On the Relation between a Taut Manifold X Modulo Δ_X and Hyperbolic Modulo Δ_X Which Is Contained in a Hypersurface of X

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Abstract

Let *X* be a n-dimensional Stein (connected complex) manifold or a compact one whose universal covering is a domain in \mathbb{C}^n or a Stein manifold. Let Δ_X be the degeneration locus of Kobayashi pseudodistance of *X* which is contained in a hypersurface *S* of *X*. Then *X* is hyperbolic modulo *S* and taut modulo *S*.

Keywords: compact or Stein manifold, taut mod Δ , degeneration locus of Kobayashi pseudodistance

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1. Introduction

In Kiernan and Kobayashi (1973) the Problem 1 was raised to determine the relationship between "taut mod Δ " and "complete hyperbolic mod Δ " and the Problem 2 asked that "hyperbolically imbedded mod Δ " imply "tautly imbedded mod Δ ".

When $\Delta \neq \emptyset$, these problems are not solved yet. In 1997 we solved the Problem 2 in a special case when $\Delta \neq \emptyset$ (Theorem and Remark in (Adachi, 1997)).

In this note we solve the Problem 1 in a case when $\Delta \neq \emptyset$ and contained in a hypersurface S of X where X is compact or open adding some conditions (Theorem 4.1 and 4.4).

For the fundamental concepts and basic properties of the Kobyashi hyperbolic geometry, see (Kobayashi,1998) for example.

2. Preliminary (1)

We denote the open unit disk in the complex plane **C** by Δ and the polydisk $\Delta \times \cdots \times \Delta$ in **C**^k by Δ^k .

Let *X* be a complex manifold and Δ_X be the degeneration locus of the Kobayashi pseudodistance on *X* (see Definition 1.1 in (Adachi,2007)). Then *X* is hyperbolic modulo Δ_X and if $\Delta_X \subset S$, *X* is hyperbolic modulo *S*.

Let $F = \{f_j\}_{j=1,2,\dots} \subset Hol(\Delta^k, X)$ with compact open topology.

Definition 2.1 We define the cluster set F(a : X) of F at a point a of Δ^k by $F(a : X) = \bigcap_{\varepsilon > 0} \bigcap_{N=1}^{\infty} \overline{\bigcup_{j \ge N} f_j(U_{\varepsilon}(a))}$ where $U_{\varepsilon}(a) = \{z \in \Delta^k; ||z - a|| < \varepsilon\}$. And $F(\Delta^k r; X) = \bigcup_{a \in \Delta^k} F(a; X)$.

If X is compact, following proposition is easy to see because F(a: X) is closed set.

Proposition 2.2 When X is compact, the set F(a: X) is compact.

Definition 2.3 A sequence $F = \{f_j\} \subset Hol(\Delta^k, X)$ is normal at $a \in \Delta^k$ if there exists a neighborhood U of a in Δ^k such that every subsequence of F has a convergent subsequence in Hol(U, X).

Clearly, we have

Proposition 2.4 *If the cluster set* F(a: X) *of a sequence* F *in* $Hol(\Delta^k, X)$ *is contained in a coordinate neighborhood of* X*, then* F *is normal at a.*

Proposition 2.5 Let $a, a_i \in \Delta^k$ with $a_i \to a$ and $f_i(a_i) \to p \notin \Delta_X$. Then F is normal at a.

Proof. The following proof is almost the same of Proposition 2.4 in (Adachi, 1997). Since Δ_X is a closed set from

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Proposition 1.5 in (Adachi, 2007), there exists a closed small neighborhood V of p in X which is biholomorphic to a closed ball such that $V \cap \Delta_X = \emptyset$. Then $d_X(p, \partial V) = \varepsilon$ where d_X is a Kobayashi pseudodistance of X and ε is a sufficiently small positive constant. If we set $U = \{z; d_{\Delta^k}(a, z) < \frac{\varepsilon}{2}\}$, it is relatively compact in Δ^k . We shall prove that $f_i(U) \subset V$ for $j \ge N$ where N is a sufficiently large integer. Then proposition is proved by Proposition 2.4.

If it is not true, we may assume that $f_{j_{\lambda}}(b_{j_{\lambda}}) \in \partial V$ where $b_{j\lambda} \in U, b_{j\lambda} \to b \in \overline{U}$ and $f_{j_{\lambda}}(b_{j_{\lambda}}) \to q \in \partial V$ since $f_{i}(a_{j}) \in V$ for sufficiently large j. This is absurd, because

$$egin{aligned} d_X(p,q) &= \lim_{j_\lambda o \infty} d_X(f_{j_\lambda}(a_{j_\lambda}),f_{j_\lambda}(b_{j_\lambda})) \ & \leq \lim_{j_\lambda o \infty} d_{\Delta^k}(a_{j_\lambda},b_{j_\lambda}) = d_{\Delta^k}(a,b) \leq rac{arepsilon}{2}. \end{aligned}$$

Corollary 2.6 If $F(a: X) \ni p \notin \Delta_X$, then F has a subsequence which converges in a neighborhood U of a in Hol(U, X).

Definition 2.7 A manifold *X* is taut modulo a closed set *S* in *X* if for each positive integer *k* and each sequence $\{f_i\} \subset Hol(\Delta^k, X)$ we have one of the following:

(a) $\{f_i\}$ has a subsequence which converges in $Hol(\Delta^k, X)$;

(b) for each compact set $K \subset \Delta^k$ and each compact set $L \subset X - S$, there exists an integer N such that $f_j(K) \cap L = \emptyset$ for $j \ge N$, that is, $F(\Delta^k: X) \subset S$ where $F = \{f_j\}$.

Corollary 2.8 (of Proposition 2.5) If $\Delta_X = \emptyset$, then F has a subsequence which converges in $Hol(\Delta^k, X)$ and consequently X is taut.

3. Preliminary (2)

Let *X* and Δ_X be the same in the section 1 and *S* be a hypersurface of *X* such as $\Delta_X \subset S$.

Lemma 3.1 Let $F = \{f_j\} \subset Hol(\Delta^k, X)$ and $D \neq \emptyset$ be a convergence open set of F in Δ^k in $Hol(\Delta^k, X)$ and $f_j \to f \in Hol(D, X)$. Then $F(a: X) \subset \Delta_X$ for every $a \in \partial D \cap \Delta^k$.

Proof. The following proof is almost the same of Lemma 3.1 in (Adachi,1997). We prove the lemma in 2 steps. Let $a \in \partial D \cap \Delta^k$ and $F(a; X) = P \cup S, P \cap \Delta_X = \emptyset, S \subset \Delta_X$.

(1) We prove that $P = \{p_0\}$ if $P \neq \emptyset$.

We assume that $P \ni p_1, p_2$ with $p_1 \neq p_2$. From Corollary 2.6, there is a neighborhood $U(a) \subset \Delta^k$ such that f_{j_λ} coverges to f_λ and f_{j_μ} converges to f_μ in Hol(U(a), X) with $f_\lambda(a) = p_1$ and $f_{\mu}(a) = p_2$. Since f_{j_λ} and f_{j_μ} coincide with f on $D \cap U(a)$, f is holomorphic on U(a) and $f(a) = p_1 = p_2$. This is absurd.

(2) We prove that $P = \emptyset$.

If $P \neq \emptyset$, there is a neighborhood U(a) of a in Δ^k where $a \in \partial D \cap \Delta^k$, $f \in Hol(U(a), X)$ and $f(a) = p_0$ from the proof of (1). And there are $a_j \in D$ such that $a_j \to a$ and $f(a_j) \to p_0$ and $b_{j_\lambda} \in \Delta^k$ such that $b_{j_\lambda} \to a$ and $f_{j_\lambda}(b_{j_\lambda}) \to q \in S$.

Let *V* be a sufficiently small neighborhood of p_0 such that $\overline{V} \cap \Delta_X = \emptyset$. Then there are $c_{j_{\lambda_\mu}} \to a$ and $f_{j_{\lambda_\mu}}(c_{j_{\lambda_\mu}}) \to r \in \partial V$.

From this situation, $F(a; X) \ni r \in \partial V$, $r \neq p_0$ and $r \notin \Delta_X$. This contradicts the claim of (1).

Lemma 3.2 Let D and f be the same of above Lemma 3.1. We assume that $\Delta_X \subset S = \{g = 0; g \in Mer(X) \text{ if } X \text{ is compact and } g \in O(X) \text{ if } X \text{ is Stein } \}$. Then one of the following two cases occurs.

(1) The set $E = D^c \cap \Delta^k$ is contained in an analytic subset Σ of Δ^k .

(2)
$$F(\Delta^k: X) \subset S$$
.

Proof. We set $\Phi = g \circ f$. Then Φ is a meromorphic or holomorphic function on D. Let P be the pole divisor of Φ where $P = \emptyset$ when X is Stein. From Lemma 3.1, $\Phi(z) \to 0$ when $z \in D$ and $z \to \partial D \cap \Delta^k$.

If we set $\Phi \equiv 0$ on E, Φ is continuous on $\Delta^k - P$ and Φ is holomorphic on D - P. From Rado's theorem Φ is holomorphic on $\Delta^k - P$. Therefore $\Phi \equiv 0$ on $\Delta^k - P$ or $\Phi \neq 0$ and $E \subset {\Phi = 0}$.

The former case, we shall prove that the case (2) in this lemma occurs. If there is a point $a \in D \cap P$ and $F(a: X) \ni q \notin \Delta_X$, there is a subsequence $\{f_{j_A}\}$ which converges to $f \in Hol(U(a), X)$ from Corollary 2.6 and f(a) = q.

This is a contradiction to the fact that there are $a_j \notin P, a_j \to a, f(a_j) \to f(a) \in S$ because $f(a_j) \in S$. Then $F(E: X) \subset S$ and $F(D) \subset S$. So $F(\Delta^k: X) \subset S$.

The latter case, $\Sigma = \{\Phi = 0\}$ is an analytic hypersurface of $\Delta^k - P$ and $\Phi(z) \to 0$ when $z \in D$ and $z \to \partial D \cap \Delta^k$, then $E \subset \Sigma$. And since $F(a; X) \subset \Delta_X$ when $a \in D \cap P$ from the proof of above, $P \subset \Sigma$ and Σ is an analytic subset of Δ^k .

4. Conclusion

Theorem 4.1 Let X be a compact manifold whose universal covering \tilde{X} is a domain in \mathbb{C}^n or a Stein manifold such as $\pi : \tilde{X} \to X$ and $\Delta_X \subset S = \{g = 0; g \in Mer(X)\}$. Let $F = \{f_j\} \subset Hol(\Delta^k, X)$. Then if $F(\Delta^k: X) \not\subset S$, F has a subsequence $\{f_{i_j}\}$ converges in $Hol(\Delta^k, X)$, that is, X is taut modulo S.

Proof. By the assumption there is a point $p \in F(\Delta^k: X) - \Delta_X$. From Corollary 2.6, there is a subsequence $\{f_{j_\lambda}\}$ and the largest open set $D \neq \emptyset$ in Δ^k such that $\{f_{j_\lambda}\}$ converges to f in Hol(D, X), that is a convergence open set of $\{f_{j_\lambda}\}$. From Lemma 3.2, there is an analytic subset Σ in Δ^k such that f_{j_λ} converges to f in $Hol(\Delta^k - \Sigma, X)$.

When \tilde{X} is a domain in \mathbb{C}^n we fix a map π^{-1} and set $g_{j_\lambda} = \pi^{-1} \circ f_{j_\lambda}, g_{j_\lambda} \in Hol(\Delta^k, \mathbb{C}^n)$ because Δ^k is simply connected. Then, by well known theorem $\{g_{j_\lambda}\}$ converges in $Hol(\Delta^k, \mathbb{C}^n)$ and $\{f_{j_\lambda}\}$ converges also in Δ^k . When \tilde{X} is a Stein manifold, there is a biholomorphic map φ of \tilde{X} to a submanifold of \mathbb{C}^N where N is a sufficiently large integer and $\varphi \circ \pi^{-1} \circ f_{j_\lambda} \in Hol(\Delta^k, \mathbb{C}^N)$ and $\{f_{j_\lambda}\}$ converges in Δ^k by the same reason above. \Box

Problem 4.2 When is the universal covering of a compact manifold *X* a domain in \mathbb{C}^n or a Stein manifold? It is well known that there are simply connected projective manifolds such as *K*3 surfaces. When above question is true if we add the condition that Δ_X is contained in a hypersurface of *X*?

Problem 4.3 When is the set Δ_X of a compact manifold *X* contained in a hypersurface of *X*?

When X is a Stein manifold, there is a biholomorphic map of X to a submanifold of \mathbb{C}^N where N is a sufficiently large integer, following theorem is easy to see from the proof of Theorem 4.1.

Theorem 4.4 Let X be a Stein manifold and $\Delta_X \subset S = \{g = 0; g \in O(X)\}$. Then X is taut modulo S.

References

- Adachi, Y. (1997). On the relation between tautly imbedded space modulo an analytic subset S and hyperbolically imbedded space modulo S. Publ. RIMS, Kyoto Univ., 33, 385-392. http://dx.doi.org/10.2977/prims/1195145321
- Adachi, Y. (2007). On a Kobayashi hyperbolic manifold N modulo a closed subset Δ_N and its applications. *Kodai Math. J.*, *30*, 131-139. http://dx.doi.org/10.2996/kmj/1175287627
- Kiernan, P., & Kobayashi, S. (1973). Holomorphic mappings into projective space with lacunary hyperplanes. *Nagoya Math. J.*, 50, 199-216.
- Kobayashi, S. (1998). Hyperbolic complex spaces, Grundlehlen der mathmatishen Wisseunschaften 318. Heiderberg, Berlin: Springer-Verlag. http://dx.doi.org/10.1007/978-3-662-03582-5