated 1478, 934 is multiplied by 314. Each square is divided into two triangles by a diagonal. The two factors are then written around the grid. In each square we write the result of the product

of the digits in the factors identifying the line and column of the square itself, writing the units into the lower triangle and the tens in the upper triangle. To obtain the result we add along the diagonals starting on the right, and adding any carry-overs into the next diagonal.



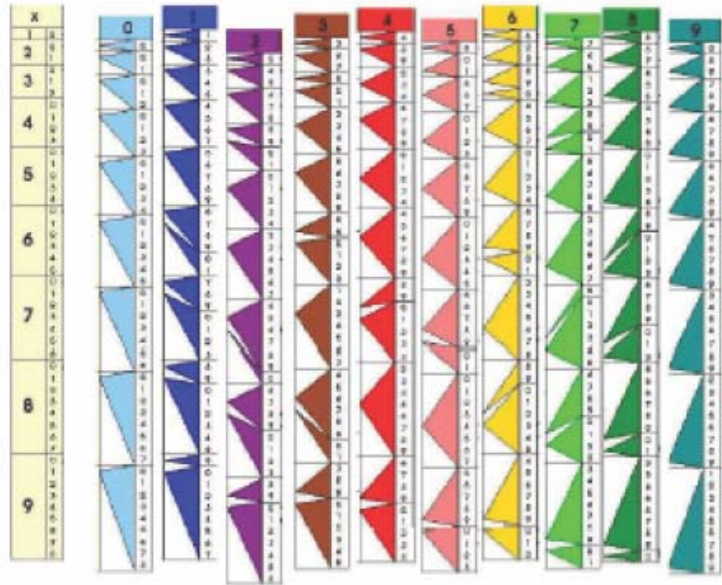
In this way we obtain 6, $4+1+2$ i.e. 7, $2+0+1+3+6$ i.e. 2 and 1 carried over and so on until the final diagonal strip which contains just 2. The result is read starting with this number and is: 293276.

Napier's bones enable multiplication by **jealousie** without having to construct the grid and, above all, without calculating or knowing the products of the individual factors, i.e. without the need to know the **multiplication** tables!

3. Multiplication Using the Genaille-Lucas Rods

The variant on Napier's bones invented by Genaille and Lucas and presented in 1885 in the book *Les réglettes multiplicatrices* makes calculations even simpler, eliminating the diagonal sums. When multiplying one multi-digit number by a single-digit number, all we now need to do is place the rods correctly and read off the result directly, without making any calculation at all.

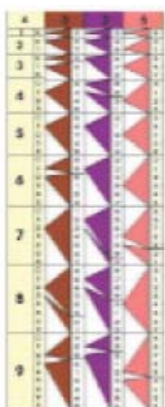
Here too there are ten rods, numbered 0 to 9, and a ruler marked \times . The rods and the ruler are divided into nine rows which get wider and wider. On the right-hand side there is a slender column with a certain number of digits: one only in the first row, two in the second, three in the third and so on, to nine in the ninth. In the rods, for each row there are one or two triangles, one side against the slender column and the apex against the side of the rod. On the ruler the rows are numbered from 1 to 9.



3.1. How They Are Used

3.1.1. Multiplying by a Single-Digit Number

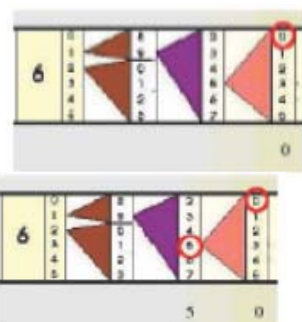
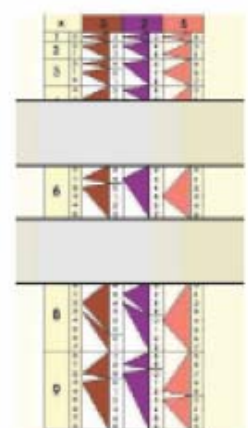
The rods have a more complex appearance than Napier's bones but they are actually easier to use.



To multiply a multi-digit number by a single-digit number, put together the rods that comprise the multi-digit number and, on the left, place the ruler with the symbol \times . To calculate, for example, 325×6 , place the rods numbered 3, 2 and 5 next to each other and the ruler on the left.

We use only the row which corresponds to the second factor, 6, so the sixth row.

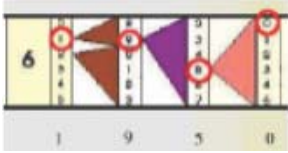
Now we just read off the result. This is given by the first number in the right hand slender column followed by the number indicated by the apex of the triangle in the next slender column (to the left) and then the other numbers indicated by the apex of the other triangles, moving from right to left.



In our example, the first number in the slender column on the right is 0 (the number highest up the column), which we write down.

The pink triangle "points to" the next number in the answer, in this case 5, so we write down 5 to the left of the 0 we previously jotted down.

The purple triangle points to 9, so we write that to the left of the 5.

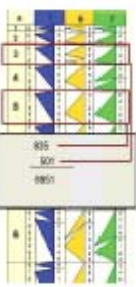


We now come to two brown triangles. We choose **the one on the top**, because the 9 we previously jotted down is along the base of the upper triangle. If we had jotted down a number along the base of the lower triangle we would have chosen the number indicated by the apex of the lower triangle. In our example, the apex of the upper triangle points to 1.

So, reading from left to right the result is: 1950.

3.1.2. Multiplying Multi-Digit Numbers

To multiply a multi-digit number by a multi-digit number, the rods are used to calculate partial products and then added.

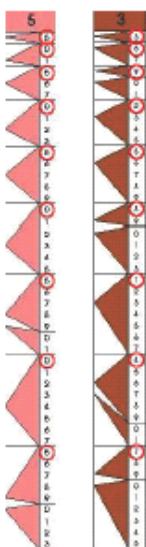


Say we want to multiply 167 by **53**. **First** we multiply 167 by 3, as described above, and note the partial result of 501.

Now we go down to the fifth row to calculate 167×5 . We note down the partial result of 835, but line up the columns shifting 835 one column to the left, i.e. writing the 5 under the 0, the 3 under the 5 and the 8 in an empty column.

Now we add the two partial results and obtain 8851.

3.2. How They Are Constructed



Like Napier's bones, here too the rods **carry the multiples** of the number the rod represents, so rod 5 carries 5, 10, 15, 20, 25, 30, 35, 40, 45. However, this time the notation is far more hidden.

Let's take a closer look at the rod numbered 5 and, in particular, **the top numbers** in the slender right-hand columns. The first row shows the number 5, the second **has the number 0 in the top position**, the third **has 5**, the fourth 0, the fifth 5, and so on. You will probably **notice these are the units of the products** 1×5 , 2×5 , 3×5 , 4×5 , 5×5 , etc. If we look at the other rods it becomes clear that indeed it is so. The rod numbered 3 in fact carries in the **top position of each row** the numbers 3, 6, 9, 2, 5, 8, 1, 4, 7, i.e. the units of the multiples of 3: 3, 6, 9, 12, 15, 18, 21, 24, 27.

So where did the tens go? **Let's** go back to rod numbered 5. We put the ruler to the left and look at the third row where we should find the result of 5×3 .



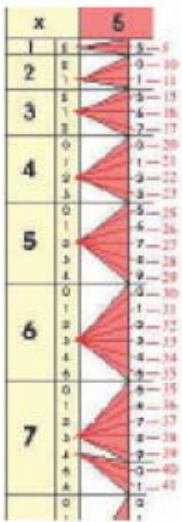
Following the general indications, as the unit number we take the top number in the slender column, which is a 5, and then we look for the number the triangle points to for the tens. Of the three possibilities, 0, 1 and 2, the triangle indicates 1, giving us the result of 15.

So the tens are the numbers the triangles point to. If we now look at the rows of rod numbered 5, we see the triangles point respectively to 0, 1, 1, 2, 2, 3, 3, 4, 4, in other words the tens of the multiples 5, 10, 15, 20, 25, 30, 35, 40, 45.

We can look at the other rods to see that the same thing happens.

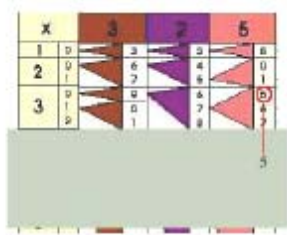
Let's go back to the rod numbered 5 next to the ruler. Why, in the third row, does the slender column of the ruler show the numbers 0, 1, 2? Because these are the only tens that are produced by multiplying the numbers from 0 to 9 by 3 since 0×3 , 1×3 , 2×3 and 3×3 are below 10 (and so have 0 tens); $4 \times 3 = 12$, $5 \times 3 = 15$ and $6 \times 3 = 18$ produce 1 ten and the remaining numbers $\times 3$ produce 2 tens.

Now we can try to understand the meaning of the other information on the rods and how they eliminate the need for calculating any sum.



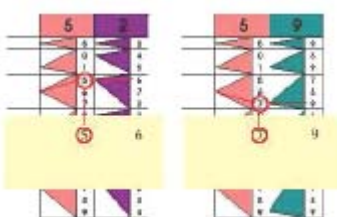
Again, we'll look at rod numbered 5, and place the ruler to the left of it in order to see where the triangles "point". Remembering what we just said, we can observe once more that the first row shows just one number, which is 5; we have 5 in the column of units and the triangle points to 0. In the second row there are two numbers; the top number is 10 (0 units and the apex of the triangle pointing to 1, i.e. 1 ten), the other number is 11 (1 unit in the column and the apex of the triangle pointing to 1 ten). In row 3 we have 15 and the numbers that follow, 16 and 17. In the fourth we have 20 and the next three, 21, 22, 23. And so on row after row, putting together units and tens. In row 7 we have 35 and the next six numbers: 36, 37, 38 and 39 with the triangle pointing to 3 tens and 40 and 41 which have units of 0 and 1 and the triangle pointing to 4 tens.

What is the purpose of these further numbers which appear on the various rows? Basically, they are the pre-calculated results of all the possible additions relating to numbers carried over which I might need to carry out for a multiplication.



To understand this better, let's say we are calculating the product of one number in which we need to calculate 5×3 , according to the instructions set out above. When we have set out the number using the rods, the rod numbered 5 will be in a certain position.

If the 5 is in the units column (for example if we have to calculate 325×3), even if I don't know any of the other numbers to be multiplied by 3 I know that the final number of the result, in the units column, will be 5. And in fact the top number in row 3 of the slender column is a 5.



If, however, 5 is in the tens column, I cannot be certain that the end result will have a 5 in the tens results because I may need to carry over a number from the units column. For example if the number in the units column is 9 I will calculate $9 \times 3 = 27$ and carry 2 over into the tens column, so there I will have $5 + 2 = 7$.

So I don't know what number I will have in the tens column. I may have to carry over 0 or 1 or 2. The carry over can be no higher than 2 because the maximum is given by $9 \times 3 = 27$. So if I start with 15, I might end to 16 or 17. And these, in

fact, are the other numbers which appear with 15, on the third row of the rod numbered 5.

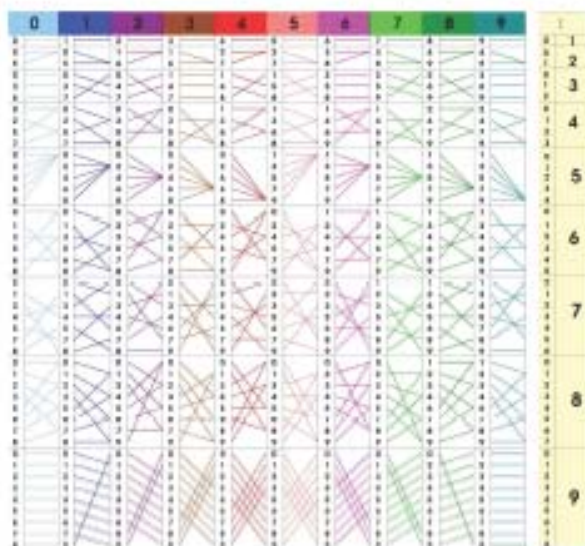
What tells me which number to choose? Simply place the rod for the units column to the right of the rod numbered 5. Given the way the rods are constructed, the number in the tens column is not given, but it is given by the apex of the triangle. So, if the preceding triangle points upwards (corresponding to 0 tens), I take the first position, in our example 15 meaning I am not adding anything; if it points to the second position (corresponding to 1 ten) I take 16, meaning I am carrying 1 over, and if it points to the third position (corresponding to 2 tens) I take 17.

Obviously the multiples of 9 give the highest carry-overs and so determine how many numbers are needed in the column of the various rows. In the third row, as we have seen, we need three numbers, 0, 1 and 2, because $9 \times 3 = 27$ (2 tens). In the fourth row we need four because $9 \times 4 = 36$ (so we need 0, 1, 2 or 3 tens), and so on until the ninth row where nine numbers are required because multiplying 9×9 , 8 tens are carried over.

4. Division Using the Genaille-Lucas Rods.

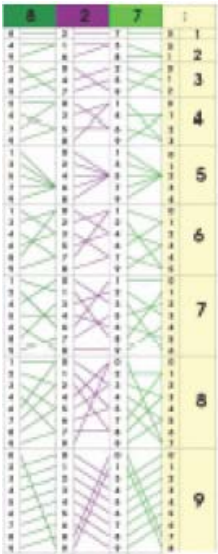
In the same publication which illustrates the multiplication rods, Genaille and Lucas also describe the rods used for division.

These are ten rods numbered from 0 to 9 and a ruler marked with the symbol \div . As with multiplication, the rods and the ruler are divided into 9 rows, which are increasingly wide. On the left there is a slender column with various numbers: there is only one number in row 1, two in row 2, three in row 3 and so on. On the rods, from each of these numbers there is a line which may be tilt up or down. On the ruler, the rows are numbered from 1 to 9.



4.1. How They Are Used

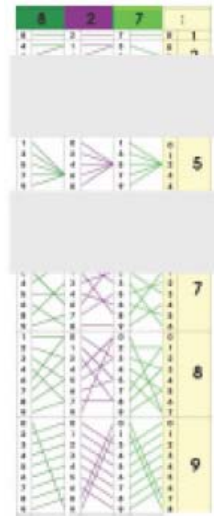
4.1.1. Dividing by a Single-Digit Number



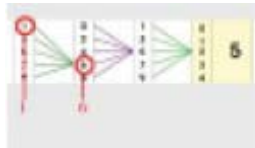
To divide a multi-digit number by a single-digit number, place the rods that make up the number to be divided next to each other and put the ruler marked with the division sign to the right. For example, to calculate $827:5$, first make the number 827 with the rods numbered 8, 2 and 7 and put the ruler to the right of the 7 rod.

Then take only the row which corresponds to the divisor, 5, so the fifth row.

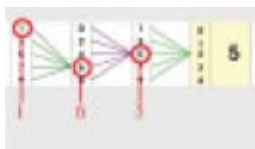
We can now read off the result. This is given by the numbers we can see starting on the left-hand rod, in which, as before, we take the number which is in the highest position in the slender column, as indicated by the line slanting to the left, and from there we move to the right, following the lines. The remainder is the number shown on the ruler.



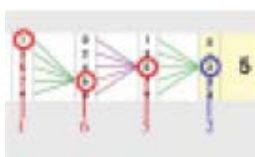
In our example the first number is 1, the top number in the left-hand column, which we can jot down on a piece of paper.



The green line now tells us the next number, which we take from the rod to the right. In our example, the green line leads to a 6.



Now we follow the purple line from the 6 to the next rod on the right, which leads to a 5.



From here a green line takes us to the ruler and the number 2, which is the remainder.

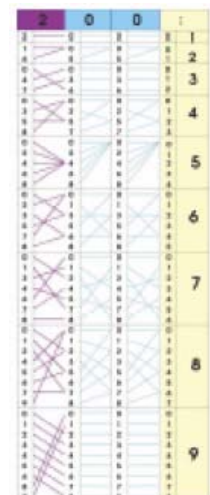
So we have $827:5 = 165$ with a remainder of 2.

4.1.2. Decimals.

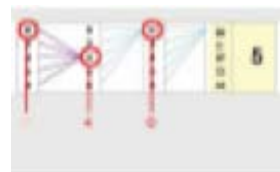
If we are not content with the result as a whole number with a remainder and want to calculate the decimals, the rods will be useful for this calculation too.

We take the remainder and add a sufficient number of zeros, as many as the decimals we want to calculate. Now we take this number and divide it by the divisor. Then we add this number to the previous whole number as decimals.

In the example above, if we want to calculate the result with two decimals, and add two zeros to 2 (the remainder), obtaining 200.



Now we divide 200 by 5 according to the instructions above. We take the rod numbered 2 and two rods numbered 0 and place the ruler on the right. On the fifth row we read off 040.



Leaving aside the first 0 (which are still units), we obtain the decimal of 40. So the full result of the previous calculation, $827:5$, is 165.40. If the ruler shows that the remainder is 0, we have come to the end of the calculation. If the remainder is not 0, the next decimal places can be determined by repeating the procedure: we take the new remainder, add as many zeros as we wish and divide the new number by the divisor.

4.2. Remarks

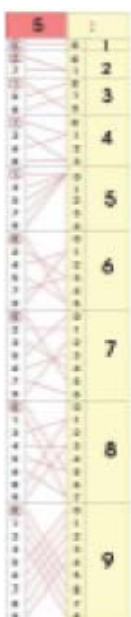
In the case of multiplication, the rods can be used to multiply multi-digit numbers by multi-digit numbers, by breaking down the calculation into partial totals. It is not possible to do anything similar to divide multi-digit numbers by multi-digit numbers, because the whole of the divisor needs to be considered.

4.3. How They Are Constructed

The way the dividing rods are constructed is in some ways a mirror-image of the multiplication rods.

As for the multiplication rods, let's look at how each rod is made by examining the rod numbered 5 and look for the quotients $5:1$, $5:2$, $5:3$, $5:4$, $5:5$, $5:6$, $5:7$, $5:8$, $5:9$. If we put the ruler to the right of the rod in order to carry out the divisions as indicated, in the first row we find the result of $5:1$, indicated by the only number in the slender column, which is a 5. In the second row we find the result of $5:2$, indicated by a 2 on the top of the slender column. In the third row, we find a 1, the result of $5:3$. And so on: every first digit in the slender column in each row indicates the quotient of 5 by the number corresponding to that row.

This is true for every rod, of course, as is inevitable if the procedure is to work for single-digit dividends.



Now we can look at the lines that start off from the quotients, with their different slants, indicating the remainder for each division on the ruler.

Here too we could wonder why the rows have columns with different numbers of digits, increasing from one to nine from the first to the last row. The numbers on the ruler represent remainders. So, clearly, on row 1 which represents division by 1 there is only one number, a 0, because the remainder of any division by 1 is 0. In the second row, corresponding to division by 2, there are two possible remainders, 0 or 1. In the third row, there are three possible remainders: 0, 1 and 2 of a division by 3. And so on until the ninth line, division by 9, where the possible remainders go from 0 to 8.

The meaning of the other numbers in the columns for each row is less obvious. We have already discussed the top number **of each row in the column**. But what about the other numbers? They are the pre-calculated results of all the possible divisions in which the rod of that number could be involved.

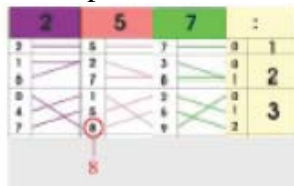
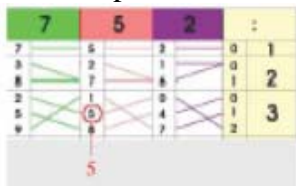
To understand this better, let's see what happens if we divide a number which includes a 5 by 3, according to the instructions.

If the 5 is in the first position on the left, i.e. it is of the highest decimal order (as in $527:3$), even if we don't know the other digits of the dividend, we know that the first digit of the result will be 1, the quotient of $5:3$. So we look at the top number in the third row and, in fact, we find 1.



If, on the other hand, the 5 is not the highest decimal order but, say, the second highest (for example in $752:3$ or $257:3$), I cannot know what the second digit of the result will be:

it will depend on the first digit i.e. it depends on the remainder left over by dividing the first digit by 3;



This should be “put in front of” the 5 before dividing by 3. In other words, in division, as is well known, to divide by 3 we do not only divide the tens of the number but also the hundreds left over.

In our example, $752:3$, to calculate the second digit of the result, we do not need to divide $5:3$ but $15:3$, because the division of 7 hundreds by 3 gives a remainder of 1. So, $15:3$ gives me 5 as the second digit in the result. If, instead of 7 hundreds, the number had had two hundreds, I would have had a remainder of 2 ($2:3$ gives 0 with a remainder of 2) and I would have divided 25 by 3 to obtain the second digit of the result, which would therefore have been 8 in the tens column.

In general, then, when **the number to be divided has a 5** in in a given position, we cannot know the digit that will be in that position in the result. But we do know, for example, if we are dividing by 3 that there are three possibilities: $5:3$, $15:3$ or $25:3$, giving a result of 1, 5 or 8, which are therefore the numbers in the column. I exclude 35, 45, 55, etc., because irrespective of the number on the left of the 5, the maximum remainder I can obtain from dividing that number by 3 is 2.

So, if we look at the fourth row of rod 5, we see the numbers 1, 3, 5, 7, 9, i.e. the quotients of $5:4$, $15:4$, $25:4$, $35:4$.

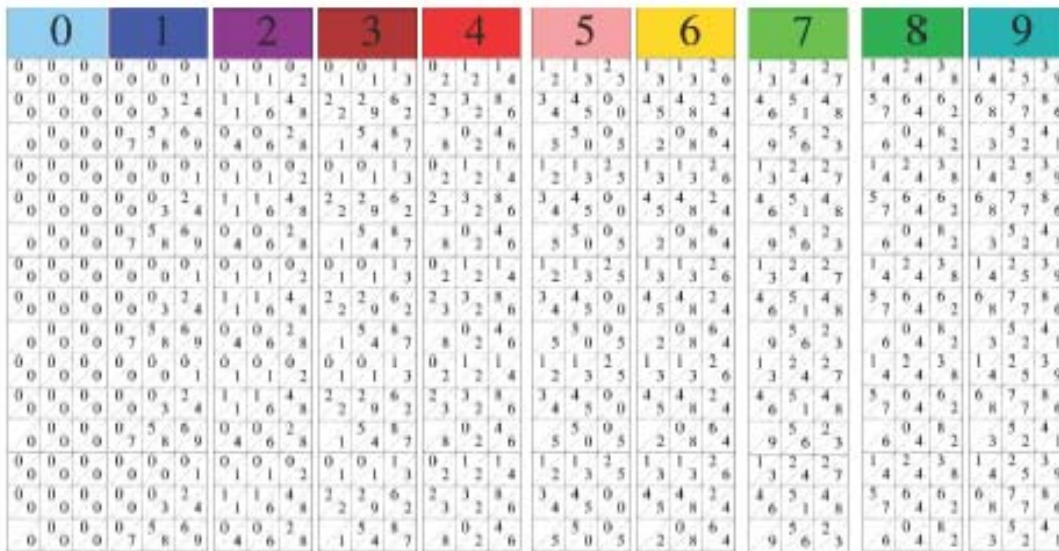
Now we know what the numbers in the columns are. **But how** do we choose the right one? The **answer comes from the lines on the rods**. The rods are constructed in such a way that the remainder of every quotient is indicated by the line tilting from the number of the preceding rod. So, automatically, if the line points to the first position (corresponding to a remainder of 0) I will simply have the quotient of the number I am looking at as the divisor. If it points to the second number (corresponding to a remainder of 1) a quotient is indicated for the number increased by 1 ten. And so on.

5. Multiplication using Napier's *Promptuarium*

In the appendix of *Rabdologiae seu Numerationis per virgulas libri duo*, which describes the rods, Napier also describes a variant which makes multiplying multi-digit numbers easier. He calls this system the *Multiplicationis Promptuarium*, by which “any multiplication, no matter how difficult and long, can be executed swiftly and with ease”.

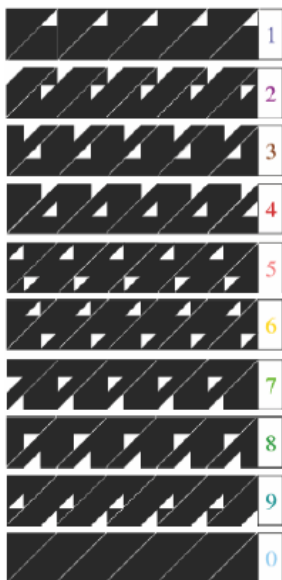
The *Promptuarium* includes two types of rods. The first are numbered from 0 to 9 and

run vertically; they have a series of rows, columns and diagonal lines full of numbers.



The others are numbered from 0 to 9, but are horizontal; they only have one number, the number of the rod, and otherwise they show diagonal lines and **triangular windows** which are repeated over and over.

5.1. How They Are Used

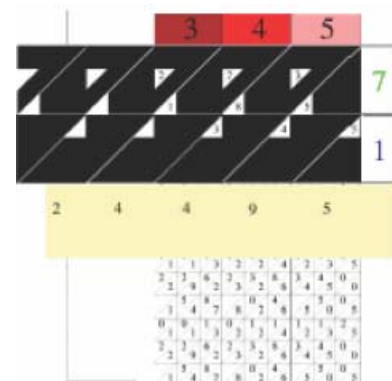


To find the product of two multi-digit numbers the first thing to do is to choose the vertical rods that represent the longest number. The other rulers are used to make the other number, one placed above the other, with the units at the bottom. These are then put on top of the other rods so the top right-hand angles are one on top of the other, excluding the numbers of the rod.

Each digit of the result **is found by means of** a diagonal strip, writing the sum of all the numbers that appear. We start with the bottom right strip (corresponding to the units) and any carry-overs are added on.

For example to multiply 345 by 71, first we set out the vertical rods marked 3, 4 and 5 making the number 345. Then, with the other rods placed one above the other, we make the number 71.

These rods are then placed on top of the others.



Now we add up the numbers inside the diagonal strips. In the right-hand strip there is only the number 5, which we jot down on a piece of paper: this will be the units. Moving to the left, in the second strip we find a 5 and a 4; so we write down 9, which will be result in the tens column. Again to the left we find a 3, an 8, and another 3; adding them together we get 14; so we write down 4 and carry 1 over to the next strip, where we find a 2 and a 1, which, with the

carry-over, makes 4. In the final strip we find only a 2 that we jot down.

So reading the result from left to right we have 24495.

5.2. Remarks

If the rods are long enough and we have plenty of them, any multi-digit numbers can be used as the first or second factor.

In the example given by Napier, he multiplies 8795036412 by 3586290741.

5.3. How They Are Constructed

Both the vertical and horizontal rods comprise square modules which are repeated. The number of modules determines the maximum number of digits in the factors.

Let's see how the modules are constructed. We start with the vertical rods with many numbers. They have modules which consist of squares of three lines and three columns. A quick look at the rods and you can see that every three lines the numbers are repeated.



So what do the numbers inside each module mean? Each module is a classic Napier's bone, only made smaller. Each module contains the multiples of the corresponding number, as in Napier's bones. In that system, the tens and units of each multiple were set out respectively in the triangle above and the triangle below each subsequent square. In the nine rods these triangles are ordered differently. If we think of each module divided into two by the diagonal, the triangles containing the units are all below the diagonal line and those containing the tens are all above the diagonal line.

For example, let's look at the first module on rod numbered 5 (the others are identical, as already pointed out).

Recombining the triangles, we find all the multiples of 5: 5, 10, 15, 20, 25, 30, 35, 40, 45. For example 35, i.e. 5×7 , can be found by putting together the number in position B7 with the number in position A7, i.e. 3 and 5.

| | | | | | |
|---|---|---|---|---|---|
| 5 | | | | | |
| 1 | 2 | 1 | 3 | 2 | 5 |
| 3 | 4 | 5 | 0 | 0 | |
| 5 | 5 | 0 | 5 | | |

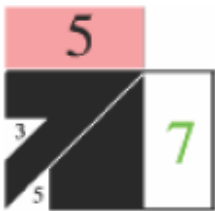
| | | | | | |
|---|---|---|---|---|---|
| 5 | | | | | |
| 1 | 2 | 1 | 3 | 2 | 5 |
| 3 | 4 | 5 | 0 | 0 | |
| 5 | 5 | 0 | 5 | | |



Now let's look at the second type of rod. Here, the square modules are repeated horizontally. Here, too, each module can be seen as a series of triangles. In each module, two triangles are blank and are actually hollowed out windows so the numbers beneath can be read. Let's look at the rod numbered 7. The triangular windows are in the position B7 and A7.

Looking at the other rods, in each one the triangular windows are in the position of the tens and units of the multiple corresponding to that number. On rod 1, where there is actually only one triangular window, because all the tens for the products of 1 are 0, the window is in position A1.

On rod 2, the windows are in positions B2 and A2. On rod 3 the positions are B3 and A3 and so on. Rod 0 has no windows, because any number multiplied by zero is zero.



When the rods are overlapped, so that the square modules fit perfectly, only the numbers of the product required are visible. So if 7 is placed above 5, only the 3 of the tens and the 5 of the units are visible, the product of 5×7 .

By placing a number of rods next to each other I obtain a series of partial products.

Since each rod is like a series of Napier's bones, all the same, I can use the second or third module, or any module, to multiply the same number by other factors.

So in 345×71 on the first row I have the products of 3×7 , 4×7 , 5×7 and on the second row I have the products of 3×1 , 4×1 and 5×1 .

The fact that the units of each product are under the diagonal dividing the module and the tens are above it, means that in the diagonal strips which are visible when one rod is placed on top of a horizontal rod or rods, each number is located in the position of others in the same decimal order. So all we have to do is add them up.

6. Obtaining Square Roots Using Napier's Bones

| quadrati | | | |
|----------|---|----|---|
| 0 | 1 | 2 | 1 |
| 0 | 4 | 4 | 2 |
| 0 | 9 | 6 | 3 |
| 1 | 6 | 8 | 4 |
| 2 | 5 | 10 | 5 |
| 3 | 6 | 12 | 6 |
| 4 | 9 | 14 | 7 |
| 6 | 4 | 16 | 8 |
| 8 | 1 | 18 | 9 |

Napier's bones can be used to calculate square roots, as Napier himself pointed out in *Rabdologiae seu Numerationis per virgulas libri duo*. The ten rods this time are used with another type of ruler, with three columns. The right-hand column shows the numbers from 1 to 9. The middle column shows the doubles of these numbers. In the left-hand column the nine squares are divided by a diagonal into two triangles, as for the rods used for the multiples; here the units are written under the diagonal line and the tens in the triangle above the line.

6.1. How They Are Used.

Let's look to see how we can obtain a square root by means of an example. Say we want to know the square root of 120419. The first thing we do is to break the number down into pairs, starting from the right:

12.04.19. We create a grid with as many columns as the pairs of numbers

| | | |
|----|----|----|
| 12 | | |
| | | |
| | 04 | |
| | | |
| | | 19 |
| | | |

and twice as many lines. We put the numbers into alternate rows and columns of the grid starting from left to right.

| | | | |
|---|----|----|----|
| | 12 | | |
| 3 | 9 | | |
| | | 04 | |
| | | | |
| | | | 19 |
| | | | |

| quadrati | | | |
|----------|---|----|---|
| 0 | 1 | 2 | 1 |
| 0 | 4 | 4 | 2 |
| 0 | 9 | 6 | 3 |
| 1 | 6 | 8 | 4 |
| 2 | 5 | 10 | 5 |
| 3 | 6 | 12 | 6 |
| 4 | 9 | 14 | 7 |
| 6 | 4 | 16 | 8 |
| 8 | 1 | 18 | 9 |

We start with the first row of the grid, where we find the number 12. We look at the numbers in the left-hand column of the ruler: 1, 4, 9, 16, We can stop before we go over 12 (we want to approximate the square root of 12 from below). In our case, we can stop at the third line, where we find 9 (the next, 16, is larger than 12). The 9 is then written below the 12, and the number 3, which corresponds to the number of the row, is written next to the 9, but outside the grid: it is the first digit of the answer we are looking for. On the same row of the ruler, in the middle column, we find the number 6. So we take Napier's bone numbered 6 and place it to the left of the ruler of squares: we will need it for the next part of the

calculation.

Now we go to the next row on our grid. In the first column we write the difference between the 12 and the 9 on the previous rows: 3, in this case. With the 04 we have already written down, we obtain the number 304.

| | | | |
|---|----|----|----|
| | 12 | | |
| 3 | 9 | | |
| | 3 | 04 | |
| 4 | 2 | 56 | |
| | | | 19 |

| | | | | |
|---|---|---|----|---|
| 0 | 0 | 0 | 2 | 1 |
| 1 | 0 | 4 | 4 | 2 |
| 1 | 8 | 9 | 6 | 3 |
| 2 | 4 | 6 | 8 | 4 |
| 3 | 0 | 5 | 10 | 5 |
| 3 | 6 | 6 | 12 | 6 |
| 4 | 2 | 9 | 14 | 7 |
| 4 | 8 | 4 | 16 | 8 |
| 5 | 4 | 1 | 18 | 9 |

Now we look at the numbers on Napier's bone numbered 6 and the left-hand column of the ruler. We need to read them remembering that numbers in the same diagonal strip should be added together. In our case, we obtain: 61, 124, 189, 256, 325, ... Here too we stop before we go over the number in the grid. In our case we stop at 256, because the next number, 325, is larger than the 304 in the grid. We write 256 below 304. 256 comes from the fourth row, so we write 4 to the left of the number outside the grid. It is the second digit of the answer. The middle column of the ruler, where on row 4 we find the number 8, tells

us now to take Napier's bone numbered 8 and to place it to the left of the ruler, moving the bone numbered 6.

| | | | |
|---|----|----|----|
| | 12 | | |
| 3 | 9 | | |
| | 3 | 04 | |
| 4 | 2 | 56 | |
| | | 48 | 19 |
| 7 | | 48 | 09 |

| | | | | |
|---|---|---|----|---|
| 0 | 0 | 0 | 2 | 1 |
| 1 | 0 | 4 | 4 | 2 |
| 1 | 8 | 9 | 6 | 3 |
| 2 | 4 | 6 | 8 | 4 |
| 3 | 0 | 5 | 10 | 5 |
| 3 | 6 | 6 | 12 | 6 |
| 4 | 2 | 9 | 14 | 7 |
| 4 | 8 | 4 | 16 | 8 |
| 5 | 4 | 1 | 18 | 9 |

We go to the next row, where we write the difference between 304 and 256, i.e. 48. With the 19 we had already written we obtain the number 4819.

Now we look at the numbers that appear on the three columns of the two bones, the 6 and the 8, and the left-hand column of the ruler. We obtain, by addition along the diagonals, 681, 1364, 2049, 2736, 3425, 4116, 4809, 4504, ... We stop at 4809 because the next number, 4504, is above the 4819 in the grid. We write it below the 4819. This identifies the seventh row so we write 7 next to 4809 outside the grid, this is the third digit of the result.

Now we have come to the end of the grid so the calculation is over.

We have found the square root of 120419 which is 347. The 10 which is obtained as the remainder, the difference between 4819 and 4809, tells us that 120419 is not a perfect square. The square root we have found is a little smaller than the true square root. In fact the square of 347 is 120409.

6.2. Remarks

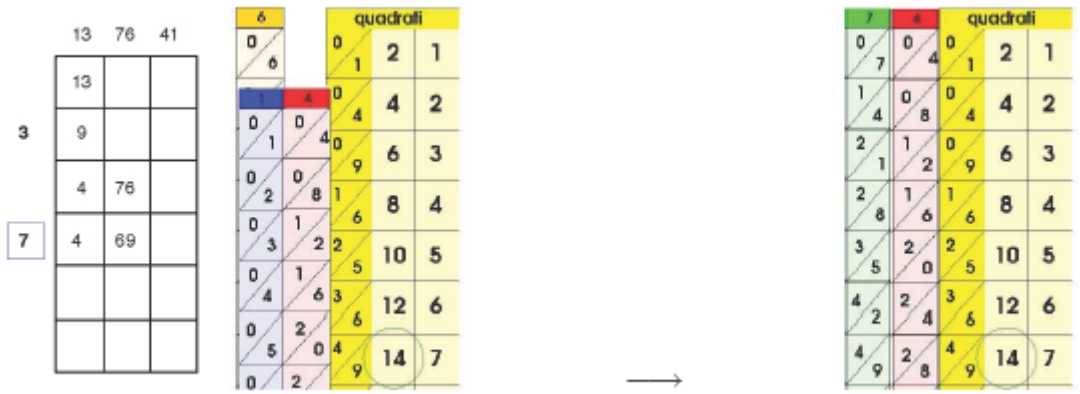
If we want to work out some decimals of the square roots, we could broaden the table with new pairs of 00 and then follow the instructions.

In the example given, following the instructions, the first two digits of the result, 3 and 4, indicated that the rods that needed to be taken were the 6 and the 8, corresponding to their doubles. If the digits in the result are higher than 4, their doubles will be a two-digit number and so will require two rods.

If the first digit of the result was 7 and its double is 14, and we would make 14 with two rods,

placing the 4 next to the ruler and rod numbered 1 to the left, and then we would proceed as before.

The square root of 137641 is 371. If we start the calculation procedure the first digit is 3 so we place the rod numbered 6 next to the ruler. Going ahead, the second digit turns out to be 7. So we move the 6 rod one box to the left and put the 14 rod. The 4 goes next to the ruler and the 1 rod should now go on the left but there is already a rod there, numbered 6 so instead we put the rod numbered $6 + 1, 7$, i.e. the sum of the two rods that should be in that position.



7. Workshop Guidelines

7.1. Before starting the workshop

The **different** forms of calculating rods presented here are suitable for various levels.

For all ages (8 - 16) they will be a pleasant discovery, making calculations simpler, as Napier wished.

If a school wants to include Napier's bones in the teaching programme, they can be taught alongside multiplication tables, which can be remembered better with the aid of Napier's bones instead of the Pythagorean table. Starting with the multiplication algorithm, the rods can be used to show combinations and how they are used, perhaps ending up with multiplication by **jealousie** and the normal procedure. Vice versa, the lesson could start with the normal procedure and go on to reveal the hidden transitions.

The variant invented by Genaille and Lucas seems rather magical. For the younger children it will be useful to learn how to use the rods, whilst the eldest will probably be interested in working out how the rods are constructed.

The same applies to the rods of the *Promptuarium*: in addition to learning how to use them it will be interesting to understand how they work, giving the classes an opportunity to see how decimal orders work which are hidden by positional representation and to understand the distribution property of the sum in relation to the product.

Finding the square root may be the opportunity to think about the squares of binomials and trinomials, etc.

7.2. Contents of the Presentations of the Workshop

In addition to the calculation rods, the material for the workshops includes a CDROM with some presentations to be projected during the workshops. These include slides in PowerPoint which contain images and brief comments explaining the exercises to be carried out. Here, we give a brief description and indication of the classes **for which** the workshops are suitable. Obviously, this is just a suggestion: the teacher is in a better position to choose the most suitable activities for his or her class.

The examples and presentations refer to a one-hour workshop. The teacher may give out further exercises according to the age and skills of the class.

Multiplying by a Single-Digit Number

From the third grade of Primary **school**.

The first slides introduce Napier's bones and the ruler. By means of a simple example, 45×3 , an explanation is given of how to use the rods to multiply a multi-digit number by a single-digit number. This is followed by examples in which the class is asked to multiply a number with three or four

digits by a single-digit number, with and without carry-over.

The **Genaille-Lucas** rods are presented. After introducing the ten rods and the ruler, an example is given repeating the calculation carried out using Napier's bones, multiplying a number with several digits by a single-digit number. This is followed by several multiplications, **with** the answers. The first factor **consists of** two, three or four digits.

Multiplying by Multi-Digit Numbers

From the sixth grade of Secondary **school**.

The file begins with the explanation of multiplication by **jealousie**, the origin of Napier's bones. After one example of multiplying with a single-digit number, which serves to memorize the method, some multiplications are carried out using a double-digit multiplication, first multiplying by **jealousie** and then doing the same calculation using the rods. This is followed by the **Genaille-Lucas** rods that are used for multi-digit calculations.

Divisions

There are two presentations, one for Primary school (divisioni1.ppt) and one for Secondary school (divisioni2.ppt). The first includes an introduction of the Genaille - Lucas dividing rods, and a few examples of division with remainders using two, three and four digits. The second presentation shows how to calculate decimals.

Promptuarium

From the fifth grade of Primary **school**.

This presentation refers to multiplication according to Napier's *Promptuarium*.

The first slides present **the two** types of rods. Multiplication by overlapping the rods is explained. The example uses a three-digit number multiplied by a two-digit number. This is followed by some examples of multiplications of three-digit numbers by two or three-digit numbers.

After illustrating the functioning of the rods the class can be asked to think about how they are constructed.

Square Roots

Third class of secondary school and high **school**.

The first slides illustrate the ruler to be used with the ten Napier's bones. **The square** roots of 1225 and 530 are calculated using Napier's bones and the ruler. At the end of the procedure a calculation is made to see if the starting number is a perfect square and what is **its** square root (exact or approximate).

Various examples and exercises follow in which the class is asked to calculate **square roots without decimals**.

The last example shows how to calculate decimals.

Contents

| | |
|--|----|
| Introduction | 3 |
| 1. Historical Remarks | 4 |
| 2. Multiplication Using Napier's Bones | 5 |
| 2.1. How They Are Used | 5 |
| 2.1.1. Multiplying by Single-Digit Numbers | 5 |
| 2.1.2. Multiplications with Carry-overs. | 7 |
| 2.1.3. Multiplying by Multi-Digit numbers | 7 |
| 2.2. How They Are Constructed. | 8 |
| 2.3. Remarks. | 8 |
| 3. Multiplication Using the GenailleLucas Rods | 9 |
| 3.1. How They Are Used | 10 |
| 3.1.1. Multiplying by Single-Digit Numbers | 10 |
| 3.1.2. Multiplying by Multi-Digit numbers | 11 |
| 3.2. How They Are Constructed. | 11 |
| 4. Divisions Using the GenailleLucas Rods | 13 |
| 4.1. How They Are Used | 14 |
| 4.1.1. Dividing by Single-Digit Numbers | 14 |
| 4.1.2. Decimal Numbers | 14 |
| 4.2. Remarks | 15 |
| 4.3. How They Are Constructed | 15 |
| 5. Multiplication Using Napier's <i>Promptuarium</i> | 17 |
| 5.1. How They Are Used | 18 |
| 5.2. Remarks | 18 |
| 5.3. How They Are Constructed | 18 |
| 6. Obtaining Square Roots Using Napier's Bones | 20 |
| 6.1. How They Are Used | 20 |
| 6.2. Remarks | 21 |
| 7. Workshop Guidelines | 23 |
| 7.1. Before Starting the Workshop | 23 |
| 7.2. Contents of the Presentations of the Workshop | 23 |

