

ON THE GEOMETRY OF SPHERES WITH POSITIVE CURVATURE

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ABSTRACT. For an n -dimensional complete connected Riemannian manifold M with sectional curvature $K_M \geq 1$ and diameter $\text{diam}(M) > \frac{\pi}{2}$, and a closed connected totally geodesic submanifold N of M , if there exist points $x \in N$ and $y \in M$ satisfying the distance $d(x, y) > \frac{\pi}{2}$, then N is homeomorphic to a sphere. We also give a counterexample in 2-dimensional case to the following problem: let M be an n -dimensional complete connected Riemannian manifold with $K_M \geq 1$ and $\text{rad}(M) > \frac{\pi}{2}$, whether does the “antipodal” map A of M restricted to a complete totally geodesic submanifold agree with the “antipodal” map of M ?

1. INTRODUCTION

Let M be an n -dimensional complete connected Riemannian manifold with sectional curvature $K_M \geq 1$. A lot of interesting results about M have been proven during the past years. In 1977 Grove and Shiohama [9] showed that M is homeomorphic to the n -dimensional sphere S^n , if the diameter of M $\text{diam}(M) > \frac{\pi}{2}$. In the paper Grove and Shiohama established critical point theory for distance functions on complete Riemannian manifolds, which serves as an very important tool in Riemannian Geometry. One can find some of them, e.g., in [1], [4], [6], [8], [9], [12].

Recall that for a compact metric space (X, d) , the radius of X at a point $x \in X$ is defined as $\text{rad}_X(x) = \max_{y \in X} d(x, y)$, and the radius of X is given by $\text{rad}(X) = \min_{x \in X} \text{rad}(x)$, which was invented in [10]. In 2002 Xia[12] showed the following result.

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Theorem 1.1. *Let M be an n -dimensional complete connected Riemannian manifold with $K_M \geq 1$ and $\text{rad}(M) > \frac{\pi}{2}$. Suppose that N is a k -dimensional closed connected totally geodesic submanifold. Then N is homeomorphic to a k -dimensional Euclidean sphere S^k .*

In Xia [12], he asked whether the condition that $\text{rad}(M) > \frac{\pi}{2}$ could be weakened to $\text{diam}(M) > \frac{\pi}{2}$.

First we defined a set

$$B := \{p \in M; \exists q \in M, \text{ such that } d(q, p) > \frac{\pi}{2}\}.$$

Now we state our first theorem as follows:

Theorem 1.2. *Let M be an n -dimensional complete connected Riemannian manifold with $K_M \geq 1$ and $\text{diam}(M) > \frac{\pi}{2}$. Suppose that N is a k -dimensional closed connected totally geodesic submanifold and $N \cap B \neq \emptyset$. Then, for any $x \in N \cap B$, $\text{rad}_N(x) \geq \text{rad}_M(x)$. Furthermore, N is homeomorphic to a k -dimensional Euclidean sphere S^k .*

This theorem partially answers Xia's question and generalizes the theorem 1.1. As a direct consequence of Theorem 1.2, we have the following corollary, which was obtained by Wang[11].

Corollary 1.3. *Let M be an n -dimensional complete connected Riemannian manifold with $K_M \geq 1$ and $\text{rad}(M) > \frac{\pi}{2}$. Suppose that N is a k -dimensional closed connected totally geodesic submanifold, then $\text{rad}(N) \geq \text{rad}(M)$.*

Recall that the proof of Corollary 1.3 relies on the fact that the ‘‘antipodal’’ map A is surjective, where the map $A: M \rightarrow M$ is defined as follows, for any $x \in M$, $A(x)$ is a point in M that is at maximal distance from x . It is not difficult to prove that A is well defined under the conditions of Corollary 1.3, i.e., for any $x \in M$, there is a unique point $A(x)$ in M such that $A(x)$ is at maximal distance from x (see Lemma 2.2); and it is also not hard to show that A is continuous and surjective (cf.[7], [12]). However under the conditions of Theorem 1.2, the map A may not be well defined there, in the proof of Theorem 1.2 we use the first variations of energy. In other words, we give new proofs of Theorem 1.1 and Corollary 1.3.

Let N be a complete totally geodesic submanifold of M . If we assume that $\text{rad}(M) > \frac{\pi}{2}$, by Theorem 1.2 (or Corollary 1.3), $\text{rad}(N) > \frac{\pi}{2}$. Hence, the ‘‘antipodal’’ map A is well defined in N . The second result of the paper is to give a counterexample in 2-dimensional case to the problem which was asked by Wang in [11]. The problem is stated as follows:

Problem 1.4. *Let M be an n -dimensional complete connected Riemannian manifold with $K_M \geq 1$ and $\text{rad}(M) > \frac{\pi}{2}$. Does the “antipodal” map A of M restricted to a complete totally geodesic submanifold agree with that of M ?*

The proof of Theorem 1.2 frequently utilizes the Toponogov comparison theorem which one can refer to [5]. In Section 2 we will prove Theorem 1.2. In Section 3 we will give the counterexample to Problem 1.4.

2. PROOF OF THEOREM 1.2

Before proving Theorem 1.2, we first give two elementary lemmas:

Lemma 2.1. *Let M be a complete Riemannian manifold and let $N \subset M$ be a closed submanifold of M . Let $p \in M$ and $p \notin N$, and let $d(p, N)$ be the distance from p to N . Then there exists a point $q \in N$ such that $d(p, q) = d(p, N)$ and that any minimizing geodesic connecting p to q is orthogonal to N .*

PROOF. the existence of q can be obtained by the compactness of N . The other assertion can be very easily obtained by using formula for the first variation of the energy of a curve (cf.[3], pp.195). \square

Lemma 2.2. *Let M be a complete Riemannian manifold with sectional curvature $K_M \geq 1$ and $p \in M$. If there exists a point $q \in M$ so that $d(p, q) > \pi/2$, then there exists a unique point $A(p)$ which is at maximal distance from p .*

PROOF. The existence obviously follows from the compactness of M . Next we show the uniqueness. If not, we let q_1 and q_2 be two different points which are at maximal distance from p , then we have

$$(2.1) \quad \pi \geq d(p, q_1) = d(p, q_2) > \pi/2.$$

The left equality follows from the well known Bonnet-Myer Theorem(cf.[3]). By the well known Berger Lemma (cf.[3]), we know that q_1 and q_2 are both critical points to p . Taking a minimal geodesic γ from q_1 to q_2 , then there exists a minimizing geodesic σ from q_1 to p such that $\angle(\gamma'(0), \sigma'(0)) \leq \frac{\pi}{2}$. Applying the Toponogov comparison theorem to the hinge (γ, σ) , we obtain

$$(2.2) \quad \begin{aligned} \cos d(p, q_2) &\geq \cos d(p, q_1) \cos d(q_1, q_2) + \\ &\quad \sin d(p, q_1) \sin d(q_1, q_2) \cdot \cos \angle(\gamma'(0), \sigma'(0)) \\ &\geq \cos d(p, q_1) \cos d(q_1, q_2). \end{aligned}$$

But we already have $d(p, q_1) = d(p, q_2) > \pi/2$, this contradicts (2.2). Hence $A(p)$ is the unique point farthest from p . \square

Now we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Let $x \in N \cap B$ and $A(x)$ is the point farthest from x . Since $\text{diam}(M) > \frac{\pi}{2}$, by Lemma 2.2 we know $A(x)$ is unique. If $A(x) \in N$, since any minimizing geodesic from x to $A(x)$ in N is also a geodesic from x to $A(x)$ in M , $\text{rad}(x)$ in N is greater than or equal to $\text{rad}(x)$ in M . Hence we only need to discuss the case that $A(x) \notin N$.

Because of compactness of N , we can take a point $y \in N$ such that

$$(2.3) \quad d(A(x), y) = \min_{z \in N} d(A(x), z).$$

case **A**: $y = x$.

Taking a point $m \in N$ different from x , let γ be a minimizing geodesic from x to $A(x)$ in M and σ be a minimizing geodesic from x to m in N . Because N is a complete totally geodesic submanifold of M , the sectional curvature $K_N \geq 1$ and σ is also a geodesic from x to m in M . By the classical Bonnet-Myer theorem one has $d^N(x, m) \leq \pi$. Under such conditions we can use the Toponogov comparison theorem to the hinge (σ, γ) . First from Lemma 2.1 we know that the angle between σ and γ is $\pi/2$, so we have

$$(2.4) \quad \begin{aligned} \cos d(m, A(x)) &\geq \cos d(x, A(x)) \cos d^N(x, m) + \\ &\quad \sin d(x, A(x)) \sin d^N(x, m) \cdot \cos \angle(\gamma'(0), \sigma'(0)) \\ &= \cos d(x, A(x)) \cos d^N(x, m). \end{aligned}$$

Since $y = x$, $d(m, A(x)) \geq d(x, A(x))$ follows from the selection of y .

Naturally we have $d(m, A(x)) > \pi/2$, hence

$$(2.5) \quad \cos d(x, A(x)) \cos d^N(x, m) < 0.$$

Since $d(x, A(x)) > \pi/2$, we have $\cos d^N(x, m) > 0$. From (2.4) we obtain

$$(2.6) \quad \cos d(m, A(x)) > \cos d(x, A(x)).$$

By the monotonicity of cosine function, we have $d(m, A(x)) < d(x, A(x))$, which contradicts the picking of y . Hence the case **A** does not happen.

case **B**: $y \neq x$.

Let γ be a minimizing geodesic from y to $A(x)$ in M and σ be a minimizing geodesic from y to x in N . Because N is a complete totally geodesic submanifold of M , the sectional curvature $K_N \geq 1$ and σ is also a geodesic from y to x in M . By the classical Bonnet-Myer theorem one has $d^N(x, y) \leq \pi$. Under such conditions we can use the Toponogov comparison theorem to the hinge (σ, γ) . First from Lemma 2.1 we know that the angle between σ and γ is $\pi/2$,

hence we obtain

$$(2.7) \quad \begin{aligned} \cos d(x, A(x)) &\geq \cos d(y, A(x)) \cos d^N(x, y) + \\ &\quad \sin d(y, A(x)) \sin d^N(x, y) \cdot \cos \angle(\gamma'(0), \sigma'(0)) \\ &= \cos d(y, A(x)) \cos d^N(x, y). \end{aligned}$$

Since $d(x, A(x)) > \pi/2$, we have

$$(2.8) \quad \cos d(y, A(x)) \cos d^N(x, y) < 0.$$

Now we discuss the problem in two cases.

case **B1**: $d^N(y, x) < \pi/2$.

From (2.8), we know $d(y, A(x)) > \pi/2$. Returning to (2.7), we have

$$(2.9) \quad \cos d(x, A(x)) > \cos d(y, A(x)).$$

By the monotonicity of cosine function, we have $d(x, A(x)) < d(y, A(x))$, which is a contradiction to the picking of y . Hence the case **B1** does not happen.

case **B2**: $d^N(y, x) > \pi/2$.

From (2.8), we know $d(y, A(x)) < \pi/2$. Returning to (2.7), we have

$$(2.10) \quad \cos d(x, A(x)) > \cos d^N(y, x).$$

By the monotonicity of cosine function, we have $d(x, A(x)) < d^N(y, x)$.

Since $\text{rad}_M(x) = d(x, A(x))$,

$$(2.11) \quad \text{rad}_N(x) > \text{rad}_M(x).$$

From the two cases above we obtain $\text{rad}_N(x) > \text{rad}_M(x)$. Since $\text{rad}_M(x) > \pi/2$, using the Grove-Shiohama diameter sphere theorem (cf.[9]) we know that N is homeomorphic to a k -dimensional Euclidean sphere S^k . \square

Remark 2.1. If $\dim M = 3$, the answer to Xia's question is affirmative. Indeed, we only need to consider N of dimension 2. By The Syngé Theorem(cf. [3]), N is homeomorphic to RP^2 if N is not simply connected. However M is homeomorphic to S^3 . By the classical topology theorem we know RP^2 can not be embedded into S^3 , so N must be simply connected. Hence, N is homeomorphic to S^2 .

3. COUNTEREXAMPLE

Before giving the counterexample, we describe our idea roughly. Let $S^2(1)$ be the standard 2-dimensional unit sphere in \mathbb{R}^3 , p be the north pole and q be the south pole. We can get a new surface M by giving a very small perturbation around the point $(1,0,0)$ of $S^2(1)$ such that the curvature does not change a lot and the length of the curve $M \cap \{x \geq 0, y = 0\}$ is less than π . Let N be the big circle $M \cap \{x = 0\}$, as in the following figure.

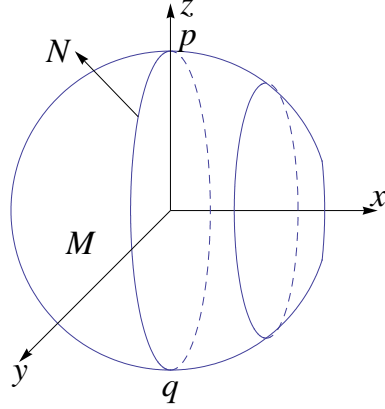


FIGURE 1

We endow the induced metric from \mathbb{R}^3 on M . It is obvious that N is a complete totally geodesic submanifold of M and q is the farthest point from p in N . But from the Berger lemma we can prove that q is not the farthest point from p in M , which gives a negative answer to the Problem 1.4. Next let us explicitly discuss the problem.

First let us recall the definition of Gromov-Hausdorff distance. Let $X, Y, X_i, i = 1, 2, 3, \dots$ be compact metric spaces. If X, Y are isometrically embedded in Z , the classical Hausdorff distance $d_H^Z(X, Y)$ satisfies

$$d_H^Z(X, Y) < \epsilon \Leftrightarrow Y \subset B(X, \epsilon), X \subset B(Y, \epsilon),$$

where $B(X, \epsilon) = \{z \in Z \mid d(z, X) < \epsilon\}$. The Gromov-Hausdorff distance d_{GH} satisfies

$$d_{GH}(X, Y) < \epsilon \Leftrightarrow d_H^Z(X, Y) < \epsilon, \text{ for some metric on } Z = X \amalg Y \text{ extending the ones on } X, Y.$$

Similarly, Gromov-Hausdorff convergence is characterized by

$$X = \lim X_i \Leftrightarrow \begin{array}{l} \text{the metrics on } X, X_i \text{ extend to a metric} \\ \text{on } Z = X \amalg X_i \text{ and } d_H^Z(X, X_i) \rightarrow 0. \end{array}$$

Now we construct a curve γ in xy -plane as follows

$$\gamma(\theta) := \begin{cases} (\cos \theta, \sin \theta) & 1 \leq \theta \leq \pi, \\ (1 + h_\lambda(\theta))(\cos \theta, \sin \theta) & 0 \leq \theta < 1, \end{cases}$$

where $h_\lambda(\theta) = -e^{\frac{\lambda}{\theta^2-1}}$, $\lambda > 0$.

Rotating γ around the x -axis we can get a closed smooth surface M_λ which is explicitly represented by the following map

$$F : [0, \pi] \times [0, 2\pi) \rightarrow \mathbb{R}^3,$$

$$(\theta, \varphi) \mapsto \begin{cases} (\cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi) & 1 \leq \theta \leq \pi, \varphi \in [0, 2\pi), \\ (1 + h_\lambda(\theta))(\cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi) & 0 \leq \theta < 1, \varphi \in [0, 2\pi). \end{cases}$$

It is not hard to see that M_λ is a smooth surface. We consider the Riemannian metric g_λ on M_λ which is given by the induced metric from \mathbb{R}^3 . Obviously the sectional curvature (i.e. Gauss curvature) $K_{(M_\lambda, g_\lambda)}(F(\theta, \varphi)) = 1$ when $\theta \geq 1$. Now let us estimate the curvature $K_{(M_\lambda, g_\lambda)}(F(\theta, \varphi))$ when $0 \leq \theta < 1$. For the sake of simplicity we replace $h_\lambda(\theta)$ by h_λ . First we know that the first and second fundamental forms of M_λ at $F(\theta, \varphi)$ are

$$\begin{aligned} I &= ((1 + h_\lambda)^2 + (h'_\lambda)^2)d\theta^2 + (1 + h_\lambda)^2 \sin^2 \theta d\varphi^2, \\ II &= ((1 + h_\lambda)^2 + (h'_\lambda)^2)^{-\frac{1}{2}} \{ -(1 + h_\lambda)^2 - 2(h'_\lambda)^2 + (1 + h_\lambda)h''_\lambda \} d\theta^2 \\ &\quad + (1 + h_\lambda) \sin \theta \{ -(1 + h_\lambda) \sin \theta + h'_\lambda \cos \theta \} d\varphi^2. \end{aligned}$$

By the standard computation(cf.[2]), the curvature $K_{(M_\lambda, g_\lambda)}(F(\theta, \varphi))$ equals to

$$\frac{-(1 + h_\lambda)^2 - 2(h'_\lambda)^2 + (1 + h_\lambda)h''_\lambda (h'_\lambda \cos \theta - (1 + h_\lambda) \sin \theta)}{((1 + h_\lambda)^2 + (h'_\lambda)^2)^2 (1 + h_\lambda) \sin \theta}.$$

Denote the expression above by $C(\lambda, \theta)$. It is easy to see that

$$h_\lambda^{(i)}(\theta) \rightarrow 0, \text{ as } \lambda \rightarrow +\infty \text{ (} i = 0, 1, 2\text{)}.$$

Hence,

$$C(\lambda, \theta) \rightarrow 1, \text{ as } \lambda \rightarrow +\infty.$$

That is,

$$K_{(M_\lambda, g_\lambda)}(F(\theta, \varphi)) \rightarrow 1, \text{ as } \lambda \rightarrow +\infty.$$

It is obvious that M_λ converge to the 2-dimensional standard unit sphere $S^2(1)$ in the sense of Gromov-Hausdorff distance. Since $\text{rad}(S^2(1)) = \pi$, we can fix a number λ_0 large enough such that

$$(3.12) \quad \begin{cases} \text{rad}(M_{\lambda_0}, g_{\lambda_0}) > \frac{3}{4}\pi \\ K_{(M_{\lambda_0}, g_{\lambda_0})} \geq \frac{4}{9}. \end{cases}$$

Considering the totally geodesic submanifold $N = M_{\lambda_0} \cap \{x = 0\}$. Choose two points $p = (0, 0, 1)$, $q = (0, 0, -1)$ on N . It is easy to see that q is the farthest point from p in N , that is $q = A(p)$ in N .

However, we claim that q is not the farthest point from p in M . To see this, firstly we can pick out a curve γ_0 from p to q , which coincides with $M|_{y=0, x \geq 0}$. The length of γ_0 is

$$\begin{aligned} L(\gamma_0) &= 2 \int_0^{\frac{\pi}{2}} |\gamma'(\theta)| d\theta = \pi - 2 + 2 \int_0^1 |\gamma'(\theta)| d\theta \\ &= \pi - 2 + 2 \int_0^1 \sqrt{(1 + h_{\lambda_0}(\theta))^2 + (h'_{\lambda_0}(\theta))^2} d\theta \\ &= \pi - 2 + 2 \int_0^1 \sqrt{1 - e^{\frac{\lambda_0}{\theta^2-1}} (2 - (1 + \frac{4\lambda_0^2\theta^2}{(\theta^2-1)^4})e^{\frac{\lambda_0}{\theta^2-1}})} d\theta. \end{aligned}$$

It is not hard to see that

$$2 - \left(1 + \frac{4\lambda_0^2\theta^2}{(\theta^2-1)^4}\right) e^{\frac{\lambda_0}{\theta^2-1}} > 0, \text{ when } \theta \in [0, 1) \text{ and } \lambda_0 \text{ is large enough.}$$

Hence

$$L(\gamma_0) < \pi - 2 + 2 \int_0^1 1 d\theta < \pi.$$

If q is the farthest point from p in M , by the well known Berger Lemma q is a critical point to p , that is, for any $v \in T_q M$, there is a minimizing geodesic γ from q to p such that the angle between $\gamma'(0)$ and v is less than or equal to $\frac{\pi}{2}$. Since the left half of M_{λ_0} (i.e., $M_{\lambda_0} \cap \{x \leq 0\}$) is just the same as that of $S^2(1)$, any geodesics in $M_{\lambda_0} \cap \{x \leq 0\}$ from p to q has length π . So they are not minimizing geodesics because we already have a curve γ_0 with the length less than π . We consider the geodesic γ_1 from q to p , which coincides with $M_{\lambda_0} \cap \{x \leq 0, y = 0\}$. From the argument above there doesn't exist any minimizing geodesic σ from q to p such that the angle between $\sigma'(0)$ and $\gamma_1'(0)$ is less than or equal to $\frac{\pi}{2}$. This contradicts that q is a critical point to p . Therefore q is not the farthest point from p in M .

Replacing the metric g_{λ_0} by $\frac{4}{9}g_{\lambda_0}$, From (3.11) we have

$$\begin{cases} \text{rad}(M_{\lambda_0}, \frac{4}{9}g_{\lambda_0}) > \frac{1}{2}\pi \\ K_{(M_{\lambda_0}, \frac{4}{9}g_{\lambda_0})} \geq 1. \end{cases}$$

After rescaling the metric, N is still a totally geodesic submanifold in M and p is still the farthest point from q in N , but we know that q is not the farthest point from p in M from the argument above. That is to say, the “antipodal” map A of M restricted to a complete totally geodesic submanifold may not agree with the “antipodal” map of M .

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