Turkish Journal of Analysis and Number Theory, 2016, Vol. 4, No. 4, 109-112
Available online at http://pubs.sciepub.com/tjant/4/4/4
©Science and Education Publishing
DOI:10.12691/tjant-4-4-4

A Nonlinear Extension of Fibonacci Sequence

M. Tamba*, Y.S. Valaulikar

Department of Mathematics, Goa University, Taleigaon Plateau, Goa, India
*Corresponding author: tamba@unigoa.ac.in

Abstract A new extension of Fibonacci sequence which yields a nonlinear second order recurrence relation is defined. Some identities and congruence properties for the new sequence are obtained.

Keywords: *fibonacci sequence, nonlinear recurrence relation, congruence properties*

Cite This Article: M. Tamba, and Y.S. Valaulikar, "A Nonlinear Extension of Fibonacci Sequence." *Turkish Journal of Analysis and Number Theory*, vol. 4, no. 4 (2016): 109-112. doi: 10.12691/tjant-4-4-4.

1. Introduction

T he well-known Fibonacci sequence F(n) is defined by

$$F(n+2)=F(n+1)+F(n),(n>0),F(0)=0,F(1)=1. \quad (1.1)$$

F(n) is called the nth Fibonacci number. The Fibonacci sequence has been extended in many ways and it has many interesting properties and applications in different fields ([5,6]). In recent years, Fibonacci numbers are also seen in many combinatorial problems ([1,3]).

In this note, we present yet another extension of Fibonacci sequence and call it Fibosenne sequence $\{P(n)\}$, defined by the relation

$$P(n)=2^{F(n)}-1 \quad n\ge 0 \qquad (1.2)$$

where $F(n)$ is the nth Fibonacci number. We call P(n) the n th Fibosenne number in view of its form like Mersenne number M_n [4]. It is clear that $P(n)=M_{F(n)}$. We shall establish various relations for $P(n)$ in line with those of $F(n)$. We shall also study some congruence properties of $P(n)$.

2. Fibosenne Sequence $\{P(n)\}$

For $n\ge 0$, let $P(n)$ denote the nth Fibosenne number. Then the following results follow immediately from definition (1.2) and from results on Fibonacci number $F(n)$ ([2,7]).

Proposition 2.1

1. For $n\ge 0$,

$$\log_2(1+P(n))=\frac{1}{\sqrt{5}}\left(\alpha^n-\beta^n\right),$$

where α,β are the roots of the equation

$$x^2=x+1.$$

2. For $n\ge 0$, $P(n)$ satisfies the nonlinear second order recurrence relation,

$$P(n+2)=P(n+1)+P(n)+P(n+1)P(n), \qquad (2.1)$$

with initial conditions $P(0)=0, P(1)=1$.

3. For $n\ge 1$,

$$P(-n)=\begin{cases} P(n), & if\ n\ is\ odd \\ \dfrac{-P(n)}{1+P(n)}, & if\ n\ is\ even. \end{cases}$$

Proof.

1. Follows from definition and Binet's formula for $F(n)$.
2. Follows from definition.
3. Follows from definition and

$$F(-n)=(-1)^{n+1}F(n).$$

The following properties can also be deduced easily .These properties are useful to find $P(n)$ using the earlier Fibosenne numbers.

Proposition 2.2 We have

1. $$(1+P(n+2))=\frac{(1+P(n))^3}{(1+P(n-2))}, \text{for } n\ge 2.$$
2. $$(1+P(n+1))=\frac{(1+P(n-2))}{(1+P(n))^2}, \text{for } n\ge 2.$$
3. For $m,n\ge 1$,

$$P(m+n)=\left\{[1+P(m)]^{F(n+1)}[1+P(n)]^{F(m-1)}\right\}-1.$$

4. For m, n ≥ 1,

5. $$P(mn)\equiv\begin{cases}\left\{\prod_{k=1}^{\frac{mn}{2}}[1+P(2k-1]\right\}-1, \\ if\ mn\ is\ even,\ mn\ge 2 \\ \left\{\prod_{k=1}^{\frac{mn-1}{2}}[1+P(2k)]\right\}-1, \\ if\ mn\ is\ odd, mn\ge 1.\end{cases}$$

Proof

1. Using definition of $P(n)$ and identity 18 for $F(n)$ given in ([2]), we get,

$$\begin{aligned}&\left(1+P(n+2)\right)\left(1+P(n)\right)\left(1+P(n-2)\right)\\&=2^{F(n+2)+F(n)+F(n-2)}\\&=2^{4F(n)}=\left(1+P(n)\right)^4.\end{aligned}$$

Hence the result follows.

2. Using (1), $(1+P(n))^2=\dfrac{\left(1+P(n+2)\right)\left(1+P(n-2)\right)}{1+P(n)}$.

By recurrence relation (2.1), we see that

$$\frac{1+P(n+2)}{1+P(n)}=1+P(n+1).$$

Combining 1. and 2. obtained in this proof gives desired result.

3. We have for $F(n)$ that

$$F(m+n)=F(m+1)F(n+1)-F(m-1)F(n-1).$$

(See [2] and [7]). Thus we arrive at the desired result.

4. Using the following identity and definition of $P(n)$ result follows.

$$\begin{aligned}F(mn)&=\sum_{k=1}^{mn/2}F(2k-1),\ mn\geq 2\ \mathit{and\ even}\\&=1+\sum_{k=0}^{mn-1}F(2k),\ mn\ \mathit{odd}.\end{aligned}$$

The following results are concerning divisibility of $P(n)$.

Proposition 2.3

1. For $n\geq 0$, $\gcd(P(n),P(n+1))$=1.
2. If $n|F(m), then\ M_n|P(m)$.
3. If $n|m, then\ P(n)|P(m)$.
4. If $p>2\ is\ a\ prime\ such\ that$

$$(p-1)\big|F(m),\ then\ p\big|P(m).$$

Proof.

1. $\gcd\big(P(0),P(1)\big)=\gcd(0,1)=1$. Suppose $n\geq 1$, $\gcd\big(P(n),P(n+1)\big)>1$. Then there is a prime p such that $p\ divides\ both$ $P(n)$ and $P(n+1)$. Hence by equation (2.1), p divides $P(n-1)$. Inductively, p divides $P(1)=1$, giving contradiction.

Proofs of (2) – *(4)* follows from the fact that, $n|m, then\ M_n|M_{m.}$

3. Some Congruence Properties of Fibosenne Sequence {P(n)}

In this section, we present some congruence properties of Fibosenne sequence. The following table gives Fibosenne numbers for $0\leq n\leq 12$.

A look at the Table 1 reveals that the last digit of P(n) follow the pattern 0113715, 113715, ... Given $m\geq 2$, we have $P(0)\equiv 0(mod\ m)$ and $P(1)\equiv 1(mod\ m)$. For $n\geq 2$, the congruence properties of P(n) can be found by using the recurrence relation (2.1). For $2\leq m\leq 10$, we show the cycles of $P(n)(mod\ m)$ in Table 2.

Table 1. $P(n)$, $0\leq n\leq 12$

n	P(n)
0	0
1	1
2	1
3	3
4	7
5	31
6	255
7	8191
8	2097151
9	17179869183
10	36028797018963967
11	618970019642690137449562111
12	22300745198530623141535718272648361505980415

Table 2. Cycles of P(n) (mod m), b(m), t(m), $2\leq m\leq 10$

m	P(n) (mod m)	Base length b(m)	Tail Period t(m)
2	(0) 1 1 1 1 1...	1	1
3	011 011 011 ...	0	3
4	(011) 3 3 3 3 3 3...	3	1
5	011321 011321 011321 ...	0	6
6	(0) 113 113 113 ...	1	3
7	01130331 01130331 ...	0	8
8	(0113) 7 7 7 7 7 7 ...	4	1
9	011374317614044671647341 ...	0	24
10	(0) 113715 113715 ...	1	6

Here we observe that P(n) modulo m show some pattern. After initial terms which we call "*Base*" there is periodic repetition of terms. We call it "*Tail*". This suggests that there is a pattern for congruence residues, which is different from that of Fibonacci sequence. For all sequences, which show such a pattern for congruence modulo m, we define the Base length as the number of terms in the base and denote it by b(m), the base length of the sequence modulo m. Similarly, we define the Tail Period as the minimum number of terms repeating in the tail of the sequence modulo m and denote it by t(m). For example when *m*=4, the sequence is 0, 1, 1,3,3,.... Here 0,1,1 is the base and 3,3,3,... the tail. There are three terms in the base and hence b(4)= 3 . In the tail number 3 repeats and so t(4)= 1. In the second column of Table 2, base is shown in brackets. If there are no initial terms forming base, then b(m) = 0 and the bracket is omitted. It is noteworthy that for m = 11, b(11) = 46 and t(11) =12.

From Table 2, for m = 10, the following result follows immediately.

Proposition 3.1. For $u\geq 0$,

$$P(6u+r)\equiv\begin{cases}1(\mathrm{mod}10),\ if\ r=1,2,5.\\3(\mathrm{mod}10),\ if\ r=3.\\5(\mathrm{mod}10),\ if\ r=6\\7(\mathrm{mod}10),\ if\ r=4\end{cases}$$

The following result also from Table 2.

Proposition 3.2

1. For $1\leq r\leq 4$ and $u\geq 0$,

$$P(4u+r)\equiv P(r)(\mathrm{mod}\,5)$$

2. For $1 \leq r \leq 3$ and $u \geq 0$,

$$P(3u+r) \equiv P(r)(\bmod 6).$$

For $n \geq 1$, let $[n]_F$ be the largest n_1 such that $F(n_1) \leq n$. For example, $[1]_F = 2, [2]_F = 4, [3]_F = 5, [4]_F = 5, \ldots$ etc.

Proposition 3.3 For $n \geq 1$, $P(k) \equiv 2^n - 1(mod 2^n)$, for $k \geq [n]_F$. In this case $b(2^n) = [n]_F$ and $t(2^n) = 1$.

Proof. For $k \geq [n]_F$.

$$\begin{aligned} P(k) &= 2^{F(k)} - 1, \\ &= 2^n \left(2^{F(k)-n}\right) - 1, F(k) - n \geq 0, \\ &\equiv -1\left(\bmod 2^n\right), \\ &\equiv 2^n - 1\left(\bmod 2^n\right). \end{aligned} \tag{3.1}$$

Given $m \geq 2$, considering the Fibonacci numbers $F(n)$ and the smallest residues $R_n(m)$ of the terms modulo m, it was observed ([9], Chap.VII) that the sequence $R_n(m)$ repeats after $\pi(m)$ term as are given in ([9]) which we tabulate here:

Table 3. Values of $\pi(m)$ for $2 \leq m \leq 30$

m		2	3	4	5	6	7	8	9	10
$\pi(m)$		3	8	6	20	24	16	12	24	60
m	11	12	13	14	15	16	17	18	19	20
$\pi(m)$	10	24	28	48	40	24	36	24	18	60
m	21	22	23	24	25	26	27	28	29	30
$\pi(m)$	16	30	48	24	100	84	72	48	14	120

$\pi(m)$ is called pisano period of $F(n)$ modulo m. From the definition of $\pi(m)$, it is clear that

Lemma 3.4 For $m \geq 2$ and $u \geq 0$,

$$F(\pi(m)u+r) \equiv F(r)(\bmod m), where\ 0 \leq r < \pi(m).$$

Using this lemma we prove the following:

Proposition 3.5 For $m \geq 2$ and $u \geq 0$, we have

$$F(\pi(m)u+r) \equiv F(r)(\bmod m), where\ 0 \leq r < \pi(m)$$

and $2^m \equiv 1(\bmod k)$, for some $k > 1$, then

$$P(\pi(m)u+r) \equiv P(r)(\bmod k).$$

Proof. We have

$$\begin{aligned} P(\pi(m)u+r) &= 2^{F(\pi(m)u+r)} - 1, \\ &= 2^{mj+F(r)} - 1, \text{ for some } j \geq 0, \\ &= (2^m)^j 2^{F(r)} - 1, \\ &\equiv 2^{F(r)} - 1(\bmod k). \end{aligned} \tag{3.2}$$

Notice that in Proposition 3.5, $2^m \equiv 1(mod\ k)$ suggest that k is odd. The case when k is power of 2 is dealt in Proposition 3.3. When k is even, but not power of 2, we have the following:

Proposition 3.6 For $\boldsymbol{m} \geq \mathbf{2}$ and $\boldsymbol{u} \geq \mathbf{0}$, if

$$F(\pi(m)u+r) \equiv F(r)(\bmod m),$$

where $0 \leq r < \pi(m)$ and $2^m \equiv 1(mod\ k)$, for some $k > 1$, then for $s \geq 1$, $P(\pi(m)u+r) \equiv N(r)(\bmod 2^s k)$, where $N(r)$ is independent of u. Here $b(2^s k) = [s]_F$.

Proof. We see for $F(\pi(m)u+r) - s \geq 0$ that

$$\begin{aligned} P(\pi(m)u+r) &= 2^{F(\pi(m)u+r)} - 1, \\ &= 2^{mj+F(r)} - 1, \text{ fors ome } j \geq 0, \\ &= 2^s \left(2^{mj+F(r)-s}\right) - 1 \\ &= 2^s(kq + r_1) - 1 \\ &= 2^s kq + 2^s r_1 - 1 \\ &\equiv 2^s r_1 - 1(\bmod 2^s k) \end{aligned} \tag{3.3}$$

where r_1 is the remainder when $2^{mj+F(r)-s}$ is divided by k.

We have the following corollaries.

Corollary 3.7. For $n \geq 0$, we have

$$P(3n+r) \equiv \begin{Bmatrix} 0(\bmod 3), if\ r = 0 \\ 1(\bmod 3), if\ r = 1, 2. \end{Bmatrix}$$

Proof. Follows from Proposition 3.5 as $\pi(2) = 3$ and taking $k = 3$.

Corollary 3.8. For $n \geq 0$,

$$P(6n+r) \equiv \begin{Bmatrix} 0(\bmod 15), if\ r = 0 \\ 1(\bmod 15), if\ r = 1, 2, 5 \\ 3(\bmod 15), if\ r = 3 \\ 7(\bmod 15), if\ r = 4 \end{Bmatrix}.$$

Proof. As $\pi(4) = 6, F(6u+r) \equiv F(r)(mod\ 4)$ so taking $k = 15$, Proposition 3.5, $P(6u+r) \equiv P(r)(mod 15)$.

When $r = 1, P(6u) \equiv 0(\text{mod } 15)$.

When $r = 0, P(6u+1) \equiv 0(\text{mod } 15)$.

Hence, by (1.2), for $r = 2, P(6u+2) \equiv 1(mod\ 15)$.

Similarly the congruence relations for $r \geq 3$ can be obtained recursively.

Corollary 3.9. For $n \geq 0$,

$$P(20n+r) \equiv \begin{Bmatrix} 0(\bmod 31), if\ r = 0, 5, 10, 15 \\ 1(\bmod 31), if\ r = 1, 2, 8, 19 \\ 3(\bmod 31), if\ r = 3, 14, 16, 17 \\ 7(\bmod 31), if\ r = 4, 6, 7, 13 \\ 15(\bmod 31), if\ r = 9, 11, 12, 18 \end{Bmatrix}$$

Proof. As $\pi(5) = 20$, taking $k = 31$, and using Proposition 3.5 and proceeding as in the proof of Corollary 3.8 the result follows.

We have the following results on $t(m)$, the tail period.

Proposition 3.10

1. If $m > 2$ is an odd integer,then $b(m) = 0$ and $t(m)|\pi(\phi(m))$, where $\phi(n)$ is Euler's ϕ -function. In particular, if $p > 2$ is prime, then $t(p)|\pi(p-1)$.
2. For $m \geq 1, t(2^m) = 1$.
3. Let $m \geq 2$ be an integer such that $2^m \equiv 1(\bmod k)$, then $t(k)|\pi(m)$.

Proof

1. $P(n) = 2^{F(n)} - 1$
$= 2^{\phi(m)+k+F(r)} - 1$

$\equiv 2^{F(r)} - 1 (\text{mod } m).$

Thus the pattern in $P(n)(\text{mod } m)$ repeats after $\pi(\phi(m))$ terms. So $\pi(\phi(m))$ must be a multiple of $t(m)$. Also since the cycle repeats right from the beginning, $b(m) = 0$.

2. For $F(k) \geq m$,

$$P(k) = 2^{F(k)} - 1 \equiv -1 (\text{mod } 2^{m)}).$$

3. $P(n) = 2^{F(n)} - 1$
$= 2^{ms+F(r)} - 1$
$\equiv 2^{F(r)} - 1 (mod\ k).$

Using result (3) in Proposition 3.10, we have the following properties.

Proposition 3.11

1. For $u \geq 2$, if $m|M_{2^u}$, then $t(m)|3(2^{u-1})$.
2. For $u \geq 1$, if $m|M_{5^u}$, then $t(m)|4(5^u)$.
3. If r is the largest integer such that $\pi(p^r)$=$\pi(p)$, and $m|M_{p^u}$ for some prime p, then t$(m)|p^{u-r}\pi(p)$, for all $u > r$.
4. If $p \neq 5$ is a prime and $m|M_p$, then $t(m)|p^2 - 1$.
5. If p is a prime of the form $10k \pm 1\ and\ m|M_p$, then $t(m)|p - 1$.
6. If prime p is of the form $10k \pm 3$ and $m|M_p$, then $t(m)|2p + 2$.

Proof

1. For $u \geq 2$, if $m|M_{2^u}\ then\ m|2^{2^u} - 1$. Hence by Proposition 3.10 (3), $t(m)|\pi(2^u)$. Result now follows by ([8], Prop 3.5).
2. Follows as in (1) above from Proposition 3.10 (3) and ([8], Prop 3.7).
3. Follows as in (1) above from Proposition 3.10 (3) and ([8], Prop 3.8).
4. If $p \neq 5$ is a prime and $m|M_p$, then $m|2^p - 1$. Then by Proposition 3.10 (3), $t(m)|\pi(p)$. Hence $\pi(m)|p^2 - 1$ by ([8], Prop 3.11).
5. Follows as in (4) above from Proposition 3.10 (3) and ([8], Prop 3.12).
6. Follows as in (4) above from Proposition 3.10 (3) and ([8], Prop 3.13).

4. Conclusion

The new sequence defined using nonlinear second order recurrence relation has most of the identities satisfied by Fibonacci sequence. However the congruence properties of this sequence are different from those of Fibonacci sequence.

References

[1] Z.Akyuz, S. Halici, *On Some Combinatorial Identities involving the terms of generalized Fibonacci and Lucas sequences*, Hacettepe Journal of Mathematics and Statistics, Volume 42 (4) (2013), 431-435.

[2] A.T. Benjmin, J.J. Quinn, Proofs that really count: The Art of Combinatorial Proof, Mathematical Association of America, Washington, D.C., 2003.

[3] A.T. Benjmin, J.J. Quinn, The Fibonacci Numbers Exposed More Discretely, Mathematics Magazine 76:3 (2003), 182-192.

[4] D. Burton, Elementary Number Theory, 6th edition, Tata McGraw-Hill, 2006.

[5] M. Edson, O. Yayenie, A new generalization of Fibonacci sequence and extended Binet's formula, Integers, Volume 9, Issue 6, Pages 639-654, ISSN (Print) 1867-0652.

[6] J. Kappraff, G.W. Adamson, Generalized Binet Formulas, Lucas polynomials and Cyclic constants, Forma,19,(2004) 355-366.

[7] M. Renault, The Fibonacci sequence under various moduli, Masters Thesis, 1996.

[8] S. Vajda, Fibonacci and Lucas numbers and the Golden section: Theory and Applications, Dover Publications, 2008.